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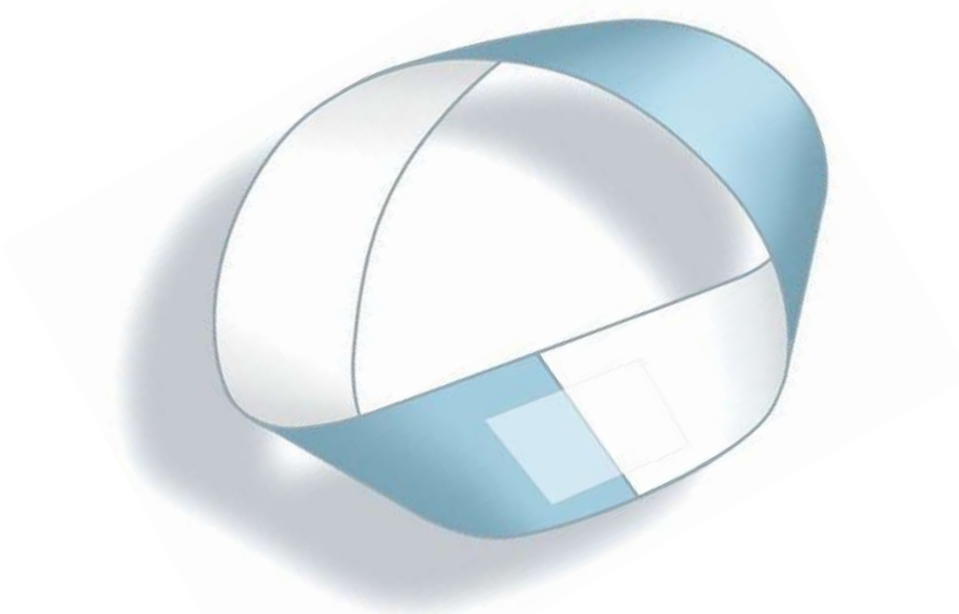

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Interval- Valued Intuitionistic Neutrosophic Sets , Interval-valued Intuitionistic Neutrosophic Soft Sets And Their Application In Decision Making Problem

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Abstract: In this paper we study the concept of interval-valued intuitionistic neutrosophic sets(IVINsets). Some definitions and operations have been proposed. The notion of . interval-valued intuitionistic neutrosophic soft sets(IVINSsets} are introduced. It is a combination of soft set and interval-valued intuitionistic neutrosophic set. Lastly, an application has been shown with the above concepts in decision making problem.

Keywords: : Interval-Valued Neutrosophic Set, Intuitionistic Neutrosophic Set, Interval-Valued Intuitionistic Neutrosophic Set , Interval-valued intuitionistic Neutrosophic Soft Set, Decision Making Problem.

AMS Classification No : 03E72, 03E75.

1. Introduction: Smarandache proposed neutrosophic logic and neutrosophic sets (*NSs*) in 1999 [9]. A *NS* is a set in which elements of the universe has respective degrees of truth, indeterminacy and falsity. They lie in the nonstandard unit interval of $]0^-, 1^+[$. The uncertainty presented here (i.e, indeterminacy factor) is independent of the truth and falsity values. Wang et al. [10] defined IVN sets and their logic operation rules in 2005. In 2009 Bhowmik and Pal [1] studied the Intuitonistic neutrosophic set and presented various properties of it. In 1999 Motodtsov [7] introduce the concept soft set which was completely a new approach for dealiy with vagueness and uncertainties. Maji [6] introduced neutrosophic soft set by the concept of neutrosophic set and soft set. The concept of intuitonistic neutrosophic soft set [2] was introduced by said and Samarandache in 2013. In 2017, interval-valued neutrosophic soft set was introduced by Deli [5]. Chinnadurai and Bobin [4] introduced Interval Valued Intuitionistic Neutrosophic Soft Set and its Application on Diagnosing Psychiatric Disorder by Using Similarity Measure in 2021.

This paper is an attempt to introduce the concept in interval-valued intuitonistic neutrosophic set and interval-valued intuitonistic neutrosophic soft set which are different from [4]. We introduce same basic definition and operations on it. We also introduce the concept of interval-valued intuitonistic neutrosophic soft set. It is a combination of the concepts of interval-valued intuitonistic set and soft set. In application is also presented. The organization of this paper is as follows: in section 2 we briefly present some basic definitions and results which will be used in the result of the paper. In section 3 *IVIN* set and *IVINS* set are defined. In section 4 an application of *IVINS* set in a decision making problem has been shown. Conclusion are there in the section 5.

2. Preliminary and Basic Definition.

In this section we recall some basic definitions and results for our future work.

Definition 2.1 [9] Let U be a universe of elemints the neutrosophic set A is an object having the form $A = \{<x, T_A(x), I_A(x), F_A(x)>: x\in U\}$. the function $T, I, F; U\to[0, 1]$ define respectively the degree of membership, the degree of indeterminacy and degree of non-membership of the element $x\in U$ to the set A. Here $^{-}0\leq T_A(x)+I_A(x)+ F_A(x)\leq 3^{+}$. From philosophical point of view, the neutrosophic set takes the value from the real standard or non-standard sub set of $]^{-}0, 1^{+}[$. But we need to take the interval [0, 1] for technical application $]^{-}0, 1^{+}[$ will be different to be apply in the real applications such as in scientific and engineering problems.

Definition 2.2 [9] A neutrosophic set A is contained in another neutrosophic set Bi.e$A\subseteq B$ if $\forall$ $x\in U$, $T_A(x)\leq T_B(x)$, $I_A(x)\leq I_B(x)$ and $F_A(x)\geq F_B(x)$.

Definition 2.3 [8] Let U be the non-empty fixed set. An interval valued neutrosophic set(IVNS) A in U is of the form $A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in U\}$

where$T_A(x) = \left[T_A^l(x), T_A^r(x)\right], I_A(x) = \left[I_A^l(x), I_A^r(x)\right]$, and $F_A(x) = [F_A^l(x), F_A^r(x)]$

Which represents the degree of membership function, indeterminacy function and non-

membership function for each part $x \in U$ in to the set A where for each element $x \in$

$U, T_A(x) \in Int[0, 1]$, $I_A(x) \in Int[0, 1]$, $F_A(x) \in Int[0, 1]$, where Int([0, 1]) denotes the set

of all closed sub intervals of [0, 1].

Definition 2.4 [8]The complement of an IVN set A = {<x, [$T_A^l(x)$, $T_A^r(x)$], [$I_A^l(x)$, $I_A^r(x)$], [$F_A^l(x)$, $F_A^r(x)$]>: $x\in U$} is denoted by

A^c={<x, [$F_A^l(x)$, $F_A^r(x)$], [$I_A^l(x)$, $I_A^r(x)$], [$T_A^l(x)$, $T_A^r(x)$]>: $x\in U$}. The maximum of an IVN set is {<x, [1, 1], [0, 0], [0, 1]> $x\in U$} and the minimum is {<x, [0, 0], [0, 0], [1, 1]> $x\in U$}.

Definition 2.5 [1,2] An element x of U is called significant with respect to neutrosophic set A of U if the degree of true-membership or indeterminacy membership or falsity membership value i.e, $T_A(x)$ or $I_A(x)$ or $F_A(x)\leq 0.5$, otherwise we call it insignificant. For neutrosophic set the true-membership, indeterminacy membership and falsity membership all cannot be significant.

An intuitinistic neutrosophic set A is defined by

A = {<x, $T_A(x)$, $I_A(x)$, $F_A(x)$>:$x\in U$}

Here min{ $T_A(x)$, $F_A(x)$} ≤ 0.5

min{ $T_A(x)$, $I_A(x)$} ≤ 0.5

min{ $F_A(x)$, $I_A(x)$} ≤ 0.5, for all $x\in U$.

also $0\leq T_A(x)+ I_A(x)+ F_A(x)\leq 2$.

Definition 2.6 [3,7] Let U is an initial universe set. E is a set of parameters. Let $P(U)$ denotes the power set of U. Consider a non-empty set A, $A\subseteq E$. A pair (F, A) is called a soft set over U. F is a mapping given by $F:A\rightarrow P(A)$.

3. Interval-valued intuitinistic neutrosophic set and Interval-valued intuitinistic neutrosophic soft set.

In this section we proposed the notion of *IVIN* set. We also study on hybrid structure involving both IVIN set and soft set. In the definitions we apply the concept of Bhowmik and Pal [1] and the concept of Saha and Said [8].

Definition 3.1 Let U be a space of points (objects) with $x \in U$. An interval-valued intuitinistic neutrosophic set (*IVINS*et) in U is characterized by truth membership function $T_A(x)$, indeterminacy membership function $I_A(x)$ and falsity membership function $F_A(x)$. For each $x \in U$, $T_A(x)$, $I_A(x)$, $F_A(x) \in$ Int [0, 1].

$A = \{<x, [T_A^l(x), T_A^r(x)], [I_A^l(x), I_A^r(x)], [F_A^l(x), F_A^r(x)]>: x \in U\}$

Here $0 \leq T_A^l(x)+I_A^l(x)+F_A^l(x) \leq 2$

And $0 \leq T_A^r(x)+I_A^r(x)+F_A^r(x) \leq 2 \ldots\ldots (A)$

With the condition

$\min\{\frac{T_A^l(x)+T_A^r(x)}{2}, \frac{F_A^l(x)+F_A^r(x)}{2}\} \leq 0.5$

$\min\{\frac{T_A^l(x)+T_A^r(x)}{2}, \frac{I_A^l(x)+I_A^r(x)}{2}\} \leq 0.5$

$\min\{\frac{I_A^l(x)+I_A^r(x)}{2}, \frac{F_A^l(x)+F_A^r(x)}{2}\} \leq 0.5$

The condition (A) can be replaced by

$0 \leq \sup T_A(x)+\sup I_A(x)+\sup F_A(x) \leq 2$.

Example 3.2 Assume that the universe of discourse $U=\{u_1, u_2, u_3\}$ where u_1, u_2 and u_3 are subsets of [0, 1] and they are obtained from some question arise of some experts and impose their opinion in three components. The interval degree of goodness, the interval degree of indeterminacy and the interval degree of poorness to explain the characteristics of the object. Suppose A is an *IVIN* set of U then

$A= \{<u_1, [0.2, 0.4], [0.4, 0.6], [0.3, 0.5]>, <u_2, [0.3, 0.5], [0.1, 0.3], [0.4, 0.8]>, <u_3, [0.4, 1], [0.2, 0.4], [0.4, 0.6]>\}$

For u_1

Here $\min\{\frac{0.2+0.4}{2}, \frac{0.3+0.5}{2}\}$ $=\min\{0.3, 0.4\}\leq 0.5$

$\min\{\frac{0.2+0.4}{2}, \frac{0.4+0.6}{2}\}$ $=\min\{0.3, 0.5\}\leq 0.5$

$\min\{\frac{0.3+0.5}{2}, \frac{0.4+0.6}{2}\}$ $=\min\{0.4, 0.5\}\leq 0.5$

Similarly for u_2 and u_3.

The maximum of an IVN set is <[1, 1], [0, 0], [0, 1]>and the minimum is < [0, 0], [0, 0], [1, 1]>. Here the truth membership interval and falsity membership interval are altered while the indeterminacy membership interval is unchanged.

Definition 3.3 Let $A = \{<x, [T_A^l(x), T_A^r(x)], [I_A^l(x), I_A^r(x)], [F_A^l(x), F_A^r(x)]>: x\in U\}$ be an *IVIN* set. Then the compliment of A is denoted by

$A^c=\{<x, [F_A^l(x), F_A^r(x)], [I_A^l(x), I_A^r(x)], [T_A^l(x), T_A^r(x)]>: x\in U\}$.

Consider the example 3.2.

Here $A= \{<x_1, [0.2, 0.4], [0.4, 0.6], [0.3, 0.5]>, <x_2, [0.3, 0.5], [0.1, 0.3], [0.4, 0.8]>, <x_3, [0.4, 0.5], [0.2, 0.4], [0.4, 0.6]>\}$.

Then $A^c=\{<x_1, [0.3, 0.5], [0.4, 0.6], [0.2, 0.4]>, <x_2, [0.4, 0.8], [0.1, 0.3], [0.3, 0.5]>, <x_3, [0.4, 0.6], [0.2, 0.4], [0.4, 0.5]>\}$. Then A^c is also an *IVIN* set.

Note: If we define the complement of A by $A^c=\{<x, [F_A^l(x), F_A^r(x)], [1-I_A^r(x), 1-I_A^l(x)], [T_A^l(x), T_A^r(x)]>: x\in U\}$ {like previous literature]

Then consider the example 3.2. Then $A^c=\{<x_1, [0.3, 0.5], [0.4, 0.6], [0.2, 0.4]>, <x_2, [0.4, 0.8], [0.1, 0.3], [0.3, 0.5]>, <x_3, [0.4, 0.6], [0.2, 0.4], [0.4, 0.5]>\}$.

Here for x_2,

$\min\{\frac{0.4+0.8}{2}, \frac{0.7+0.9}{2}\} = \min\{0.6, 0.8\} \nleq 0.5$

So A^c is not an *IVIN* set.

Definition 3.4 Let U be an initial universe $A \subseteq E$ be a set of parameters. Let $IVIN(U)$ denotes the set of all interval-valued intuionistic neutrosophic sets of U. The collection of (F, A) performed to be the interval-valued Intuitonisticneutrosophic soft set over U. Here F is a mapping given by $F{:}A \rightarrow IVIN(U)$.

Example 3.5 Let U be the set of hours under consideration. E is the set of parameters for qualities. Each parameter is an interval-valued intuitonistic neutrosophic word or sentence involving interval-valued intuitionistic neutrosophic words.

Consider $E=\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ the set of parameters. There are five elements in the universe U. $U=\{u_1, u_2, u_3, u_4, u_5\}$. $A \subseteq E$ where $A=\{e_1, e_2, e_3, e_4\}$ e_1 stands for expensive, e_2 stands for green surrounding, e_3 stands for made of wooden, e_4 stands for cheap.

Suppose that

F(expensive) = {<u_1, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]>, <u_2, [0.3, 0.5], [0.5, 0.9], [0.1, 0.3]>, <u_3, [0.5, 0.7], [0.1, 0.3], [0.2, 0.4]>, <u_4, [0.5, 0.9], [0.2, 0.4], [0.1, 0.3]>, <u_5, [0.6, 1], [0.1, 0.3], [0.2, 0.4]>}.

F(green surrounding) = {<u_1, [0.4, 0.8], [0.2, 0.4], [0.4, 0.6]>, <u_2, [0.5, 0.9], [0.3, 0.5], [0.2, 0.4]>, <u_3, [0.7, 0.9], [0, 0.2], [0.1, 0.3]>, <u_4, [0.5, 0.9], [0, 0.2], [0.2, 0.4]>, <u_5, [0.6, 1], [0.2, 0.4], [0.3, 0.5]>}.

F(made of wooden) = {<u_1, [0.5, 0.9], [0.3, 0.5], [0.2, 0.4]>, <u_2, [0.4, 0.8], [0, 0.2], [0.1, 0.3]>, <u_3, [0.4, 1], [0.1, 0.3], [0.4, 0.6]>, <u_4, [0.3, 0.7], [0.1, 0.3], [0.4, 0.8]>, <u_5, [0.4, 1], [0.2, 0.4], [0.1, 0.3]>}.

F(Cheap) = {<u_1, [0.6, 1], [0, 0.2], [0.3, 0.5]>, <u_2, [0.2, 0.6], [0.1, 0.3], [0.4, 0.8]>, <u_3, [0.2, 0.4], [0.4, 0.8], [0.3, 0.5]>, <u_4, [0.2, 0.6], [0.6, 1], [0.4, 0.6]>, <u_5, [0.2, 0.4], [0.4, 0.6], [0.4, 1]>}.

Definition 3.6 Let A and B are two *IVIN* sets. A = {<x, [$T_A^l(x)$, $T_A^r(x)$], [$I_A^l(x)$, $I_A^r(x)$], [$F_A^l(x)$, $F_A^r(x)$]>: $x\in X$} where $0\leq \sup T_A(x)+\sup I_A(x)+\sup F_A(x)\leq 2$ and B = {<x, [$T_B^l(x)$, $T_B^r(x)$], [$I_B^l(x)$, $I_B^r(x)$], [$F_B^l(x)$, $F_B^r(x)$]>: $x\in X$} where $0\leq \sup T_B(x)+\sup I_B(x)+\sup F_B(x)\leq 2$. Then A is contained in Bi.e$A\subseteq B$ if and only if

$$T_A^l(x)\leq T_B^l(x),\ T_A^r(x)\leq T_B^r(x), \text{ for all } x\in X$$

$$I_A^l(x)\geq I_B^l(x),\ I_A^r(x)\geq I_B^r(x), \text{ for all } x\in X$$

$$F_A^l(x)\geq F_B^l(x),\ F_A^r(x)\geq F_B^r(x), \text{ for all } x\in X$$

Definition 3.7 The union of two *IVIN* sets A and B is defined as $C=A\cup$B, where true-membership, indeterminacy membership and falsity membership functions related n those of A and B by

$$T_C^l(x)=\max\{T_A^l(x),T_B^l(x)\},\ T_C^r(x)=\max\{T_A^r(x),T_B^r(x)\}, \text{ for all } x\in X$$

$$I_C^l(x)=\min\{I_A^l(x),I_B^l(x)\},\ I_C^r(x)=\min\{I_A^r(x),I_B^r(x)\}, \text{ for all } x\in X$$

$$F_C^l(x)=\min\{F_A^l(x),F_B^l(x)\},\ F_C^r(x)=\min\{F_A^r(x),F_B^r(x)\}, \text{ for all } x\in X$$

Definition 3.8 The intersection of two *IVIN* sets A and B is defined as $D=A\cap$B, where true-membership, indeterminacy membership and falsity membership functions related n those of A and B by

$$T_D^l(x)=\min\{T_A^l(x),T_B^l(x)\},\ T_D^r(x)=\min\{T_A^r(x),T_B^r(x)\}, \text{ for all } x\in X$$

$$I_D^l(x)=\max\{I_A^l(x),I_B^l(x)\},\ I_D^r(x)=\max\{I_A^r(x),I_B^r(x)\}, \text{ for all } x\in X$$

$$F_D^l(x)=\max\{F_A^l(x),F_B^l(x)\},\ F_D^r(x)=\max\{F_A^r(x),F_B^r(x)\}, \text{ for all } x\in X$$

Example 3.9 Let A and B are two *IVIN* sets of U. A = {(<u_1, [0.2, 0.4], [0.4, 0.6], [0.3, 0.5]>), (<u_2, [0.3, 0.5], [0.1, 0.3], [0.4, 0.8]>), (<u_3, [0.4, 1], [0.2, 0.4], [0.4, 0.6]>)} and B = {(<u_1, [0.6, 1], [0.0, 0.2], [0.3, 0.5]>), (<u_2, [0.2, 0.6], [0.1, 0.3], [0.4, 0.8]>), (<u_3, [0.2, 0.4], [0.4, 0.8], [0.3, 0.5]>)}

Where $U=\{u_1, u_2, u_3\}$ be the universe then $A \cup B$

$=\{(<u_1, [0.6, 0.1], [0.0, 0.2], [0.3, 0.5]>), (<u_2, [0.3, 0.6], [0.1, 0.3], [0.4, 0.8]>), (<u_3, [0.4, 1], [0.2, 0.4], [0.3, 0.5]>)\}$

Here for u_1, $\min(\frac{0.6+1}{2}, \frac{0+0.2}{2}) = \min(0.8, 0.1) = 0.1 \leq 0.5$

$\min(\frac{0.6+1}{2}, \frac{0.3+0.5}{2}) = \min(0.8, 0.4) = 0.4 \leq 0.5$

$\min(\frac{0.3+0.5}{2}, \frac{0+0.2}{2}) = \min(0.4, 0.1) = 0.1 \leq 0.5$

Similarly for u_2 and u_3

So $A \cup B$ is also a IVIN set.

$A \cap B=\{(<u_1, [0.2, 0.4], [0.4, 0.6], [0.3, 0.5]>), (<u_2, [0.2, 0.5], [0.1, 0.3], [0.4, 0.8]>), (<u_3, [0.2, 0.4], [0.4, 0.8], [0.4, 0.6]>)\}$

Here $A \cap B$ is also an IVIN set.

Definition 3.10: Comparison matrix: The comparison matrix is a matrix whose rows are labeled by the object names of the universe such as $u_1, u_2, \ldots u_n$ and the columns are labeled by the parameters $e_1, e_2, \ldots, e_m$. The entries are calculated by $c_{ij}=a+b-c$, where a is the integer calculated as low many times $\frac{T^l_{u_i}(e_j)+T^r_{u_i}(e_j)}{2}$ exceeds or equal to $\frac{T^l_{u_k}(e_j)+T^r_{u_k}(e_j)}{2}$ for $u_i \neq u_k$,

$\forall\, u_k \in U\ \{where\ T_{U_i}(e_j) = [T^l_{u_i}(e_j), T^r_{u_i}(e_j)]\}$

b is the integer calculated as how many times $\frac{I^l_{u_i}(e_j)+I^r_{u_i}(e_j)}{2}$

exceeds or equal to $\frac{I^l_{u_k}(e_j)+I^r_{u_k}(e_j)}{2}$ for $u_i \neq u_k$, $\forall\, u_k \in U$

$\{where\ I_{U_i}(e_j) = [I^l_{u_i}(e_j), I^r_{u_i}(e_j)]\}$

c is the integer calculated as how many times $\frac{F^l_{u_i}(e_j)+F^r_{u_i}(e_j)}{2}$ exceeds or equal to $\frac{F^l_{u_k}(e_j)+F^r_{u_k}(e_j)}{2}$ for $u_i \neq u_k$, $\forall\, u_k \in U$

$\{where\ F_{U_i}(e_j) = [F^l_{u_i}(e_j) + F^r_{u_i}(e_j)]\}$.

4. An application of IVIN soft set in decision making problem.

Let us consider the example 3.5. Then the tabular representation of the IVIN soft set is (F, A) is given by the following table.

e_1 expensive	e_2 green surrounding	e_3 made of wooden	e_4 cheap
[0.4, 0.6], [0.5, 0.7], [0.2, 0.4]	[0.4, 0.6], [0.2, 0.4], [0.4, 0.6]	[0.5, 0.9], [0.3, 0.5], [0.2, 0.4]	[0.6, 1], [0, 0.2], [0.3, 0.5]
[0.3, 0.5], [0.5, 0.9], [0.1, 0.3]	[0.5, 0.9], [0.3, 0.5], [0.2, 0.4]	[0.4, 0.8], [0, 0.2], [0.1, 0.3]	[0.2, 0.6], [0.1, 0.3], [0.4, 0.8]
[0.5, 0.7], [0.1, 0.3], [0.2, 0.4]	[0.7, 0.9], [0, 0.2], [0.1, 0.3]	[0.4, 1], [0.1, 0.3], [0.4, 0.6]	[0.2, 0.4], [0.4, 0.8], [0.3, 0.5]
[0.5, 0.9], [0.2, 0.4], [0.1, 0.3]	[0.5, 0.9], [0, 0.2], [0.2, 0.4]	[0.3, 0.7], [0.1, 0.3], [0.4, 0.8]	[0.2, 0.6], [0.6, 1], [0.4, 0.6]
[0.6, 1], [0.1, 0.3], [0.2, 0.4]	[0.6, 1], [0.2, 0.4], [0.3, 0.5]	[0.4, 1], [0.2, 0.4], [0.1, 0.3]	[0.2, 0.4], [0.4, 0.6], [0.4, 1]

The aim is to find out the most suitable house with the choice parameter for X. The algorithm for most appropriate selection of an object will be as follows:

1. Input the IVIN soft set (F, E)
2. Input A, The choice of parameters of μ_r X which is a subset of E.
3. Consider the IVIN soft set (F, A) and write it in the tabulated form
4. Compute the comparison matrix of the IVIN soft set (F, A)

5. Compute the score S_i of $u_i \forall\, i$
6. Find S_k = max S_i
7. If k has more than one value then μ_r may be chosen.

:- Table 1 is how converted to Table 2 in the following way

Table 2

U	e_1	e_2	e_3	e_4
u_1	(0.5, 0.6, 0.3)	(0.5, 0.3, 0.5)	(0.7, 0.4,0.3)	(0.8,0.1,0.4)
u_2	(0.4, 0.7, 0.2)	(0.7,0.4,0.3)	(0.6,0.1,0.2)	(0.4,0.2,0.6)
u_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)	(0.3,0.6,0.4)
u_4	(0.7,0.3,0.2)	(0.7,0.1,0.3)	(0.5,0.2,0.6)	(0.4,0.8,0.5)
u_5	(0.8,0.2,0.3)	(0.8,0.3,0.4)	(0.7,0.3,0.2)	(0.3,0.5,0.7)

The comparison matrix of the IVIN soft set *(F,A)* is represented by the following tabular form

Table 3

U	e_1	e_2	e_3	e_4
u_1	1+3-4=0	0+3-4=-1	4+4-2=6	4+0-1=3
u_2	0+4-1=3	2+4-2=4	4+0-1=3	3+1-3=1
u_3	2+1-4=-1	4+1-0=5	4+2-3=3	1+3-1=3
u_4	3+2-1=4	2+1-2=1	0+2-4=-2	3+4-2=5
u_5	4+1-4=1	4+3-3=4	4+3-1=6	1+2-4=-1

Table-4

We calculate the score for each u_i

U	Score S_i
u_1	8
u_2	11
u_3	10
u_4	8
u_5	10

Clearly the maximum score is 11. Hence the last decision for X is to select u_2.

5. Conclusions

In this paper we study the notion of IVINset and IVINsoft set. We have also defined some operations on IVINsets. Finally we present an application of IVINsoft set in a decision making problem.

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Some New Integral's Representations for the Gauss's Hypergeometric Function with Applications

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Abstract: In this work, we establish novel representations for the Gauss hypergeometric function in different limits, expanding the known methods for expressing and evaluating this function. These representations are derived using some basic important integrations. The obtained results extend classical identities and offer valuable insights into the structure and properties of the hypergeometric function, with potential implications for various areas of mathematical analysis, physics and engineering. Additionally, a few special examples have also been given.

2020 Mathematical subject classification;33C05, 33C20

Keywords: Hypergeometric function, Gauss's formula, Whipple's formula, Definite Integration.

1. INTRODUCTION

The Gauss hypergeometric function plays a fundamental role in various branches of mathematics and physics, including differential equations, special functions, and applied sciences. We are motivated by recent advancements in integrals involving the product of two generalized hypergeometric functions, as well as double integrals involving generalized hypergeometric functions, as obtained by Basnet et al [1], [2]. In this work, we present novel integral representations for the Gauss hypergeometric function, derived using advanced techniques in integration. By bridging theoretical insights with computational efficiency, this study aims to enrich the existing framework of hypergeometric functions and contribute to their broader applicability in interdisciplinary research.

In 1812, Gauss systematically discussed the series [3] [4] [5]

$$1+\frac{a.b}{c}\frac{x}{1!}+\frac{a\,(a+1).b(b+1)}{c\,(c+1)}\frac{x^2}{2!}+\cdots \qquad (1.1)$$

The series (1.1) is denoted by ${}_2F_1$ (a, b; c; x) or ${}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix};\quad x\right]$ that is

$${}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix};\quad x\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{x^n}{n!} \tag{1.2}$$

In this case, a and b are the numerator parameters, while c is the denominator parameter. For $c \neq 0, -1, -2, \ldots$ and if either a or b is a negative integer, the series in equation (1.2) will terminate The Gauss hypergeometric series (1.2) is [3] [4] [6] [5]

i. converges if $|x| < 1$, diverges if $|x| > 1$,
ii. when $x = 1$, converges if $\mathrm{Re}(c - a - b) > 0$, diverges if $\mathrm{Re}(c - a - b) \leq 0$,
iii. when $x = -1$, converges absolutely if $\mathrm{Re}(c - a - b) > 0$, converges but not absolutely if $-1 \leq Re(c - a - b) > 0$, diverges if $\mathrm{Re}(c - a - b) \leq -1$.

The generalized hypergeometric function with p numerator and q denominator parameters is defined as [3] [4] [7]

$${}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{matrix};\quad x\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\,\frac{x^n}{n!}\,. \tag{1.3}$$

Pochhammer symbol: In series (1.2) and (1.3) where $(a)_n$ denotes the Pochhammer symbol with its usual representation in terms of Gamma function defined by [3] [5]

$$(a)_n = \prod_{k=1}^{n}(a + k - 1), (a)_0 = 1,\ (1)_n = n!, (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \text{ and}$$

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a}{2} + \frac{1}{2}\right)_n. \tag{1.4}$$

where n is a non- negative integer. Pochhammer symbol was introduced by the German mathematician Leo Pochhammer (1841 – 1920). Hypergeometric function reduces to the gamma function, results are very important from the application point view.

Gamma function: The Gamma function, denoted by Γ(x), exists for positive, negative, and complex values of x, except at $x = 0, -1, -2, -3, \ldots$ and it is defined by [3]

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt,\quad \mathcal{R}(x) > 0 \text{ and } \frac{\Gamma(z)}{\Gamma(z- n)} = (-1)^n\,\frac{\Gamma(-z+n+1)}{\Gamma(-z+1)}. \tag{1.5}$$

Beta Function: Beta function of m and n is denoted by $B(m,n)$ and defined by [3]

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \mathcal{R}(m) > 0,\ \mathcal{R}(n) > 0.$$

2. SOME CLASSICAL SUMMATION FORMULAS AND IMPORTANT INTEGRALS:

This section explores some fundamental summation formulas and important integrals, highlighting their mathematical significance and applications.

2.1. Some important and applicable definite integrals are [8]

$$\int_0^1 [(1+t)^{x-1}(1-t)^{y-1} + (1-t)^{x-1}(1+t)^{y-1}]\,\mathrm{dx} = 2^{x+y-1}B(x,y)\,. \quad (2.1)$$

$$\int_0^1 x^{c-1}(1-x)^{d-1}(1+b\,t)^{-c-d}\mathrm{dx} = (1+b)^{-c}B(c,d)\,. \quad (2.2)$$

$$\int_b^a (t-b)^{x-1}(a-t)^{y-1} = (a-b)^{x+y-1}B(x,y)\,. \quad (2.3)$$

$$\int_p^q (x-p)^{c-1}(q-x)^{d-1}(x-r)^{-c-d}dx = \frac{(q-p)^{c+d-1}}{(q-r)^c(p-r)^d}B(c,d)\,. \quad (2.4)$$

$$\int_{-1}^1 (1+x)^{2c-1}(1-x)^{2d-1}(1+x^2)^{-c-d}\,dx = 2^{c+d-2}B(c,d)\,. \quad (2.5)$$

2.2. **Classical summation formula:**

There are many classical summation formulas in hypergeometric functions; however, we list only a few that are used in our main results [3] [7] [9].

Gauss's summation formula: If $Re(c-a-b) > 0$, ${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix};\ 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. (2.6)

Whipple's summation formula: When $a+b=1$ and $e+f=2c+1$,

$${}_3F_2\left[\begin{matrix} a,\ b, c \\ e, f \end{matrix};\ 1\right] = \frac{\pi\ \Gamma(e)\,\Gamma(f)}{2^{2c-1}\,\Gamma\left(\frac{a+e}{2}\right)\Gamma\left(\frac{a+f}{2}\right)\Gamma\left(\frac{b+e}{2}\right)\Gamma\left(\frac{b+f}{2}\right)}\,. \quad (2.7)$$

3. MAIN RESULTS:

In this section, we shall establish some new integrals representations for the Gauss hypergeometric function in different limits of integrals are asserted in the following Theorem.

Theorem 3.1: The following integrals representation for the Gauss hypergeometric function hold true

i. $\int_0^1 \left[(1+x)^{c-\frac{3}{2}}(1-x)^{c-1} + (1-x)^{c-\frac{3}{2}}(1+x)^{c-1}\right] \times {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \quad (1-x^2)\right] dx$

$$= \pi\sqrt{2} \frac{\Gamma(c)\ \Gamma\left(c-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{4}\right)\Gamma\left(c+\frac{1}{4}\right)}{\Gamma\left(2c-\frac{1}{2}\right)\Gamma\left(\frac{a+c}{2}-\frac{1}{8}\right)\Gamma\left(\frac{a+c}{2}+\frac{1}{8}\right)\Gamma\left(\frac{b+c}{2}-\frac{1}{8}\right)\Gamma\left(\frac{b+c}{2}+\frac{1}{8}\right)} \left(Re(c) > \frac{1}{2}\right). \tag{3.1}$$

$ii.$ $\int_0^1 [x^{c-1}(1-x)^{d-1}(1+bx)^{-c-d-n}]\ {}_2F_1\left[\begin{matrix} a, b \\ d \end{matrix}; \quad 1-x\right] dx$

$= \frac{B(c,d)}{(1+b)^c} \frac{\Gamma(c+d)\,\Gamma(c+d-a-b)}{\Gamma(c+d-a)\,\Gamma(c+d-b)}$ $(\,x > -1,\ Re(c), Re(d) > 0$ and $n = 0, 1, 2, \dots)$. (3.2)

Proof (i): Let left-hand side of (3.1) be I, we have

$$I = \int_0^1 \left[(1+x)^{c-\frac{3}{2}}(1-x)^{c-1} + (1-x)^{c-\frac{3}{2}}(1+x)^{c-1}\right]\ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \quad (1-x^2)\right] dx$$

$$= \int_0^1 \left[(1+x)^{c-\frac{3}{2}}(1-x)^{c-1} + (1-x)^{c-\frac{3}{2}}(1+x)^{c-1}\right] \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{(1-x^2)^n}{n!} dx.$$

Changing the order of integration and summation, which is justified due to the uniform convergence of the series and absolute convergent of the integral, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n n!} \int_0^1 \left[(1+x)^{c+n-\frac{1}{2}-1}(1-x)^{c+n-1} + (1-x)^{c+n-\frac{1}{2}-1}(1+x)^{c+n-1}\right] dx$$

Using integration (2.1), we obtain

$$I = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n n!}\ 2^{2c+2n-\frac{3}{2}} \quad B\left(c+n-\frac{1}{2}, c+n\right)$$

Using (1.4) and after a little simplification, we obtain

$$I = 2^{2c-3/2}\frac{\Gamma(c)\,\Gamma(c-1/2)}{\Gamma(2c-1/2)}\ {}_3F_2\left[\begin{matrix}a, b,\ c-\frac{1}{2};\\ c-\frac{1}{4}, c+\frac{1}{4}\end{matrix}\ 1\right].$$

We now observe that the ${}_3F_2$ can now be evaluated with the help of Whipple's formula (2.7) and we arrive at the right-hand side of (3.1). This completes the proof of the result (3.1) asserted in the theorem 3.1 (i).

Similarly, to prove theorem 3.1 (ii), we use integration (2.2) and Gauss's summation theorem (2.6) and followed by a method similar to that used in theorem 3.1(i).

Theorem 3.2: The following integrals representation for the Gauss hypergeometric function hold true for $Re(1-a-b-d)>0$ and $r<p<q$

i. $$\int_p^q (x-p)^{c-1}(q-x)^{d-1}\ {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix};\ -\frac{x-p}{q-x}\right]dx$$

$$= (q-p)^{c+d-1}B(c,d)\ \frac{\Gamma(1-d)\,\Gamma(1-a-b-d)}{\Gamma(1-a-d)\,\Gamma(1-b-d)}\ . \tag{3.3}$$

ii. $$\int_p^q (x-p)^{c-1}(q-x)^{d-1}(x-r)^{-c-d}\ {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix};\ -\frac{(q-r)(x-p)}{(p-r)(q-x)}\right]dx\ .$$

$$= B(c,d)\,\frac{(q-p)^{c+d-1}}{(q-r)^c(p-r)^d}\ \frac{\Gamma(1-d)\,\Gamma(1-d-a-b)}{\Gamma(1-d-a)\,\Gamma(1-d-b)}\ . \tag{3.4}$$

Proof (i): Let left - hand side of (3.3) be I, we have

$$I = \int_p^q (x-p)^{c-1}(q-x)^{d-1}\ {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix};\ -\frac{x-p}{q-x}\right]dx$$

$$I = \int_p^q (x-p)^{c-1}\left(q-x\right)^{d-1}\ \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\ \frac{(-1)^n(x-p)^n}{(q-x)^n\,n!}\,dx$$

Changing the order of integration and summation, which is justified due to the uniform convergence of the series and absolute convergent of the integral, we have

$$I = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\ \frac{(-1)^n}{n!}\int_p^q (x-p)^{c+n-1}\,(q-x)^{d-n-1}\ dx$$

Using integration (2.3), we obtain

$$I = (q - p)^{c+d-1} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(-1)^n}{n!} \frac{\Gamma(c+n)\Gamma(d-n)}{\Gamma(c+d)}.$$

Using (1.5) then (1.3), we obtain

$$I = (q - p)^{c+d-1} B(c, d)\, {}_2F_1\left[\begin{matrix} a, b \\ 1-d \end{matrix}; \quad 1\right].$$

We now observe that the ${}_2F_1$ can now be evaluated with the help of Gauss's formula (2.6) and we arrive at the right-hand side of (3.3). This completes the proof of the result (3.3) asserted in the theorem 3.2 (i).

Similarly, to prove theorem 3.2 (ii), we use integration (2.4) and Gauss's summation theorem (2.6) and followed by a method similar to that used in theorem 3.2(i).

Theorem 3.3: The following integral representation for the Gauss hypergeometric function holds true for $Re(1-a-b-d) > 0, Re(c) > 0, Re(d) > 0$

$$\int_{-1}^{1} (1+x)^{2c-1} (1-x)^{2d-1} (1+x^2)^{-c-d}\, {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \quad \left(\frac{1+x}{1-x}\right)^2\right] dx =$$

$$2^{c+d-2} B(c, d) \frac{\Gamma(1-d)\,\Gamma(1-a-b-d)}{\Gamma(1-a-d)\,\Gamma(1-b-d)}. \qquad (3.5)$$

Proof : Let left-hand side of (3.5) be I, we have

$$I = \int_{-1}^{1} (1+x)^{2c-1} (1-x)^{2d-1} (1+x^2)^{-c-d}\, {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \quad -\left(\frac{1+x}{1-x}\right)^2\right] dx$$

$$I = \int_{-1}^{1} (1+x)^{2c-1} (1-x)^{2d-1} (1+x^2)^{-c-d} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(1+x)^{2n}(-1)^n}{(1-x)^{2n}\, n!} dx.$$

Changing the order of integration and summation, which is justified due to the uniform convergence of the series and absolute convergent of the integral, we have

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n (b)_n}{(c)_n\, n!} \int_{-1}^{1} (1+x)^{2c+2n-1} (1-x)^{2d-2n-1} (1+x^2)^{-c-d}\, dx.$$

Using the integration result (2.5), followed by applying (1.2) and (1.5), we observe that the ${}_2F_1$ function can now be evaluated using Gauss's formula (2.6), leading us to the right-hand side of (3.5).This completes the proof of the result (3.5) asserted in the theorem 3.3.

4. SPECIAL EXAMPLES

In this section, we will mention a few special cases of our main findings**.**

4.1. In (3.1), if we put , $a = \frac{1}{4}, b = \frac{3}{4}$ and $c = \frac{3}{2}$, then we obtain the following result:

$$\int_0^1 \left[\sqrt{1-x} + \sqrt{1+x}\right] {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{3}{4}; \\ \frac{3}{2}\end{matrix} \quad (1-x^2)\right] dx = \frac{\pi}{\sqrt{2}}. \tag{4.1}$$

4.2. In (3.2), if we put $a = b = 1, c = 2,\ d = 3$, then we obtain the following result:

$$\int_0^1 [x\ (1-x)^2 (1+x)^{-5-n}]\ {}_2F_1\left[\begin{matrix}1, 1; \\ 1\end{matrix} \quad 1-x\right] dx = \frac{1}{36}. \tag{4.2}$$

4.3. In (3.3), if we put $c = \frac{3}{2}$, $a = b = d = \frac{1}{4}$, then we obtain the following result:

$$\int_p^q (x-p)^{1/2} (q-x)^{-3/4}\ {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4}; \\ \frac{3}{2}\end{matrix} \quad -\frac{x-p}{q-x}\right] dx = \frac{2}{3\sqrt{\pi}} (q-p)^{3/4}\, \Gamma^2\left(\frac{1}{4}\right). \tag{4.3}$$

4.4. In (3.4), if we put $c = \frac{5}{4}$, $a = b = d = \frac{1}{4}$, then we obtain the following result:

$$\int_p^q (x-p)^{\frac{1}{4}} (q-x)^{\frac{-3}{4}} (x-r)^{\frac{-3}{2}}\ {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4}; \\ \frac{5}{4}\end{matrix} \quad -\frac{(q-r)(x-p)}{(p-r)(q-x)}\right] dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{q-p}}{(q-r)^{5/4}(p-r)^{1/4}}\, \Gamma^2\left(\frac{1}{4}\right). \tag{4.4}$$

4.5. In (3.5), if we put $c = \frac{5}{4}$, $a = b = d = \frac{1}{4}$, then we obtain the following result:

$$\int_{-1}^1 (1+x)^{\frac{3}{2}} (1-x)^{\frac{-1}{2}} (1+x^2)^{\frac{-3}{2}}\ {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4}; \\ \frac{5}{4}\end{matrix} \quad \left(\frac{1+x}{1-x}\right)^2\right] dx = \frac{1}{2\sqrt{\pi}} \Gamma^2\left(\frac{1}{4}\right). \tag{4.5}$$

5. CONCLUSION:

In conclusion, this study presents three new theorems providing integral representations for the Gauss hypergeometric function, extending its theoretical scope and potential applications. Each theorem is accompanied by special cases, which illustrate specific instances of these results. These findings not only enhance the understanding of hypergeometric functions but also open pathways for their application in fields such as mathematical physics and engineering. The work offers both theoretical insights and practical tools, laying a foundation for further research in this area.

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Characterizations of Multi Fermatean Fuzzy Lie Ideals

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Abstract: In the present paper, the concept of Multi Fermatean fuzzy (briefly, MFF) Lie sub algebras and Multi Fermatean fuzzy (briefly,MFF) Lie ideals of Lie algebra are introduced. Some of their fundamental properties and operations like intersection and Cartesian product are investigated. Moreover, the relationship between Multi Fermatean fuzzy Lie sub algebras and Multi Fermatean Lie ideals are also established. Finally, the images and the inverse images of both Multi Fermatean fuzzy Lie sub algebras and Multi Fermatean fuzzy Lie ideals under Lie homomorphisms are also studied.

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Key words and phrases: Lie algebra, Lie sub algebra, Multi Fermatean fuzzy set, Fermatean fuzzy sub algebra, Multi Fermatean fuzzy Lie sub algebras, Multi Fermatean fuzzy Lie ideals.

1. Introduction

From historical point of view it is true fact that the modern age is the age of artificial intelligence(AI). In the modern age, science and technology significantly deal with intricate phenomena and processes for which there is inadequate information. In such kind of situations, mathematical models are created for dealing with different types of systems that have uncertain and imprecise components. Many of these models, like Fuzzy sets, soft sets, neutrosophic sets, intuitionistic fuzzy Sets, Fermatean fuzzy sets, Pythagorean fuzzy sets, multi Pythagorean fuzzy sets and many more sets are built on the extensions of standard set theory. In this paper we cover the core features of a Multi Fermatean Fuzzy Lie sub algebra of Lie algebra.

Most probably numerous domains along with signal processing, artificial intelligence, multi agent systems, computer networks, robotics, genetic algorithms, expert systems, neural networks, decision making, medical diagnosis and automata theory shall be benefited with the acquired outcomes. The idea of Lie algebra was first introduced by Sophus Lie (1842–1899) in an effort to categorise certain smooth subgroups of general linear groups [8]. Lie algebras, so named because they were invented by Sophus Lie, are a specific instance of general linear algebra. Following the introduction of this theory, Lie groups were used in mathematics and physics to categorise smooth subgroups. In [[1],[11],[12],[17],[18], [19]], Lie sub algebras and their properties were created and explored in more detail. L.A. Zadeh [10] proposed the concept of fuzzy set (FS) in situations that are vague, imprecise and uncertain. As a generalisation of fuzzy set, K. Atanassov [9] created intuitionistic fuzzy set (IFS) in 1986. His theory thereafter became widely acknowledged as an essential resource in the fields of science, technology, engineering, medicine, etc. In 1995, neutrosophic set (in short,NS) was introduced by F. Smarandache [4] as a generalization of not only intuitionistic fuzzy set but also of inconsistent intuitionistic fuzzy set, Pythagorean fuzzy set, Farmatean fuzzy set, spherical fuzzy set, n-Hyperspherical fuzzy set and so on. In order to create model for vague and imprecise information, the model of Fermatean fuzzy sets (FFS) was introduced by Senapati and Yager [20] which is someway different from IFS model since it involves the condition $0 \leq T^3 + F^3 \leq 1$, where T and F stand for membership and non-membership function respectably. In decision making problems, Fermatean fuzzy set model (FFS model) has significant application proposed by Zamana, F. Ghania, A. Khana, S. Abdullaha and F. Khan [13]. The application range of solving real life problems such as decision making problems in FFS model is popularly increasing than that of the IFS model because number of pairs satisfying the condition $0 \leq T^3 + F^3 \leq 1$ is higher than that of the condition $0 \leq T + F \leq 1$. Smarandache [3] subsequently proposed the model of multi Fermatean Fuzzy set which is the genralization of Fermatean Fuzzy set model.

The present paper is organized in the following manner: Section 1 represents the introduction and literature review of neutrosophic logic.

Section 2 focuses into common definitions and preliminaries. Section 3 describes the concept of Multi Fermatean Fuzzy Lie sub algebras and Multi Fermatean Fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized cartesian product are investigated. Moreover, the relationship between Multi Fermatean fuzzy Lie sub algebras and Multi Fermatean fuzzy Lie ideals are also established. In section 4, we investigate the images and the inverse images of Multi Fermatean fuzzy Lie sub algebras and Multi Fermatean fuzzy Lie ideals under Lie homomorphisms. In section 5, we give the conclusion of the newly defined concept of Multi Fermatean fuzzy Lie sub algebras and Multi Fermatean fuzzy Lie ideals.

2. Prelimineries

This section consists of some common notations and definitions which have been involved in the course of the paper. A Lie algebra is a vector space L over the field F(F = R or C) on which $L \times L \rightarrow L$ defined by $(\gamma, \mu)=[\gamma, \mu]$ for all $\gamma, \mu \in L$ where $[\gamma, \mu]$ is called Lie Bracket satisfying the following conditions: **(1)** $[\gamma, \mu]$ is bilinear **(2)** $[\gamma, \gamma]=0$, for all $\gamma \in L$ **(3)** $[[\gamma, \mu], \lambda]+[[\mu, \lambda], \gamma]+[[\lambda, \gamma], \mu]=0$, for all $\gamma, \mu, \lambda \in L$ (This is called Jacobi identity).

Throughout the paper L will denote Lie algebra and also we note that the operation Lie bracket [.,.] is neither associative nor commutative i.e,$[[\gamma, \mu], \lambda] \neq [[\mu, \lambda],\gamma]$ and $[\gamma, \mu] \neq [\mu, \gamma]$. But the operation Lie bracket [.,.] is anti-commutative i.e. $[\gamma, \mu] = -[\mu, \gamma]$. A subspace H of L is called a Lie sub algebra if it is closed under [. , .]. A subspace I of L is called a Lie ideal if $[I, L] \subset I$. It is always true that every Lie ideal is Lie sub algebra.

Definition 2.1. [10] Let U be a Universe of discourse. Then the fuzzy set (briefly, FS) on U is described as F= $\{(x, \mu(x))$: $x \in X$, where, $\mu(x) \in [0,1]$, denotes the degree of membership of $x \in X\}$.

Definition 2.2. [9] Let U be a Universe of discourse. Then the Intuitonistic fuzzy set on U is described as F= $\{(x, \mu(x), \nu(x))$: $x \in X$, where, $\mu(x), \nu(x) \in [0,1]$, indicating the degrees of membership and non-membership respectively such that $0 \leq \mu_A(x)+\nu_A(x) \leq 1\}$.

Definition 2.3. [20] Let U be a Universe of discourse. Then the Fermatean fuzzy set on U is described as F= {(x, μ(x), ν(x)): x∈X, where, μ(x), ν(x) ∈ [0,1], indicating the degrees of membership and non-membership respectively satisfying $0 \leq (\mu_F(x))^3 + (\nu_F(x))^3 \leq 1$}.

Definition 2.4. [4] Let X be a Universe of discourse. Then the neutrosophic set is defined by N= {(x, τ (x), λ(x), η(x)), x∈X, where, τ , λ, η ∈ [0,1], indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy 0 ≤ inf(τ) + inf(λ)+ inf(η) ≤ sup(τ) + sup(λ)+sup(η) ≤3}.

Definition 2.5. [3] Let X be a universe of discourse. Then, a Multi Fermatean Fuzzy set (shortly, multi PFS) on X is defined by M= {(x, x($T_1,T_2,...,T_p$; $F_1,F_2,...,F_s$)): x∈X, where p and s are integers ≥ 3 with p+s = n ≥3, and at least one of p and s is ≥2, in order to ensure the existence of multiplicity of at least one Fermatean component: truth or falsehood; all subsets $T_1,T_2,...,T_p$; $F_1,F_2,...,F_s \subseteq [0,1]$; $0 \leq (T_i)^3 + (F_k)^3 \leq 1$, for all i=1,2,3,.....,p and k=1,2,3,....,s.

3. Properties of Multi Fermatean fuzzy Lie algebra

In this section, we first define Fermatean fuzzy Lie sub algebras and Fermatean fuzzy Lie ideals and then the notion of Multi Fermatean fuzzy Lie sub algebras and Multi Fermatean fuzzy Lie ideals are initiated. Some characterizations, counter examples, and basic properties are also investigated.

Definition 3.1. A Fermatean Fuzzy set (briefly, PFS) W= (τ, η) on L is said to be Fermatean Fuzzy subalgebra if, ∀ γ,μ ∈L; ∀c∈ F, the following assumptions hold good:
(1) $\tau_W(\gamma+\mu) \geq \mathrm{Min}\{\tau_W(\gamma),\tau_W(\mu)\}$, $\eta_W(\gamma+\mu) \leq \mathrm{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$
(2) $\tau_W(c\gamma) \geq \tau_W(\gamma)$, $\eta_W(c\gamma) \leq \eta_W(\gamma)$
(3) $\tau_W[\gamma,\mu] \geq \mathrm{Min}\{\tau_W(\gamma),\tau_W(\mu)\}$, $\eta_W[\gamma,\mu] \leq \mathrm{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$.

Definition 3.2. A Fermatean Fuzzy set (briefly,PFS) W= (τ, η) on L is said to be Fermatean Fuzzy Lie ideal if, ∀ γ,μ∈L; ∀c∈ F, the conditions (1) and (2) of definition (3.1) and the condition: $\tau_W([\gamma,\mu]) \geq \tau_W(\gamma)$, $\eta_W([\gamma,\mu]) \leq \eta_W(\gamma)$, holds good.

Definition 3.3. Let N_l ={1,2,....,l}, N_n ={1,2,....,n} and L be a Lie Algebra of vectors over the field F. A Multi Fermatean Fuzzy set W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) on L is said to be Multi Fermatean Fuzzy Lie sub algebra over L if, ∀ γ,η ∈L; ∀ i∈N_l ,∀ k∈N_n, ∀c∈ F, the following assumptions hold good:

(1) τ_W^i(γ+μ) ≥ Min{ τ_W^i(γ),τ^iw(μ)}, η_W^k(γ+μ) ≤ Max{ η_W^k(γ), η_W^k(μ)}
(2) τ_W^i(cγ) ≥ τ_W^i(γ), η_W^k(cγ) ≤ η_W^k(γ)
(3) τ_W^i[γ,μ] ≥ Min{τ_W^i (γ), τ_W^i (μ)}, η_W^k[γ,μ] ≤ Max{ η_W^k(γ), η_W^k(μ)}

Definition 3.4. Let N_l={1,2,....,l}, N_n={1,2,....,n} and L be a Lie Algebra of vectors over the field F. A Multi Fermatean Fuzzy set W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) on L is said to be Multi Fermatean Fuzzy Lie ideal if,∀ γ,η ∈L; ∀ i∈N_l ,∀ k∈N_n, ∀c∈ F, the conditions **(1)-(2)** of definition **(3.3)** along with the following conditions are satisfied: ∀ γ,η ∈L, **(4)** τ_W^i[γ,μ] ≥ τ_W^i (γ), η_W^k [γ,μ] ≤ η_W^k (γ). It follows from condition **(2)** that **(5)** τ_W^i (0) ≥ τ_W^i (γ), η_W^k(0) ≤ η_W^k(γ) (6) τ_W^i (-γ) ≥ τ_W^i (γ), η_W^k(-γ) ≤ η_W^k(γ).

Theorem 3.5. Let M=(τ_M^1 , τ_M^2,...,τ_M^l; η_M^1 ,η_M^2,...,η_M^n) be a Multi Fermatean Fuzzy Lie ideal over L and let N_l={1,2,....,l}, N_n={1,2,....,n}, then τ_M^i(0)= Sup{τ_M^i(γ): γ∈L} and η_M^k (0)= Inf{ η_M^k (γ): γ∈L},∀ i∈N_l ,∀ k∈N_n.

Proof. From condition (5) of definition (3.4), we have, τ_M^i(0) ≥ τ_M^i(γ)...........(1), η_M^k(0) ≤ η_M^k(γ).............(2) As γ runs over L, the results follow just taking supremum on both sides of above inequality (1) and infimum on (2).

Theorem 3.6. Let M = (L, $\{\tau_M^i\}_{i=1}^{i}$, $\{\eta_M^k\}_{k=1}^{n}$) be a Multi Fermatean fuzzy Lie ideal over L. Then for each ψ, σ ∈[0,1] satisfying τ_M^i (0) ≥ ψ, η_M^k (0) ≤ σ and 0 ≤ ψ+ σ ≤ 1, the (ψ, σ)-level subset $L_M^{(\psi,\sigma)}$ is a Multi Fermatean fuzzy Lie ideal of L.

Proof. Straight forward from definition.

Theorem 3.7. If δ is a fixed element of L and M = (L, $\{\tau_M^i\}_{i=1}^l$, $\{\eta_M^k\}_{k=1}^n$) is a Multi Fermatean Fuzzy Lie ideal of L. Then the set defined by $M^\delta = \{\gamma \in L: \tau_M^i(\gamma) \geq \tau_M^i(\delta), \eta_M^k(\gamma) \leq \eta_M^k(\delta)\}$ is a Multi Fermatean Fuzzy Lie ideal of L.

Proof. Suppose that $\gamma,\mu \in M^\delta$, $i \in N_l$, $k \in N_n$. Then $\forall \gamma,\mu \in M^\delta$, $\forall\, i \in N_l$, $\forall\, k \in N_n$
$\tau_M^i(\gamma+\mu) \geq \mathrm{Min}\{\tau_M^i(\gamma), \tau_M^i(\mu)\} \geq \tau_M^i(\delta)$;
$\eta_M^k(\gamma+\mu) \leq \mathrm{Max}\{\eta_M^k(\gamma), \eta_M^k(\mu)\} \leq \eta_M^k(\delta)$.
This implies that $\gamma+\mu \in M^\delta$.
Now, $\forall \gamma \in M^\delta, \forall\, i \in N_l, \forall\, k \in N_n, \forall c \in F$,
$\tau_M^i(c\gamma) \geq \tau_M^i(\gamma) \geq \tau_M^i(\delta)$;
$\eta_M^k(c\gamma) \leq \eta_M^k(\gamma) \leq \eta_M^k(\delta) \Rightarrow c\gamma \in M^\delta$.
Also for every $\gamma \in M^\delta$ and for every $\mu \in M^\delta$, $\forall\, i \in N_l$, $\forall\, k \in N_n$,
$\tau_M^i[\gamma,\mu] \geq \mathrm{Min}\{\tau_M^i(\gamma), \tau_M^i(\mu)\} \geq \tau_M^i(\delta)$; $\eta_M^k[\gamma,\mu] \leq \mathrm{Max}\{\eta_M^k(\gamma), \eta_M^k(\mu)\} \leq \eta_M^k(\delta)$, which shows that $[\gamma,\eta] \in M^\delta$. Hence, M^δ is a Multi Fermatean Fuzzy Lie ideal of L.

Theorem 3.8. If M = (L, $\{\tau_M^i\}_{i=1}^l$, $\{\eta_M^k\}_{k=1}^n$) is a Multi Fermatean Fuzzy Lie ideal of L. Then the set defined by $M^0 = \{\gamma \in L: \tau_M^i(\gamma) \geq \tau_M^i(0), \eta_M^k(\gamma) \leq \eta_M^k(0), \forall\, i \in N_l, k \in N_n\}$ is a Multi Fermatean Fuzzy Lie ideal of L.

Proof. Straight forward.

Theorem 3.9. Let W = (L, $\{\tau_W^i\}_{i=1}^l$, $\{\eta_W^k\}_{k=1}^n$) be a Multi Fermatean Fuzzy Lie sub algebra of a Lie algebra L and $R \subseteq L \times L$ be a binary relation on L defined by $R = \{(\gamma,\mu) \in L \times L \mid \tau_W^i(\gamma-\mu) = \tau_W^i(0), \eta_W^k(\gamma-\mu) = \eta_W^k(0), \gamma,\mu \in L, i \in N_l, k \in N_n\}$, then R is a congruence relation on L.

Proof. First of all, we need to prove that the relation R is equivalence relation on L.
Now **(i) Reflexivity**: Since $\forall \gamma \in L$, $\tau_W^i(\gamma-\gamma) = \tau_W^i(0)$, $\eta_W^k(\gamma-\gamma) = \eta_W^k(0)$, thus, $(\gamma,\gamma) \in R$, $\forall \gamma \in L, \forall\, i \in N_l, k \in N_n$, and consequently R is reflexive relation on L.

(ii) Symmetric: Let, $(\gamma,\mu) \in R$. Then $\tau_W^i(\gamma-\mu) = \tau_W^i(0) \Rightarrow \tau_W^i(-(\mu-\gamma)) \geq \tau_W^i(\mu-\gamma) = \tau_W^i(0)$, $\eta_W^k(\gamma-\mu) = \eta_W^k(0)$, $\Rightarrow \eta_W^k(-(\mu-\gamma)) \leq \eta_W^k((\mu-\gamma)) = \eta_W^k(0)$, Thus, $(\mu,\gamma) \in R$, $\forall\ \gamma, \mu \in L, \forall\ i \in N_l$, $k \in N_n$, so that R is symmetric relation on L.

(iii) Transitive: Let, (γ,μ), $(\mu,\sigma) \in R$. Then $\tau_W^i(\gamma-\mu) = \tau_W^i(0)$, $\tau_W^i(\mu-\sigma) = \tau_W^i(0)$ $\eta_W^k(\gamma-\mu) = \eta_W^k(0)$, $\eta_W^k(\mu-\sigma) = \eta_W^k(0)$ From which we have, $\tau_W^i(\gamma-\sigma) = \tau_W^i\{(\gamma-\mu)+(\mu-\sigma)\} \geq \text{Min}\ \{\tau_W^i(\gamma-\mu), \tau_W^i(\mu-\sigma)\} = \tau_W^i(0)$, $\eta_W^k(\gamma-\sigma) = \eta_W^k\{(\gamma-\mu)+(\mu-\sigma)\} \leq \text{Max}\ \{\eta_W^k(\gamma-\mu), \eta_W^k(\mu-\sigma)\} = \eta_W^k(0)$. Hence, $(\gamma,\sigma) \in L, \forall\ i \in N_l$, $k \in N_n$ and consequently, R is transitive relation on L. Hence, R is an equivalence relation on L.

We now verify that R is an congruence relation on L and for that let us take (γ,μ), $(\mu,\sigma) \in R$. Then $\tau_W^i(\gamma-\mu) = \tau_W^i(0)$, $\tau_W^i(\mu-\sigma) = \tau_W^i(0)$, $\eta_W^k(\gamma-\mu) = \eta_W^k(0)$, $\eta_W^k(\mu-\sigma) = \eta_W^k(0)$.
Now if $\gamma_1, \gamma_2, \mu_1, \mu_2 \in R$, then we must have,
$\tau_W^i\{(\gamma_1+\gamma_2)-(\mu_1+\mu_2)\} = \tau_W^i\{(\gamma_1-\mu_1)+(\gamma_2-\mu_2)\} \geq \text{Min}\ \{\tau_W^i(\gamma_1-\mu_1), \tau_W^i(\gamma_2-\mu_2)\} = \tau_W^i(0)$,
$\eta_W^k\{(\gamma_1+\gamma_2)-(\mu_1+\mu_2)\} = \eta_W^k\{(\gamma_1-\mu_1)+(\gamma_2-\mu_2)\} \leq \text{Max}\ \{\eta_W^k(\gamma_1-\mu_1), \eta_W^k(\gamma_2-\mu_2)\} = \eta_W^k(0)$,
$\tau_W^i(c\gamma_1-c\mu_1) = \tau_W^i\{c(\gamma_1-\mu_1)\} \geq \tau_W^i(\gamma_1-\mu_1) = \tau_W^i(0)$,
$\eta_W^k(c\gamma_1-c\mu_1) = \eta_W^k\{c(\gamma_1-\mu_1)\} \geq \eta_W^k(\gamma_1-\mu_1) = \eta_W^k(0)$,
$\tau_W^i\{[\gamma_1,\gamma_2]-[\mu_1,\mu_2]\} = \tau_W^i[(\gamma_1-\mu_1),(\gamma_2-\mu_2)] \geq \text{Min}\ \{\tau_W^i(\gamma_1-\mu_1), \tau_W^i(\gamma_2-\mu_2)\} = \tau_W^i(0)$,
$\eta_W^k\{[\gamma_1+\gamma_2]-[\mu_1+\mu_2]\} = \eta_W^k[(\gamma_1-\mu_1),(\gamma_2-\mu_2)] \leq \text{Max}\ \{\eta_W^k(\gamma_1-\mu_1), \eta_W^k(\gamma_2-\mu_2)\} = \eta_W^k(0)$,
Thus, $(\gamma_1+\gamma_2)\ R\ (\mu_1+\mu_2)$, $c\gamma_1 R c\mu_1$ and $[\gamma_1,\gamma_2]R[\mu_1,\mu_2]$.
Hence, R is a congruence relation on L.

Theorem 3.10. Every Multi Fermatean Fuzzy Lie ideal is Multi Fermatean Fuzzy Lie sub algebra. Proof. Straight forward from definition.

The converse of the above theorem (**3.10**) is not true which can be seen from the following example:

Example 3.11. Suppose that F=R, the set of real numbers and L = {(α,β,γ): α,β,γ∈R} be the Lie algebra. Let us define the mapping L×L→L by [u,v]=u×v+v, where × denotes the vector product (or cross product). Consider the Multi Fermatean fuzzy set W= (τ_W^1 , τ_W^2,..., τ_W^4; η_W^1 ,η_W^2,...,η_W^5) : L→ [0,1]×[0,1]×[0,1]×[0,1]×[0,1]×[0,1]×[0,1]×[0,1]×[0,1] described by

$$\tau_W^i\ (\alpha,\beta,\gamma)=\begin{cases}\frac{0.9}{i}, & if\ \alpha=\beta=\gamma=0\\ \frac{0.5}{i}, & if\ \alpha\neq 0, \beta=\gamma=0\\ 0, & otherwise\end{cases}, \text{ for i=1,2,3,4.}$$

$$\eta_W^k\ (\alpha,\beta,\gamma)=\begin{cases}0, & if\ \alpha=\beta=\gamma=0\\ \frac{0.7}{k}, & if\ \alpha\neq 0, \beta=\gamma=0\\ \frac{1}{k}, & otherwise\end{cases}, \text{ for k=1,2,3,4,5.}$$

Then it is easy to verify that W is a Multi Fermatean Fuzzy Lie sub algebra of L but it is not Multi Fermatean Fuzzy Lie ideal of L because for all i=1,2,3,4, τ_W^i [(2,0,0),(i,-i,i)]= τ_W^i(i,-3i,-i)=0< τ_W^i (2,0,0).

Theorem 3.12. Let N_l={1,2,....,l}, N_n={1,2,....,n} and L be a Lie Algebra of vectors over the field F. The necessary and sufficient condition for a Multi Fermatean fuzzy set W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) to be a Multi Fermatean fuzzy Lie sub algebra over L is that, ∀r∈[0,1], non-empty upper r-level cut $U_r(\tau_W^i)$ = {γ ∈ L : τ_W^i (γ) ≥ r, ∀ i∈N_l} and non-empty lower r-level cut $V_r(\eta_W^k)$ ={γ ∈L: η_W^k (γ) ≤ r, ∀ k∈N_n} are Lie sub algebra over L.

Proof. Suppose that W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) is a Multi Fermatean Fuzzy Lie sub algebra over L and r∈[0,1] is such that $U_r(\tau_W^i)$ ≠ φ. Let γ,η ∈ $U_r(\tau_W^i)$. Then ∀ γ,η∈L; ∀ i∈N_l ,∀ k∈N_n, ∀c∈ F, τ_W^i (γ+μ) ≥ Min{τ_W^i(γ),τ_W^i(μ)}≥r,
η_W^k (γ+μ) ≤ Max{η_W^k(γ), η_W^k(μ)} ≤ r
τ_W^i (cγ) ≥ τ_W^i(γ)≥r , η_W^k(cγ) ≤ η_W^k(γ) ≤ r,
τ_W^i[γ,μ] ≥ Min{τ_W^i(γ),τ_W^i(μ)}≥ r,
η_W^k[γ,μ] ≤ Max{η_W^k(γ), η_W^k(μ)} ≤ r.

Thus, γ+μ,cγ, [γ,μ] ∈ $U_r(\tau_W^i)$, γ+μ,cγ, [γ,μ] ∈ $V_r(\eta_W^k)$. Hence, $U_r(\tau_W^i)$ and $V_r(\eta_W^k)$ constitute Lie sub algebra over L.
Conversely, suppose that ∀ i∈N_l and ∀ r ∈ [0,1], $U_r(\tau_W^i) \neq \varphi$ is a Lie sub algebra over L and if possible suppose that τ_W^i (γ+μ) < Min{τ_W^i(γ),τ_W^i(μ)} for some γ,μ ∈ L. If we choose $r_0 = \frac{1}{2}$ (τ_W^i(γ+μ) + Min{τ_W^i(γ),τ_W^i(μ)}), the by properties of inequality we must have, τ_W^i (γ+μ) < r_0<Min{τ_W^i(γ),τ_W^i(μ)}. This implies that, γ+μ ∉ $U_r(\tau_W^i)$, γ,μ ∈ $U_r(\tau_W^i)$, which is a contradiction. Hence τ_W^i(γ+μ) ≥ Min{τ_W^i(γ),τ_W^i(μ)}, ∀ γ,μ ∈ L. In a similar manner we can prove that τ_W^i(cγ) ≥ τ_W^i(γ) and τ_W^i[γ,μ] ≥ Min{τ_W^i(γ),τ_W^i(μ)}, ∀c∈ F, ∀ i∈N_l . The proof is similar for the case $V_r(\eta_W^k)$. This completes the proof.

Theorem 3.13. Let N_l={1,2,....,l}, N_n={1,2,....,n} and L be a Lie Algebra of vectors over the field F. If V = (τ_V^1 , τ_V^2,...,τ_V^l; η_V^1 ,η_V^2,...,η_V^n) and W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) are two Multi Fermatean Fuzzy Lie sub algebra over L, then their intersection V∩W= H = (τ_H^1 , τ_H^2,...,τ_H^l; η_H^1 ,η_H^2,...,η_H^n) is also Multi Fermatean Fuzzy Lie sub algebra over L.

Proof. Suppose that γ,μ ∈ L be arbitrary. Then ∀ i∈N_l ,∀ k∈N_n, ∀c∈ F, we have,
τ_H^i (γ+μ) = Min{τ_V^i (γ+μ), τ_W^i (γ+μ)}
≥ Min{Min{τ_V^i(γ),τ_V^i(μ)},Min{τ_W^i (γ),τ_W^i(μ)}}
=Min{Min{τ_V^i (γ),τ_W^i (γ)},Min{τ_V^i(μ),τ_W^i(μ)}}
=Min{τ_H^i(γ),τ_H^i(μ)}
η_H^k (γ+μ) = Max{η_V^k (γ+μ), η_W^k (γ+μ)}
≤ Max{Max{η_V^k(γ),η_V^k(μ)},Max{η_W^k(γ),η_W^k(μ)}}
=Max{Max{η_V^k (γ), η_W^k (γ)},Max{η_V^k (μ), η_W^k (μ)}}
=Max{η_H^k(γ),η_H^k(μ)}
τ_H^i (cγ) = Min{τ_V^i (cγ), τ_W^i(cγ)}≥ {Min{τ_V^i (γ), τ_W^i(γ)} = τ_H^i (γ)
η_H^k (cγ) = Max{η_V^k(cγ), η_W^k (cγ)} ≤ {Max{η_V^k (γ), η_W^k (γ)} =η_H^k (γ)
τ_H^i [γ,μ] = Min{τ_V^i [γ,μ], τ_W^i [γ,μ]}
≥ Min {Min{τ_V^i (γ),τ_V^i(μ)}, Min{τ_W^i (γ),τ_W^i(μ)}}
= Min {Min{τ_V^i (γ), τ_W^i (γ)},Min {τ_V^i(μ), τ_W^i (μ)}}
= Min {τ_H^i (γ),τ_H^i(μ)}

η_H^k [γ, μ] = Max{ η_V^k [γ, μ], η_W^k [γ, μ]} ≤ Max {Max{η_V^k (γ),η_V^k(μ)}, Max{η_W^k (γ), η_W^k (μ)}}
= Max {Max{η_V^k(γ), η_W^k (γ)},Max {η_V^k(μ), η_W^k (μ)}} = Max {η_H^k (γ), η_H^k(μ)}.
Hence, V∩W= H is Multi Fermatean Fuzzy Lie sub algebra over L.

Definition 3.14. Let N_l={1,2,....,l}, N_n={1,2,....,n} and L be a Lie Algebra of vectors over the field F. If V = (τ_V^1 , τ_V^2,...,τ_V^l; η_V^1 ,η_V^2,...,η_V^n) and W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) are two Multi Fermatean Fuzzy sets on L, then the product H= V×W defined on L ×L will be known as generalized Cartesian product if
(V×W) (γ,μ) =[(τ_V^1 , τ_V^2,...,τ_V^l; η_V^1 ,η_V^2,...,η_V^n) × (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n)] (γ,μ)
= (τ_H^1(γ,μ), τ_H^2(γ,μ),...,τ_H^l(γ,μ); η_H^1(γ,μ), η_H^2(γ,μ),...,η_H^n(γ,μ)), ∀ (γ,μ) ∈ L ×L,
where, τ_H^i (γ,μ)=(τ_V^i ×τ_W^i)(γ,μ)= Min { τ_V^i (γ), τ_W^i (μ)}, ∀ i∈N_l .
η_H^k (γ,μ)=(η_V^k ×η_W^k)(γ,μ)= Max{ η_V^k(γ), η_W^k(μ)}, ∀ k∈N_n. Evidently the generalized Cartesian product (V×W) is Multi Fermatean Fuzzy set on L ×L if 0 ≤ {τ_H^i (γ,μ)}^{3} +{η_H^k(γ,μ)}^{3} ≤ 1 i.e., 0 ≤ {Min{τ_V^i (γ), τ_W^i (μ)}}^{3} + {Max{η_V^k(γ), η_W^k(μ)}}^{3} ≤ 1, where, i∈N_l ,k∈N_n.

Theorem 3.15. Let L be the Lie Algebra of vectors over the field F. If V = (τ_V^1 , τ_V^2,...,τ_V^l; η_V^1 ,η_V^2,...,η_V^n) and W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) are two Multi Fermatean Fuzzy Lie sub algebra of L, then the generalized Cartesian product (V×W) is Multi Fermatean Fuzzy Lie sub algebra of L×L.

Proof. Let N_l={1,2,....,l}, N_n={1,2,....,n} and L be a Lie Algebra of vectors over the field F. Then, ∀ i∈N_l ,∀ k∈N_n ∀ γ=(γ_1,γ_2), μ=(μ_1,μ_2) ∈L ×L and c∈F, we have,
(τ_V^i ×τ_W^i)(γ+μ) =(τ_V^i ×τ_W^i) ((γ_1,γ_2)+(μ_1,μ_2)) =(τ_V^i× τ_W^i) ((γ_1 + μ_1),(γ_2 + μ_2))
= Min { τ_V^i(γ_1 + μ_1), τ_W^i(γ_2 + μ_2) }
≥ Min {Min { τ_V^i(γ1), τ_V^i(μ_1) },Min { τ_W^i(γ_2), τ_W^i(μ_2)}}
=Min {Min { τ_V^i(γ_1), τ_W^i(γ_2) },Min { τ_V^i(μ_1), τ_W^i(μ_2)}}
=Min { (τ_V^i× τ_W^i) (γ_1 , γ_2),(τ_V^i× τ_W^i) (μ_1 , μ_2) }
=Min { (τ_V^i× τ_W^i) (γ),(τ_V^i× τ_W^i) (μ) }

$(\eta_V^k \times \eta_W^k)(\gamma + \mu) = (\eta_V^k \times \eta_W^k)((\gamma_1, \gamma_2) + (\mu_1, \mu_2)) = (\eta_V^k \times \eta_W^k)((\gamma_1 + \mu_1), (\gamma_2 + \mu_2$

$= \text{Max}\{\eta_V^k(\gamma_1 + \mu_1), \eta_W^k(\gamma_2 + \mu_2)\}$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_V^k(\mu_1)\}, \text{Min}\{\eta_W^k(\gamma_2), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \text{Max}\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2), (\eta_V^k \times \eta_W^k)(\mu_1, \mu_2)\}\} = \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$

$(\tau_V^i \times \tau_W^i)(c\,\gamma) = (\tau_V^i \times \tau_W^i)\{c(\gamma_1, \gamma_2)\} = (\tau_V^i \times \tau_W^i)(c\,\gamma_1, c\,\gamma_2)$

$= \text{Min}\{\tau_V^i(c\,\gamma_1), \tau_W^i(c\,\gamma_2)\} \geq \text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}$

$= (\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2) = (\tau_V^i \times \tau_W^i)(\gamma)$

$(\eta_V^k \times \eta_W^k)\{(c\,\gamma)\} = (\eta_V^k \times \eta_W^k)\{c(\gamma_1, \gamma_2)\} = (\eta_V^k \times \eta_W^k)(c\,\gamma_1, c\,\gamma_2)$

$= \text{Max}\{\eta_V^k(c\,\gamma_1), \eta_W^k(c\,\gamma_2)\} \leq \text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}$

$= (\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2) = (\eta_V^k \times \eta_W^k)(\gamma)$

$(\tau_V^i \times \tau_W^i)[\gamma, \mu] = (\tau_V^i \times \tau_W^i)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \{\text{Min}\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}$

$= \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2)\}, \{(\tau_V^i \times \tau_W^i)(\mu_1, \mu_2)\}$

$= \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$

$(\eta_V^k \times \eta_W^k)[\gamma, \mu] = (\eta_V^k \times \eta_W^k)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \{\text{Max}\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2)\}, \{(\eta_V^k \times \eta_W^k)(\mu_1, \mu_2)\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}.$

Hence, (V×W) is Multi Fermatean Fuzzy Lie sub algebra of L×L.

4. Multi Fermatean fuzzy Lie algebra homomorphisms

In this section, the properties of Multi Fermatean Fuzzy Lie sub algebras and Multi Fermatean Fuzzy Lie ideals under homomorphisms of Lie algebras are investigated. Also some of their preservation aspects are examined.

Definition 4.1. Let L_1 and L_2 be two Lie algebras over the field F. A linear transformation ϕ: $L_1 \rightarrow L_2$ is said to be Lie homomorphism if the relationship $\phi([\gamma,\mu])=[\phi(\gamma),\phi(\mu)]$ is true, $\forall\ \gamma,\mu \in L_1$.

Definition 4.2. Let L_1 and L_2 be two Lie algebras over the field F. Then a Lie homomorphism ϕ: $L_1 \rightarrow L_2$ is said have natural extension ϕ: $I^{L1} \rightarrow I^{L2}$ if for all W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) $\in I^{L1}$ and $\mu \in L_2$, the followings hold: $\phi(\tau_W^i)(\mu)=$ Sup{$\tau_W^i(\gamma)$: $\gamma\in\phi^{-1}(\mu),\gamma\in L_1$}, for all i=1,2,3,...,l. $\phi(\eta_W^k)(\mu)=$ Inf{$\eta_W^k(\gamma)$: $\gamma\in \phi^{-1}(\mu),\gamma\in L_1$}, for all k=1,2,3,...,n.

Theorem 4.3. Let W = (τ_W^1 , τ_W^2,...,τ_W^l; η_W^1 ,η_W^2,...,η_W^n) $\in I^{L1}$ be Multi Fermatean Fuzzy Lie sub algebras and ϕ: $L_1 \rightarrow L_2$ be Lie homomorphism between L_1 and L_2. Then ϕ (W) is Multi Fermatean Fuzzy Lie sub algebras of L_2.

Proof. Suppose that $\mu_1, \mu_2 \in L_2$. Then $\{\gamma: \gamma\in \phi^{-1}(\mu_1+\mu_2)\} \supseteq \{\gamma_1+\gamma 2:\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$.
Now, for all i=1,2,3,...,l, we have,
$\phi(\tau_W^i)(\mu_1+\mu_2)=$ Sup{$\tau_W^i(\gamma)$: $\gamma\in \phi^{-1}(\mu_1+\mu_2),\gamma\in L_1$}
$\geq \{\tau_W^i(\gamma_1+\gamma_2) :\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$
$\geq$ Sup{ Min{ $\tau_W^i(\gamma_1)$, $\tau_W^i(\gamma_2)$} :$\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)$}
= Min{Sup{$\tau_W^i(\gamma_1)$:$\gamma_1\in \phi^{-1}(\mu_1)$}, Sup{$\tau_W^i(\gamma_2)$:$\gamma_2\in \phi^{-1}(\mu_2)$}}
= Min{$\phi(\tau_W^i)(\mu_1)$,$\phi(\tau_W^i)(\mu_2)$}
and for all k=1,2,3,...,n, we have,
$\phi(\eta_W^k)(\mu_1+\mu_2)=$ Inf{$\eta_W^k(\gamma)$: $\gamma\in \phi^{-1}(\mu_1+\mu_2),\gamma\in L_1$}
$\leq \{\eta_W^k(\gamma_1+\gamma_2) :\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$
$\leq$ Inf{ Max{ $\eta_W^k(\gamma_1)$,$\eta_W^k(\gamma_2)$} :$\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)$}
=Max{Inf{$\eta_W^k(\gamma_1)$:$\gamma_1\in \phi^{-1}(\mu_1)$}, Inf{$\eta_W^k(\gamma_2)$:$\gamma_2\in \phi^{-1}(\mu_2)$}}

= Max{$\phi(\eta_W^k)(\mu_1)$,$\phi(\eta_W^k)(\mu_2)$}.
For $\mu \in L_2$ and c$\in$ F, we have, $\{\gamma: \gamma\in \phi^{-1}(c\mu_1)\} \supseteq \{c\gamma:\gamma\in \phi^{-1}(\mu)\}$.
Now, for all i=1,2,3,...,l, we have,
$\phi(\tau_W^i)(c\mu)=$ Sup{$\tau_W^i(c\gamma)$: $\gamma\in \phi^{-1}(\mu),\gamma\in L_1$}

$\geq$ Sup{$\tau_W^i(c\gamma)$: $\gamma\in \phi^{-1}(c\mu),\gamma\in L_1$}
$\geq$ Sup{$\tau_W^i(\gamma)$: $\gamma\in \phi^{-1}(\mu),\gamma\in L_1$}
= $\phi(\tau_W^i)(\mu)$.

Similarly, for all k=1,2,3,...,m, we can we prove that
$\phi(\eta_W^k)(c\mu)=\mathrm{Inf}\{\eta_W^k(c\gamma): \gamma\in \phi^{-1}(\mu),\gamma\in L_1\}$
$\leq \mathrm{Inf}\{\eta_W^k(c\gamma): \gamma\in \phi^{-1}(c\mu),\gamma\in L_1\}$
$\leq \mathrm{Inf}\{\eta_W^k(\gamma): \gamma\in \phi^{-1}(\mu),\gamma\in L_1\}$
$=\phi(\eta_W^k)(\mu)$.
For, $\mu_1, \mu_2 \in L_2$, then $\{\gamma: \gamma\in \phi^{-1}(\mu_1+\mu_2)\} \supseteq \{\gamma_1+\gamma_2:\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$.
Now, for all i=1,2,3,...,l, we have,
$\phi(\tau_W^i)([\mu_1,\mu_2])=\mathrm{Sup}\{\tau_W^i(\gamma): \gamma\in \phi^{-1}([\mu_1,\mu_2]),\gamma\in L_1\}$
$\geq \mathrm{Sup}\{\tau_W^i([\gamma_1,\gamma_2]) :\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$
$\geq \mathrm{Sup}\{ \mathrm{Min}\{ \tau_W^i(\gamma_1),\tau_W^i(\gamma_2)\} :\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$
$=\mathrm{Min}\{\mathrm{Sup}\{\tau_W^i(\gamma_1) :\gamma_1\in \phi^{-1}(\mu_1)\}, \mathrm{Sup}\{\tau_W^i(\gamma_2) :\gamma_2\in \phi^{-1}(\mu_2)\}\}$
$=\mathrm{Min}\{\phi(\tau_W^i)(\mu_1),\phi(\tau_W^i)(\mu_2)\}$.
Similarly, for all k=1,2,3,...,m, we can we prove that
$\phi(\eta_W^k)([\mu_1,\mu_2])=\mathrm{Sup}\{\eta_W^k(\gamma): \gamma\in \phi^{-1}([\mu_1,\mu_2]),\gamma\in L_1\}$
$\geq \mathrm{Sup}\{\eta_W^k([\gamma_1,\gamma_2]) :\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$
$\geq \mathrm{Sup}\{ \mathrm{Min}\{ \eta_W^k(\gamma_1), \eta_W^k(\gamma_2)\} :\gamma_1\in \phi^{-1}(\mu_1),\gamma_2\in \phi^{-1}(\mu_2)\}$
$=\mathrm{Min}\{\mathrm{Sup}\{\eta_W^k(\gamma_1) :\gamma_1\in \phi^{-1}(\mu_1)\}, \mathrm{Sup}\{\eta_W^k(\gamma_2) :\gamma_2\in \phi^{-1}(\mu_2)\}\}$
$=\mathrm{Min}\{\phi(\eta_W^k)(\mu_1), \phi(\eta_W^k)(\mu_2)$.
Hence, ϕ (W) is Multi Fermatean Fuzzy Lie sub algebras of L_2.

Theorem 4.4. Let $W = (\tau_W^1, \tau_W^2,...,\tau_W^l; \eta_W^1, \eta_W^2,...,\eta_W^n) \in I^{L1}$ be Multi Fermatean Fuzzy Lie ideal and $\phi: L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then $\phi(W)$ is Multi Fermatean Fuzzy Lie ideal of L_2.

Proof. The proof is similar to the proof of Theorem 4.3

Theorem 4.5. Let $W = (\tau_W^1, \tau_W^2,...,\tau_W^l; \eta_W^1, \eta_W^2,...,\eta_W^n)\in I^{L2}$ be Multi Fermatean Fuzzy Lie sub algebras and $\phi: L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then $\phi^{-1}(W)$ is Multi Fermatean Fuzzy Lie sub algebra of L_1.

Proof. Suppose that $\mu_1, \mu_2 \in L_1$. Now, for all i=1,2,3,...,l; k=1,2,3,...,n; we have,
$\phi^{-1}(\tau_W^i)(\mu_1+\mu_2)= \tau_W^i[\phi(\mu_1+\mu_2)]$
$=\tau_W^i[\phi(\mu_1)+\phi(\mu_2)]$
$\geq \text{Min }\{\tau_W^i(\phi(\mu_1)), \tau_W^i(\phi(\mu_2))\}$
$=\text{Min }\{\phi^{-1}(\tau_W^i)(\mu_1), \phi^{-1}(\tau_W^i)(\mu_2)\}$
$\phi^{-1}(\eta_W^k)(\mu_1+\mu_2) = \eta_W^k[\phi(\mu_1+\mu_2)]$
$=\eta_W^k[\phi(\mu_1)+\phi(\mu_2)]$
$\leq \text{Max }\{\eta_W^k(\phi(\mu_1)), \eta_W^k(\phi(\mu_2))\}$
$=\text{Max }\{\phi^{-1}(\eta_W^k)(\mu 1), \phi^{-1}(\eta_W^k)(\mu_2)\}$.
For all $\mu \in L_1$ and $c \in F$, we have,

$\phi^{-1}(\tau_W^i)(c\mu)= \tau_W^i[\phi(c\mu)] =\tau_W^i[c\phi(\mu)] \geq \tau_W^i(\phi(\mu)) = \phi^{-1}(\tau_W^i)(\mu)$.
$\phi^{-1}(\eta_W^k)(c\mu)= \eta_W^k[\phi(c\mu)] =\eta_W^k[c\phi(\mu)] \leq \eta_W^k(\phi(\mu)) = \phi^{-1}(\eta_W^k)(\mu)$.
For all $\mu_1, \mu_2 \in L_1$, we have,
$\phi^{-1}(\tau_W^i)[\mu_1,\mu_2]= \tau_W^i(\phi[\mu_1,\mu_2]) =\tau_W^i[\phi(\mu 1),\phi(\mu_2)]$
$\geq \text{Min }\{\tau_W^i(\phi(\mu_1)), \tau_W^i(\phi(\mu_2))\}$
$=\text{Min }\{\phi^{-1}(\tau_W^i)(\mu_1), \phi^{-1}(\tau_W^i)(\mu_2)\}$
$\phi^{-1}(\eta_W^k)[\mu_1,\mu_2] = \eta_W^k(\phi[\mu_1,\mu_2])=\eta_W^k[\phi(\mu_1),\phi(\mu_2)]$
$\leq \text{Max }\{\eta_W^k(\phi(\mu_1)), \eta_W^k(\phi(\mu_2))\}$
$=\text{Max }\{\phi^{-1}(\eta_W^k)(\mu_1), \phi^{-1}(\eta_W^k)(\mu_2)\}$.
Hence, $\phi^{-1}(W)$ is multi Fermatean Fuzzy Lie sub algebra of L_1.

Theorem 4.6. Let $W = (\tau_W^1, \tau_W^2,...,\tau_W^l;\ \eta_W^1, \eta_W^2,...,\eta_W^n) \in I^{L2}$ be Multi Fermatean Fuzzy Lie ideal and $\phi: L_1 \rightarrow L_2$ be Lie homomorphism between L_1 and L_2. Then $\phi^{-1}(W)$ is Multi Fermatean Fuzzy Lie ideal of L_1.

Proof. The proof is similar to the proof of theorem (4.5)

5. Conclusion

In this paper we introduce the concept of Multi Fermatean Fuzzy Lie sub algebras and Multi Fermatean fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized Cartesian product of Multi Fermatean Fuzzy Lie sub algebras are investigated. Moreover, the relationship between Multi Fermatean Fuzzy Lie sub algebras and Multi Fermatean Fuzzy Lie ideals are established. Lastly, the image and the inverse image of Multi Fermatean Fuzzy Lie sub algebras (Multi Fermatean Fuzzy Lie ideals) under Lie homomorphisms are also studied. In future the proposed work shall be extended with the help of multi spherical set and so on in Lie algebras. The proposed work is applicable in any multi criterion decision making problem, pattern recognition and classification problems especially problems with more than one decision makers. Therefore, this new theory will be a useful tool in decision and ranking problems such as robot selection, green supplier's selections, solid waste landfill site section problems etc. In the near future we give some application of the proposed theory to some multi criterion decision making problems such as medical diagnosis and pattern recognitions.

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On Statistical Weak Convergence in Pringsheim's Sense of Double Sequences

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Abstract. In this article, we introduce the notion of statistical weak convergence in Pringsheim's sense of double sequence in a normed linear space X. We discuss the algebra of weak statistical limit of double sequence. Further, we also discuss the existence of weak statistical limit and inclusion property.

Keywords: Weak Convergence, Statistical Convergence, Double Sequence

AMS Classification: 40A05; 54A20; 17C65; 46B10;

1. Introduction

The idea of statistical convergence is the generalization of the usual convergence. Independently, Fast [6] and Schoenberg [13] proposed the idea of statistical convergence. It was further examined from the perspective of sequence space and connected to summability theory by a number of authors, including Buck [4], Fridy [7], Gokhan et al. [8], Tripathy [14], and others. The concept of asymptotic density of subsets of the natural number set $\mathbb{N}$ is necessary for the idea to be feasible. The following definitions are due to Tripathy [14].

The density $\delta(D)$ of a subset D of $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$\delta(D) = \lim_{p,q \to \infty} \frac{1}{pq} \left\{ \sum_{n=1}^{p} \sum_{k=1}^{q} \chi_D(n,k) \, , \chi_D(n,k) \right\}$ exists. $\delta(D^c) = 1 - \delta(D)$.

If a double sequence (a_{nk}) fails to satisfy a property over a subset of $\mathbb{N} \times \mathbb{N}$ of density zero, then (a_{nk}) is said to satisfy property P for almost all n and k, written as $a.a.n$ and k.

It was Pringsheim [12] who first proposed the idea of double sequence. Furthermore, it is found in Bromwich [3]. Moricz [10], Basarir and Sonalcan [1], and others have all examined the double sequence from various angles. Pringsheim [12] defined the convergence of double sequence as following.

A double sequence (a_{nk}) is considered to converge if, $\lim_{n,k \to \infty} a_{nk} = L$, where $n, k \to \infty$, independent of one another.

Connor et al. [5] have created a novel concept, weak statistical convergence, to characterize Banach spaces with separable duals. Weak statistical convergence was also used by Pehlivan and Karaev [11] to justify a result of Gokhberg and Krein [9] on compact operators. Bhardwaj and Bala [2] established that the notion of weak statistical convergence is a generalization of the notion of weak convergence and that the notions of the norm and weak statistical convergence are comparable in finite dimensional normed spaces.

2. Statistical Weak Convergence in Pringsheim's Sense Double Sequence

Definition 2.1. A double sequence (a_{nk}) is said to be weak statistically convergent in Pringsheim's sense to limit $L \in X$ if $\lim_{p,q \to \infty} \frac{1}{pq} |n \le p; k \le q : f(a_{nk} - L) \ge \varepsilon| = 0, f \in X'$. i.e., for all $f \in X'$, $f(a_{nk}) - f(L) < \varepsilon$, for $a.a.n$ &k. It is denoted by $a_{nk} \xrightarrow{w-stat} L$.

Theorem 2.1. The weak statistical limit of a double sequence if it exists is unique.

Proof: Consider that $a_{nk} \xrightarrow{w-stat} L$ and $a_{nk} \xrightarrow{w-stat} M$, where $L \neq M$ and $L, M \in X$.

Then, for all $f \in X'$ and $L, M \in X$,

$|f(a_{nk}) - f(L)| < \frac{\varepsilon}{2}$, for $a.a.n$ &k and, $|f(a_{nk}) - f(M)| < \frac{\varepsilon}{2}$, for $a.a.n$ &k.

We have, $|f(L) - f(M)| = |f(L) - f(a_{nk}) + f(a_{nk}) - f(M)|$

$$= |f(L) - f(a_{nk})| + |f(a_{nk}) - f(M)|$$

$$\geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, a.a.n \,\&k$$

$$\geq \varepsilon, a.a.n \,\&k.$$

This implies, $|f(L) - f(M)| \to 0$.

This implies, $f(L) = f(M)$. Hence, the weak statistical limit is unique.

Theorem 2.2. A double sequence which is convergent in Pringsheim's sense is weak statistical convergent in Pringsheim's sense but not conversely.

Proof: Let (a_{nk}) be convergent in Pringsheim's sense. Then, for every $\varepsilon > 0$ and for all $n, k \geq n_0$ we have, $|a_{nk} - L| < \varepsilon$. i.e., $|a_{nk} - L| \to 0$. ... ... (1)

Also, $|f(a_{nk}) - f(L)| = |f(a_{nk} - L)| \to 0$, for $a.a.n$&k (using 1)

This implies, $|f(a_{nk}) - f(L)| \to 0$, for $a.a.n$&k. Hence, (a_{nk}) is weak statistically convergent in Pringsheim's sense.

The following example shows that the converse of the theorem is not true.

Example 2.1. Consider $X = \ell_p, 1 < p < \infty$. Let $(a_{nk}) \in \ell_p$, defined by $a_j^{nk} = e_{nk}$, for all $j \leq n, k$. Let $f \in (\ell_p)'$,where $(\ell_p)'$ is ℓ_q with $\frac{1}{p} + \frac{1}{q} = 1$, defined by $f(a_{nk}) = \sum_{j=1}^{\infty} a_j^{nk} b_j$. Then, $|f(a_{nk})| = \left|\sum_{j=1}^{\infty} a_j^{nk} b_j\right| \leq \left|\left(\sum_{j=1}^{\infty} a_j^{nk}\right)^p\right|^{\frac{1}{p}} \left|\left(\sum_{j=1}^{\infty} b_j\right)^q\right|^{\frac{1}{q}} \to 0,$ as $n, k \to \infty$. This implies that $a_{nk} \xrightarrow{w} 0$ and hence, $a_{nk} \xrightarrow{w-stat} 0$. However, $a_{nk} \to 0$, as $n, k \to \infty$. Hence, the result.

Theorem 2.3. Consider two double sequences (a_{nk}) and (b_{nk}) in X and $a_{nk} \xrightarrow{w-stat} L$, $b_{nk} \xrightarrow{w-stat} M$, where $L, M \in X$. Then the following conditions holds true:

(a) $a_{nk} + b_{nk} \xrightarrow{w-stat} L + M$

(b) $a_{nk} - b_{nk} \xrightarrow{w-stat} L - M$

(c) $a_{nk}. b_{nk} \xrightarrow{w-stat} L. M$

(d) $\frac{a_{nk}}{b_{nk}} \xrightarrow{w-stat} \frac{L}{M}$, provided neither b_{nk} nor M are zero element of X.

(e) $\alpha a_{nk} \xrightarrow{w-stat} \alpha L$, α is a scalar.

Proof: (a) Given that $a_{nk} \xrightarrow{w-stat} L$ and $b_{nk} \xrightarrow{w-stat} M$. Then, for every $\varepsilon > 0$, $|f(a_{nk}) - f(L)| < \frac{\varepsilon}{2}$, for $a.a.n\&k$ and $|f(b_{nk}) - f(M)| < \frac{\varepsilon}{2}$, for $a.a.n\&k$ and for all $f \in X', L, M \in X$.

For all $f \in X'$, $|\{f(a_{nk}) + f(b_{nk})\} - \{L + M\}| = |\{f(a_{nk}) - L\} + \{f(a_{nk}) - M\}|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \text{for } a.a.n\&k$$

$$< \varepsilon, \text{for } a.a.n\,\&k.$$

This implies, $|\{f(a_{nk}) + f(b_{nk})\} - \{L + M\}| < \varepsilon$, for $a.a.n\&k$.

(b) Given that $a_{nk} \xrightarrow{w-stat} L$ and $b_{nk} \xrightarrow{w-stat} M$. Then, for every $\varepsilon > 0$, $|f(a_{nk}) - f(L)| < \frac{\varepsilon}{2}$, for $a.a.n\&k$ and $|f(b_{nk}) - f(M)| < \frac{\varepsilon}{2}$, for $a.a.n\&k$ and for all $f \in X', L, M \in X$.

For all $f \in X'$, $|\{f(a_{nk}) - f(b_{nk})\} - \{L - M\}| = |\{f(a_{nk}) - L\} + \{f(a_{nk}) - M\}|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \text{for } a.a.n\&k$$

$$< \varepsilon, \text{for } a.a.n\,\&k.$$

This implies, $|\{f(a_{nk}) - f(b_{nk})\} - \{L - M\}| < \varepsilon$, for $a.a.n\&k$.

(c) Given that $a_{nk} \xrightarrow{w-stat} L$ and $b_{nk} \xrightarrow{w-stat} M$. Then, for every $\varepsilon > 0$, $|f(a_{nk}) - f(L)| < \varepsilon_1$, for $a.a.n\&k$ and $|f(b_{nk}) - f(M)| < \varepsilon_2$, for $a.a.n\&k$ and for all $f \in X', L, M \in X$.

For all $f \in X'$, we have

$$\begin{aligned}
|f(a_{nk}b_{nk}) - f(L.M)| &= |f(a_{nk}b_{nk}) - f(a_{nk})f(M) + f(a_{nk})f(M) - f(LM)| \\
&= |f(a_{nk}b_{nk}) - f(a_{nk})f(M) + f(a_{nk})f(M) - f(LM)| \\
&= |f(a_{nk})f(b_{nk}) - f(a_{nk})f(M) + f(a_{nk})f(M) - f(L)f(M)| \\
&= |f(a_{nk})\{f(b_{nk}) - f(M)\} + f(M)\{f(a_{nk}) - f(L)\}| \\
&\leq |f(a_{nk})|\,|\{f(b_{nk}) - f(M)\}| + |f(M)|\,|\{f(a_{nk}) - f(L)\}| \\
&< |f(a_{nk})|\varepsilon_2 + |f(M)|\varepsilon_1 \text{for } a.a.n\&k \qquad \ldots\ldots\ldots\ldots\ldots.. (2)
\end{aligned}$$

We have,

$$|f(a_{nk})| = |f(a_{nk}) - f(L) + f(L)|$$

$$\leq |f(a_{nk}) - f(L)| + |f(L)|$$

$$< \varepsilon_1 + |f(L)|, \text{for } a.a.n\&k \quad \ldots\ldots\ldots\ldots(3)$$

i.e., Considering $\varepsilon_1 < 1$, we have $|f(a_{nk})| < 1 + |f(L)|$, for $a.a.n\&k$. Then, $|f(a_{nk}b_{nk}) - f(L.M)| < \{1 + |f(L)|\}\varepsilon_2 + |f(M)|\varepsilon_1$, for $a.a.n\&k$ (using Equation (3) in (2)).

Choosing $\varepsilon_2 = \frac{\varepsilon}{2|f(L)+1|}$ and $\varepsilon_1 = \frac{\varepsilon}{2|f(M)|}$, $|f(a_{nk}b_{nk}) - f(L.M)| < \{1 + |f(L)|\}\frac{\varepsilon}{2|f(L)+1|} + |f(M)|\frac{\varepsilon}{2|f(M)|}$, for $a.a.n\&k$. This implies, $|f(a_{nk}b_{nk}) - f(L.M)| < \varepsilon$, for $a.a.n\&k$. Hence, $a_{nk}.b_{nk} \xrightarrow{w-stat} L.M$.

(d) Given that $a_{nk} \xrightarrow{w-stat} L$ and $b_{nk} \xrightarrow{w-stat} M$. Then, for every $\varepsilon > 0$, $|f(a_{nk}) - f(L)| < \varepsilon_1$, for $a.a.n\&k$ and $|f(b_{nk}) - f(M)| < \varepsilon_2$, for $a.a.n\&k$ and for all $f \in X', L, M \in X$.

For all $f \in X'$, $\left|f\left(\frac{a_{nk}}{b_{nk}}\right) - f\left(\frac{L}{M}\right)\right| = \left|\frac{f(a_{nk})}{f(b_{nk})} - \frac{f(L)}{f(M)}\right|$

$$= \left|\frac{f(M)f(a_{nk}) - f(L)f(b_{nk})}{f(M)f(b_{nk})}\right|$$

$$= \left|\frac{f(M)f(a_{nk}) - f(L)f(M) + f(L)f(M) - Lf(b_{nk})}{f(M)f(b_{nk})}\right|$$

$$= \left|\frac{f(M)\{f(a_{nk}) - f(L)\} + f(L)\{f(b_{nk}) - f(M)\}}{f(M)f(b_{nk})}\right|$$

$$\leq \frac{|f(a_{nk})-f(L)|}{|f(b_{nk})|}+\frac{|f(L)||f(b_{nk})-f(M)|}{|f(M)||f(b_{nk})|}$$

$$< \frac{\varepsilon_1}{|f(b_{nk})|}+\frac{|f(L)|\varepsilon_2}{|f(M)||f(b_{nk})|} \text{ , for } a.a.n\&k \dots\dots\dots\dots\dots\dots..(4)$$

Now, $|f(M)| = |f(M) - f(b_{nk}) + f(b_{nk})| \leq |f(b_{nk}) - f(M)| + |f(b_{nk})|$

$< \varepsilon_2 + |f(b_{nk})|$, for $a.a.n\&k$.

i.e., $|f(b_{nk})| > |f(M) - \varepsilon_2|$, for $a.a.n\&k$. Let $\varepsilon_2 > \frac{|f(M)|}{2}$, we have, $|f(b_{nk})| > \left|\frac{f(M)}{2}\right|$, for $a.a.n\&k$. i.e., $\frac{1}{|f(b_{nk})|} < \left|\frac{2}{f(M)}\right|$, for $a.a.n\&k$.

Equation (4) implies, $\left|f\left(\frac{a_{nk}}{b_{nk}}\right) - f\left(\frac{L}{M}\right)\right| < \frac{2\varepsilon_1}{|f(M)|} + \frac{|f(L)|2\varepsilon_2}{|f(M)|^2}$, for $a.a.n\&k$…(5). Choosing $\varepsilon_1 = \frac{\varepsilon|f(M)|}{4}$ and $\varepsilon_2 = \frac{\varepsilon|f(M)|^2}{4|f(L)|}$, for $a.a.n\&k$.

Hence, Equation (5) implies that $\left|f\left(\frac{a_{nk}}{b_{nk}}\right) - f\left(\frac{L}{M}\right)\right| < \varepsilon$, for $a.a.n\&k$. i.e., $\frac{a_{nk}}{b_{nk}} \xrightarrow{w-stat} \frac{L}{M}$.

(e) Given that $a_{nk} \xrightarrow{w-stat} L$. Then, for every $\varepsilon > 0$, $|f(a_{nk}) - f(L)| < \varepsilon$, for $a.a.n\&k$, for all $f \in X', L \in X$. i.e., $|f(a_{nk}) - f(L)| \to 0$.

For all $f \in X'$ and any scalar α,

$|f(\alpha a_{nk}) - f(\alpha L)| = |\alpha f(a_{nk}) - \alpha f(L)| < \alpha|f(a_{nk}) - f(L)| \to 0.$

i.e., $\alpha a_{nk} \xrightarrow{w-stat} \alpha L$.

3. Conclusions

In this article, we study statistical convergence of double sequence from the point of view of weak convergence. We also established the algebra of limits for addition, subtraction, multiplication, division and absolute homogeneity of weak statistical convergence of double sequence. Further, we demonstrate that a double sequence which is convergent in Pringsheim's sense is weak statistical convergence but not vice-versa and provide example to support the assertion.

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Class of Difference Double Sequences of Interval Numbers

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Absract: In this paper we are have introduce the classes of double sequence ${}_2\bar{\ell}(p)(\Delta_m)$, ${}_2\bar{c}(p)(\Delta_m)$ and ${}_2\bar{c}_0(p)(\Delta_m)$ of interval numbers in terms of the difference operator Δ_m ($m \geq 0$ be an integer). We investigate some algebraic and topological properties like solid, monotone, symmetric, convergence free etc.

Keywords: Interval numbers; Solid; monotone; Convergence free.

AMS Subject Classification No.: 40C05, 40J05, 46A45.

1. Introduction

The concept of interval arithmetic was suggested by Dwyer [15] in 1951. It has been further studied and developed by Moore [8], Moore and Yang [9] and others ([15], [16], [17] and [20]). Chiao [13] studied on sequence of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryilmaz [14] studied bounded and convergent sequence spaces of interval numbers and proved completeness of the spaces. Recently Esi [1-8], Esi and Braha [18], Esi and Esi [19], Esi and Catalbas [21] studied strongly almost-convergence and statistically almost-convergence of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can be considered as a set and we denote the set of all real valued closed intervals by $\mathbb{R}$. Any elements of $\mathbb{R}$ is a closed interval, denoted by $\bar{x}$, defined by $\bar{x} = \{x \in \mathbb{R}: a \leq x \leq b\}$. An interval number $\bar{x}$ is a closed subset of real numbers [15]. Let x_l and x_r be first and last points of interval number $\bar{x}$, respectively. For $x_1, x_2 \in \mathbb{R}$, we have the following arithmetic operations

(i) $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$

(ii) $\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R}: x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$

(iii) $\alpha\, \bar{x} = \{x \in \mathbb{R}: \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ $\alpha \geq 0$

(iv) $\alpha\bar{x} = \{x \in \mathbb{R}: \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$, for $\alpha < 0$

(v) $$\bar{x}_1\bar{x}_2 = \left\{\begin{matrix} x \in R: \\ min\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\} \leq x \leq \\ max\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\} \end{matrix}\right\}$$

The set of all interval numbers $\mathbb{R}$ is a complete metric space with respect to the metric defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1_l} - x_{2_l}|,\ |x_{1_r} - x_{2_r}|\}.$$

In the special case of point interval i.e. $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Consider the transformation $f: N \to \mathbb{R}$, by $k \to f(k) = \bar{x}$, where $x = (x_k)$, then $\bar{x} = (\bar{x}_k)$ is called a sequence of interval numbers. The term $\bar{x}_k$ is called k^{th} term of sequence, $\bar{x} = (\bar{x}_k)$. We denote the set of all sequence of interval numbers with real terms by $\bar{w}$.

2. Preliminaries

Consider the transformation $f : N \times N \to \mathbb{R}$ by $f(n, k) = (\bar{x}_{nk})$, then $\bar{x} = (\bar{x}_{nk})$ is called a double sequence of interval numbers. The $\bar{x}_{nk}$ is the $(n, k)^{th}$ term of the sequence (x_{nk}).

Definition 2.1. An interval valued double sequence $(\bar{x}_{nk})$ is said to be bounded if there exists a positive number B such that $\bar{d}(\bar{x}_{nk}, \bar{0}) < B$ for all $n, k \in N$.

We denote the set of all bounded double sequences of interval numbers by ${}_2\overline{\ell}_\infty$ and the classes of all double sequence of interval numbers by ${}_2\bar{w}$.

Definition 2.2. A sequence $\bar{x} = (\bar{x}_{nk})$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each $\epsilon > 0$ there exists a positive integer k_o such that $d(\bar{x}_{nk}, \bar{x}_o) < \epsilon$ for all $k \geq k_0$ and we denote it by $\lim_{n,k} \bar{x}_{nk} = \bar{x}_o$. This imply that

$$\lim_{n,k} \bar{x}_{nk} = \bar{x}_o \Leftrightarrow \lim_{n,k} x_{nk_l} = x_{0_l} \text{ and } \lim_{n,k} \bar{x}_{nk_l} = x_{0_r}.$$

We denote the set of all convergent double sequence of interval numbers are denoted by ${}_2\bar{c}$.

Definition 2.3. A sequence $\bar{x} = (\bar{x}_{nk})$ of interval numbers is said to be interval valued Cauchy sequence if for every $\epsilon > 0$ there exists a positive integer k_o such that $d(\bar{x}_{nk}, \bar{x}_{mp}) < \epsilon$ for all $n \geq m \geq k_o$, $k \geq p \geq k_o$.

Definition 2.4. An interval valued double sequence space $\bar{E}$ is said to be solid if $\bar{y} = (\bar{y}_{nk}) \in \bar{E}$ whenever $|\bar{y}_{nk}| \leq |\bar{x}_{nk}|$, for all $n, k \in N$ and $\bar{x} = (\bar{x}_{nk}) \in \bar{E}$.

Definition 2.5. An interval valued sequence space $\bar{E}$ is said to be monotone if $\bar{E}$ contains the canonical pre- image of all its step spaces.

Definition 2.6. An interval valued sequence space $\bar{E}$ is said to be convergence free if $\bar{y} = (\bar{y}_{nk})$ $\in\bar{E}$ whenever $\bar{x} = (\bar{x}_{nk}) \in\bar{E}$ and $\bar{x}_{nk} = 0$ implies $\bar{y}_{nk} = 0$.

Throughout the paper, $p = (p_{nk})$ is a sequence of bounded strictly positive numbers, for all n, $k\in N$.

Esi [1] define the following interval valued sequence space:

$$\bar{\ell}(p) = \left\{\bar{x} = (\bar{x}_k): \sum_{k=1}^{\infty}\left[d\left(\bar{x}_{k,}\bar{0}\right)\right]^{p_k} < \infty\right\}.$$

if $p_k = 1$, for all $k\in N$, then we have

$$\bar{\ell} = \left\{\bar{x} = (\bar{x}_k): \sum_{k=1}^{\infty}\left[d\left(\bar{x}_{k,}\bar{0}\right)\right] < \infty\right\}.$$

Kizmaz [12] defined the difference sequence space for crisp set. This concept was further generalized by Tripathy and Esi [12] as follows.

Let $m \geq 0$ be an integer then

$Z_1(\Delta_m) =\{(x_k)\in {}_2\bar{w}: (\Delta_m x_k) \in Z_1\}$, for $Z_1 = \ell_\infty$, c and c_0, where $\Delta_m x_k = x_k - x_{k+m}$, for all $k\in N$. They showed that these are Banach spaces under the norm

$\|x\|_{\Delta_m} = \sum_{r=1}^{m}|x_r| + \sup_k|\Delta_m x_k|$. For $m = 1$, the sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are studied by Kizmaz [12].

In this paper we introduce the double sequence space of interval number studied with the generalized difference operator as follows:

Let $\bar{x}$ = ($\bar{x}_{nk}$) be a double sequence of interval numbers and p = (p_{nk}) is a sequence of bounded positive numbers then for an integer $m \geq 0$ we define

$Z(\Delta_m) = \{(\bar{x}_{nk}) \in {}_2\bar{w} : (\Delta_m \bar{x}_{nk}) \in Z\}$, for $Z = {}_2\bar{\ell}(p)$, ${}_2\bar{c}(p)$ and ${}_2\bar{c}_0(p)$, where $\Delta_m \bar{x}_{nk} = \bar{x}_{nk} - \bar{x}_{k+m}$, for all n, $k \in N$.

3. Main Results

Theorem 3.1 The classes of sequences ${}_2\bar{\ell}(p)(\Delta_m)$, ${}_2\bar{c}(p)(\Delta_m)$ and ${}_2\bar{c}_0(p)(\Delta_m)$ are solid and hence monotone.

Proof: Let $\bar{x} = (\bar{x}_{nk}) \in {}_2\bar{\ell}(p)(\Delta_m)$ and $\bar{y} = (\bar{y}_{nk}) \in {}_2\bar{\ell}(p)(\Delta_m)$ be interval valued double sequences such that $|\bar{y}_{nk}| \leq |\bar{x}_{nk}|$, for all n, $k \in N$. Then

$$\sum_{k=1}^{\infty} \left[d\left(\Delta_m \bar{x}_{nk,} \bar{0}\right)\right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} [d(\Delta_m \bar{y}_{nk}, \bar{0})]^{p_k} \leq \sum_{k=1}^{\infty} \left[d\left(\Delta_m \bar{x}_{nk,} \bar{0}\right)\right]^{p_k} < \infty.$$

Thus $\bar{y} = (\bar{y}_{nk}) \in {}_2\bar{\ell}(p)(\Delta_m)$.

Hence ${}_2\bar{\ell}(p)(\Delta_m)$ is a solid sequence space.

This completes the proof.

Theorem 3.2 The classes of sequences ${}_2\bar{\ell}(p)(\Delta_m)$, ${}_2\bar{c}(p)(\Delta_m)$ and ${}_2\bar{c}_0(p)(\Delta_m)$ are not convergence free.

Proof: Let $m = 2$, we consider the interval sequence $\bar{x} = (\bar{x}_{nk})$ as follows

$$x = \bar{x}_{nk} = \left[\frac{-1}{(n+k)^2}, 0\right],\ \Delta_2\bar{x} = \left[\frac{-1}{(n+k)^2}, \frac{1}{(n+k+2)^2}\right], \text{ for all } n, k \in N.$$

Then, for $p_{nk} = 1$, for all $n, k \in N$, we have

$$\sum_{n,k=1}^{\infty}\left[d\left(\Delta_2\bar{x}_{nk}, \bar{0}\right)\right] < \sum_{n,k=1}^{\infty}\left(\frac{1}{(n+k)^2}\right) < \infty.$$

Thus $\bar{x} = (\bar{x}_{nk}) \in {}_2\bar{\ell}(p)(\Delta_m)$.

Now let us define $\bar{y} = (\bar{y}_{nk})$ as follows

$$\bar{y}_{nk} = [-(n+k)^2, 0], \text{ then } \Delta_2\,\bar{y}_{nk} = [-(n+k)^2,\ (n+k+2)^2], \text{ for all } n,$$

$k \in N$.

Then

$$\sum_{n,k=1}^{\infty}\left[d\left(\Delta_2\ \bar{y}_{nk}, \bar{0}\right)\right] \leq \sum_{n,k=1}^{\infty}(n+k+2)^2 = \infty.$$

Therefore $\bar{y} = (\bar{y}_{nk})) \notin {}_2\bar{\ell}(p)(\Delta_m)$.

Hence the class of sequence ${}_2\bar{\ell}(p)(\Delta_m)$ is not convergence free.

This completes the proof.

Theorem 3.3 The classes of sequence ${}_2\bar{\ell}(p)(\Delta_m)$, ${}_2\bar{c}(p)(\Delta_m)$ and ${}_2\bar{c}_0(p)(\Delta_m)$ are sequence algebra.

Proof. We prove for the sequence space ${}_2\overline{\ell}(p)(\Delta_m)$, and for the other classes the proof follows similarly. Consider the interval valued double sequences $\bar{x} = (\bar{x}_{nk}) \in {}_2\overline{\ell}(p)(\Delta_m)$, and $\bar{y} = (\bar{y}_{nk})$ $\in {}_2\overline{\ell}(p)(\Delta_m)$.

Then we have

$$\sum_{k=1}^{\infty}\left[d\left(\Delta_m \bar{x}_{nk,}\bar{0}\right)\right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty}\left[d\left(\Delta_m \bar{y}_{nk,}\bar{0}\right)\right]^{p_k} < \infty.$$

Now we have

$$\sum_{k=1}^{\infty}[d(\Delta_m(\bar{x}_{nk}\otimes \bar{y}_{nk}),\bar{0})]^{p_k} \leq \sum_{k=1}^{\infty}\left[d\left(\Delta_m \bar{x}_{nk,}\bar{0}\right)(\left(\Delta_m \bar{y}_{nk,}\bar{0}\right)\right]^{p_k}$$

$$< \sum_{k=1}^{\infty}\left[d\left(\Delta_m \bar{x}_{nk,}\bar{0}\right)\right]^{p_k}\sum_{k=1}^{\infty}\left[d\left(\Delta_m \bar{x}_{nk,}\bar{0}\right)\right]^{p_k}$$

$$< \infty.$$

Thus $(\bar{x}_{nk}\otimes \bar{y}_{nk}) \in {}_2\overline{\ell}(p)(\Delta_m)$.

This completes the proof.

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Difference Gai Sequences of Interval numbers with Orlicz Function

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Abstract: In this article we are going to study difference Gai sequence of interval number using Orlicz function. We investigate the completeness property, solidity, symmetricity and convergence free.

Key words: Gai sequence, completeness property, solidity, symmetricity and convergence free.

Subject Classification: 2000 Mathematics 40C05,40J05, 46A45.

1. Introduction

From the 19th century real and complex numbers are used to construct many mathematical structure.Nowadays fuzzy numbers or interval numbers have also being used to replace these real or complex number. It was Dwyer [6] who introduced the concept of interval arithmetic in the year 1951. Later on Moore [8] in 1959, Moore and Young [9] in 1962 have developed interval arithmetic to a formal system. Chiao introduced sequence of interval numbers and defined usual convergence of sequence of interval numbers. Recently Baruah and Dutta [22] introduced new types of difference class of interval numbers where they studied the completeness property, solidness, convergence free for the classes of sequence using difference operator $\ell(p)(\Delta_m)$, $c(p)(\Delta_m)$and $c_0(p)(\Delta_m)$.

2.Definition and Preliminaries:

The closed interval of real number y can be defined as $c \leq y \leq d$ .This closed interval can also be called an interval number. A real interval number can also be considered as a set and we denote the set of all real valued closed interval by IR. Any member of IR is said to be closed interval. If $\bar{y}$ be the member of IR then $\overline{y}$ ={$y \in$ IR $: c \leq y \leq d$}.

Let y_l and y_r be first and last points of $\bar{y}$ interval number respectively. For y_1, $y_2 \in$IR, we have

$$\overline{y_1} = \overline{y_2} \Leftrightarrow y_{1_l} = y_{2_l}, y_{1_r} = y_{2_r},$$

$$\bar{y}_1 + \bar{y}_2 = \{y \in \text{ IR} : y_{1_l} + y_{2_l} \le y \le y_{1_r} + y_{2_r}\}$$

and if $\alpha \ge 0$, then $\alpha\bar{y} = \{y \in \text{ IR}: \alpha y_{1_l} \le y \le \alpha y_{1_r}\}$

and $\alpha < 0$, then

$$\alpha\bar{y} = \{y \in \text{ IR}: \alpha y_{1_r} \le y \le \alpha y_{1_l}\},$$

$$\bar{y}_1\bar{y}_2 = \begin{Bmatrix} x \in \text{ IR} : min\{y_{1_l}y_{2_l}, \ y_{1_l}y_{2_r}, \ \ y_{1_r}y_{2_l}, \ \ y_{1_r}y_{2_r}\} \le y \le \\ max\{y_{1_l}y_{2_l}, \ y_{1_l}y_{2_r}, \ \ y_{1_l}y_{2_l}, \ \ y_{1_r}y_{2_r}\} \end{Bmatrix}$$

The set of all interval numbers IR is a complete metric space defined by

$$d(\bar{y}_1, \ \bar{y}_2) = \max\{|y_{1_l} - y_{2_l}|, \ |y_{1_r} - y_{2_r}|\}$$

In the special case $\bar{y}_1 = [a, a]$ and $\bar{y}_2 = [b, b]$, we obtain usual metric of IR.

Let us define transformation f : N $\rightarrow$IR by $\lambda \rightarrow f(\lambda) = \bar{y}$, $y = (y_\lambda)$. Then $\bar{y} = (\bar{y}_\lambda)$ is called sequence of interval numbers. The $\bar{y}_\lambda$ is called λ^{th} term of sequence $\bar{y} = (\bar{y}_\lambda) \in w^i$ denotes the set of all interval numbers with real terms and the algebraic properties of w^i.

Now we give new definitions for interval sequences as follows:

A sequence space $\bar{E}$with interval value is considered as solid if $\bar{y}=(\bar{y}_\lambda)\in\bar{E}$ whenever$|\bar{y}_\lambda| \le |\bar{x}_\lambda|$,or all $\lambda\in$N and $\bar{x} = (\bar{x}_\lambda)\in\bar{E}$.

A sequence space $\bar{E}$ with interval values is considered to be monotone if $\bar{E}$contains the canonical pre_image of all its step spaces.

A sequence space $\bar{E}$ with interval values is convergence free defined as follows if $\bar{y}=(\bar{y}_\lambda) \in\bar{E}$ whenever $\bar{x} = (\bar{x}_\lambda) \in\bar{E}$ and $\bar{x}_\lambda = 0$implies $\bar{y}_\lambda = 0$.

Now we give the definition of convergence of interval numbers:

Definition 2.1. A sequence $\bar{x}$= ($\bar{x}_{ƛ}$) of interval numbers is said to be convergent to the interval number $\bar{x}_o$if for each ε >0 there exists a positive integer $ƛ_o$ such that d ($\bar{x}_{ƛ}$, $\bar{x}_o$) <ε for all $ƛ$ ≥$ƛo$ and we denote it by $\lim_{ƛ} \bar{x}_{ƛ}$= $\bar{x}_o$.Thus, $\lim_{ƛ} \bar{x}_{ƛ}$= $\bar{x}_o$⇔ $\lim_{ƛ} x_{ƛ_l}$= $x_{0_l}$and$\lim_{ƛ} \bar{x}_{ƛ}$= x_{0_r}

Burton and Coleman [3] defined quasi Cauchy sequence following ways

If $ƛx_{ƛ}$is a null sequence where $ƛx_{ƛ} = x_{ƛ} - x_{ƛ+1}$ then a sequence $\bar{x}$= ($\overline{x_{ƛ}}$) of points in IR is referred to as quasi Cauchy sequence.

Throughout the paper, p=($p_{ƛ}$) is a sequence of bounded non-negative real numbers. Esi [11] define the following interval valued sequence space:

$$\ell_\infty(p) = \left\{\bar{x} = (\overline{x}_{ƛ}): \sum_{ƛ=1}^{\infty} [d(\overline{x_{ƛ}}, \overline{0})]^{p_{ƛ}} < \infty\right\}$$

and if $p_{ƛ}$ = 1 for all $ƛ$∈ N, then we have

$$\ell_\infty = \left\{\overline{x} = (\overline{x}_{ƛ}): \sum_{ƛ=1}^{\infty} [d(\overline{x}_{ƛ,}\overline{0})] < \infty\right\}$$

A continuous, convex, non-decreasing function ß is called an Orlicz function if for $x > 0$ such that $ß(0) = 0$ and $ß(x) > 0$ for $x > 0$ and $ß(x) \to \infty$ as $x \to \infty$.

A function is referred to as a modulus function if the convexity of the Orlicz function is substituted by $ß(x + y) \leq ß(x) + ß(y)$.

An Orlicz function $ß$ is said to satisfy $ƛ_2$ condition for all values u,if there exits $Ƙ$>0, such that $ß(2u) \leq Ƙß(u), u \geq 0$.

Lindenstrauss and Tzafriri [32] used the idea of Orlicz function to construct the sequence space

$$\ell_{ß} = \left\{x_{ƛ} \in w: \sum_{ƛ=1}^{\infty} ß\left(\frac{|x_{ƛ}|}{\rho}\right) < \infty: \text{for some } \rho > 0\right\}$$

The space $\ell_{ß}$ becomes a Banach space, with the norm

$$\|x\| = \left\{ inf\, \rho > 0: \sum_{ƛ=1}^{\infty} ß\left(\frac{|x_{ƛ}|}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz sequence space

A complex sequence whose $ƛ^{th}$ term is $x_{ƛ}$ denoted by $(x_{ƛ})$ or simply x. Let Z_1 be the set of all finite sequence. Let ℓ_{∞}, c and c_0 be the sequence space of bounded, convergent and null sequences of $x = (x_{ƛ})$ respectively. In respect of ℓ_{∞}, c and c_0 we have $\|x\| = \sup_{ƛ} |x_{ƛ}|$, where $x = (x_{ƛ}) \in \ell_{\infty} \subset c \subset c_0$. A sequence $x = (x_{ƛ})$ is called analytic sequence if $\sup_{ƛ} |x_{ƛ}|^{\frac{1}{ƛ}} < \infty$. The vector space of all analytic sequences will be denoted by Z_2. Asequence $x = (x_{ƛ})$ is called gai sequence if $(ƛ!\, |x_{ƛ}|)^{\frac{1}{ƛ}} \to 0$, as $ƛ \to \infty$. The vector space of all gai sequences will be denoted by Z_3

Definition 2.2: An interval number based sequence $\bar{x}= (\bar{x}_{ƛ})$ is referred to as gai sequence if $(ƛ!\, |x_{ƛ}|)^{\frac{1}{ƛ}} \to 0$, as $ƛ \to \infty$.
A sequence $x = (x_{ƛ})$ is called analytic sequence of interval number if $\sup_{ƛ} |x_{ƛ}|^{\frac{1}{ƛ}} < \infty$.
Kizmaz [5] defined the difference sequence space for crisp set. This concept further generalized by Tripathy and Esi [19] as follows:

Let $m \geq 0$ be an integer then $Z_1(Ꜫ_m) =\{(x_{ƛ}) \in w: (Ꜫ_m x_{ƛ}) \in Z_1\}$, for $Z_1=\ell_{\infty}$, c and c_0. Where $Ꜫ_m x_{ƛ} = x_{ƛ} - x_{ƛ+m}$, for all $ƛ \in N$ and they showed that these are Banach spaces under the norm $\|x\|_{Ꜫ_m} = \sum_{r=1}^{m} |x_r| + \sup_{ƛ} |Ꜫ_m x_{ƛ}|$. For m =1, the spaces $\ell_{\infty}(Ꜫ)$, $c(Ꜫ)$ and $c_0(Ꜫ)$ are studied by Kizmaz [5].

In this paper using the difference operator Ƶ_m and Orlicz functions we introduce the following sequence space

$$\text{ʻ}\Upsilon(\text{Ƶ}_m) = \left\{(x_\lambda) \in W : ß\left(d\left(\lambda\,(|\text{Ƶ}_m x_\lambda|)^{\frac{1}{\lambda}}, 0\right)\right) \to 0, \text{ as } \lambda \to \infty\right\}$$

3. Results:

Theorem 3.1: The spaces$\text{ʻ}\Upsilon(\text{Ƶ}_m)$is complete metric space with the following metric

$$\rho(x, \mathrm{y}) = d(x_1, y_1) + \sup_{\lambda}\left[ß\left(d\left(\left(\lambda\,(|\text{Ƶ}_m x_\lambda|, |\text{Ƶ}_m y_\lambda|)\right)^{\frac{1}{\lambda}}\right)\right)\right]$$

Proof: Let(x^i) be a Cauchy sequence in $\text{ʻ}\Upsilon(\text{Ƶ}_m)$ such that

$(x^i) = (x^i_\lambda) = (x^i_1, x^i_2, x^i_3, \ldots\ldots) \in \text{ʻ}\Upsilon(\text{Ƶ}_m)$, for each $i \in N$. Then for a given ε >0, there exists$n_0 \in N$.

$$\rho(x^i, x^j) = d(x^i_1, x^j_1) + \sup_{\lambda}\left[ß\left(d\left(\left(\lambda\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right)\right)^{\frac{1}{\lambda}}\right)\right)\right] < \varepsilon$$

for all $i, j \geq n_0$

Then $d(x^i_1, x^j_1) < \varepsilon,$ for all $i, j \geq n_0$ . . . . (1.1)

$ß\left[d\left(\left(\lambda\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right)\right)^{\frac{1}{\lambda}}\right)\right] < \varepsilon,$ for all $i, j \geq n_0$ . . . . (1.2)

$$\Rightarrow d\left(\left(\lambda\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right)\right)^{\frac{1}{\lambda}}\right) < \varepsilon, \text{for all } i, j \geq n_0$$

$$\Rightarrow \lambda\, d\left(\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right)^{\frac{1}{\lambda}}\right) < \varepsilon, \text{for all } i, j \geq n_0$$

$$\Rightarrow d\left(\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right)^{\frac{1}{\lambda}}\right) < \varepsilon, \text{for all } i, j \geq n_0$$

$$\Rightarrow \left(d\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right)\right)^{\frac{1}{\lambda}} < \varepsilon, \text{for all } i, j \geq n_0$$

$$\Rightarrow d\left(\left|\text{Ƶ}_m x^i_\lambda\right|, \left|\text{Ƶ}_m x^j_\lambda\right|\right) < \varepsilon, \text{for all } i, j \geq n_0$$

$$\Rightarrow d\left(\text{Ƶ}_m x^i_\lambda, \text{Ƶ}_m x^j_\lambda\right) < \varepsilon, \text{for all } i, j \geq n_0$$

Now $\left(x_1^i\right)$ and $\left(ƛ_m x_ƛ^i\right)$ for all $ƛ \in N$ are Cauchy sequence in IR. Since IR is complete, so $\left(x_1^i\right)$ and $\left(ƛ_m x_ƛ^i\right)$ for all $ƛ \in N$ are convergent in IR.

Let $\lim_{i\to\infty} x_1^i = x_1$ . . . . (1.3)

and $\lim_{i\to\infty} ƛx_1^i = z_ƛ$ for all $ƛ \in N$ . . . . (1.4)

From (1.3) and (1.4) we have

$\lim_{i\to\infty} x_ƛ^i = x_ƛ$ for all $ƛ \in N$.

Now fix $i \geq n_0$ and let $j \to \infty$ in (1.1) and (1.2)

We have

$$d\left(x_1^i, x_1\right) < \varepsilon \text{ and } d\left(\left(ƛ\left(\left|ƛ_m x_ƛ^i\right|, |ƛ_m x_ƛ|\right)\right)^{\frac{1}{ƛ}}\right) < \varepsilon, \text{for all } i \geq n_0 \quad ...(1.5)$$

which gives

$$\rho\left(x^i, x\right) < \varepsilon \text{ for all } i \geq n_0$$

ie $x^i \to x$, as $i \to \infty$

Now we shall show that $x \in ՙ\Upsilon(ƛ_m)$.

From (1.5) for all $i \geq n_0$

$$d\left(\left(ƛ\left(\left|ƛ_m x_ƛ^i\right|, |ƛ_m x_ƛ|\right)\right)^{\frac{1}{ƛ}}\right) < \varepsilon,$$

Again for all $i \in N$,

$$x^i = \left(x_ƛ^i\right) \in ՙ\Upsilon(ƛ_m)$$

$$\Rightarrow d\left(\left(\left|ƛ_m x_ƛ^i\right|, \overline{0}\right)^{\frac{1}{ƛ}}\right) < \infty,$$

Now for all $i \geq n_0$ we have

$$\Rightarrow d\left(\left(|\mathrm{ᴧ}_m x_ƛ|, \overline{0}\right)^{\frac{1}{ƛ}}\right) < d\left(\left(|\mathrm{ᴧ}_m x_ƛ|, \left|\mathrm{ᴧ}_m x_ƛ^i\right|\right)^{\frac{1}{ƛ}}\right) + \left(\left(\left|\mathrm{ᴧ}_m x_ƛ^i\right|, \overline{0}\right)^{\frac{1}{ƛ}}\right) < \infty,$$

Hence $x \in$ ʿϒ$(\mathrm{ᴧ}_m)$. This proves the completeness of ʿϒ$(\mathrm{ᴧ}_m)$.

Theorem 3.2: The spaces ʿϒ$(\mathrm{ᴧ}_m)$ is closed under the operation of addition and scalar multiplication

Proof :

(i) Let $x_ƛ \in$ ʿϒ$(\mathrm{ᴧ}_m)$ and $c \in$ IR

Then

$$ß\left[d\left((ƛ\, c|\mathrm{ᴧ}_m x_ƛ|)^{\frac{1}{ƛ}}, \overline{0}\right)\right]$$

$$\leq max(1, c^{\frac{1}{ƛ}})ß\left[d\left((ƛ|\mathrm{ᴧ}_m x_ƛ|)^{\frac{1}{ƛ}}, \overline{0}\right)\right]$$

$$= max(1, c^{\frac{1}{ƛ}}).\overline{0}$$

$$= \overline{0}$$

(ii) Let $x_ƛ, y_ƛ \in$ ʿϒ$(\mathrm{ᴧ}_m)$ then

$$ß\left[d\left((ƛ\,(|\mathrm{ᴧ}_m x_ƛ| \oplus |\mathrm{ᴧ}_m y_ƛ|))^{\frac{1}{ƛ}}, \overline{0}\right)\right]$$

$$\leq Ҝ\left[ß\left[d\left((ƛ\,|\mathrm{ᴧ}_m x_ƛ|)^{\frac{1}{ƛ}}, \overline{0}\right)\right] + ß\left[d\left((ƛ\,|\mathrm{ᴧ}_m y_ƛ|)^{\frac{1}{ƛ}}, \overline{0}\right)\right]\right]$$

$$= \overline{0}$$

Hence $x_ƛ \oplus y_ƛ \in$ ʿϒ$(\mathrm{ᴧ}_m)$

Theorem 3.3: The spaces $\Upsilon(\Delta_m)$ is Solid.

Proof: Let $x = (x_\lambda) \in \Upsilon(\Delta_m)$ and $y = (y_\lambda) \in \Upsilon(\Delta_m)$ be fuzzy sequence such that $|y_\lambda| \leq |x_\lambda|$ for all $\lambda \in N$. Then

$$ß\left[d\left((\lambda|\Delta_m x_\lambda|)^{\frac{1}{\lambda}}, \overline{0}\right)\right] \to 0, \text{as } \lambda \to \infty$$

and

$$ß\left[d\left((\lambda|\Delta_m y_\lambda|)^{\frac{1}{\lambda}}, \overline{0}\right)\right] \leq ß\left[d\left((\lambda|\Delta_m x_\lambda|)^{\frac{1}{\lambda}}, \overline{0}\right)\right] \to 0, \text{as } \lambda \to \infty$$

Thus $y = (y_\lambda) \in \Upsilon(\Delta_m)$ is solid. This completes the proof.

Theorem 3.4: The spaces $\Upsilon(\Delta_m)$ is not Symmetric.

Proof : Example 1. Let $m = 2$ and

Consider the sequence

$$x_\lambda = (\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \ldots\ldots)$$

Then $x_\lambda \in \Upsilon(\Delta_m)$.

Now if y_λ be the rearrangement of x_λ defined by

$$y_\lambda = (\overline{1}, \overline{7}, \overline{3}, \overline{2}, \overline{5}, \overline{4}, \overline{6}, \ldots\ldots)$$

Then $y_\lambda \notin \Upsilon(\Delta_m)$.

Therefore, the spaces $\Upsilon(\Delta_m)$ is not Symmetric.

Theorem 3.5: The spaces ${}^{\backprime}\Upsilon(ƛ_m)$ is not convergence free

Proof: **Example 2.** For each $ƛ \in N$ and m=1 let us consider the sequences

$x_ƛ = \left[-\frac{1}{ƛ}, \frac{1}{ƛ}\right]$and

$$x_{ƛ+1} = \left[-\frac{1}{ƛ+1}, \frac{1}{ƛ+1}\right]$$

$$ƛx_ƛ = \left[-\frac{1}{ƛ} - \frac{1}{ƛ+1}, \frac{1}{ƛ} + \frac{1}{ƛ+1}\right]$$

$$\lim_{ƛ\to\infty} ƛx_ƛ = 0$$

Thus $x_ƛ \in {}^{\backprime}\Upsilon(ƛ_m)$.

But

$$y_ƛ = [-ƛ, ƛ] \text{ and } \quad y_{ƛ+1} = [-(ƛ+1), (ƛ+1)]$$

Then

$$ƛy_ƛ = [-(2ƛ+1), (2ƛ+1)]$$

Therefore $y_ƛ \notin {}^{\backprime}\Upsilon(ƛ_m)$
Hence the sequence space${}^{\backprime}\Upsilon(ƛ_m)$is not convergence frees.
Which complete the result.

4. Conclusions: We have examined some features of Difference Gai Sequences of Interval numbers with the Orlicz function in this article. Using the Orlicz function, we examine the Completeness, Solidness, Symmetricity and Convergence free of Difference Gai Sequences of Interval numbers. This concept will benefit the workers.

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Difference Lacunary Weak Convergence of Sequences Defined by Orlicz Function

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Abstract: This article introduces the concept of difference lacunary weak convergence of sequences defined by an Orlicz function. We examine various algebraic and topological properties and establish several inclusion relations among these spaces.

AMS Subject Classification No: 40A05 ,40A30, 40F05, 42A61
Keywords : Weak convergence, Orlicz function, Lacunary sequence

1.Introduction

Freedman et al. [6] did the first research on lacunary sequences. They investigated strongly Cesàro summable and strongly lacunary convergent sequences, taking consideration of a general lacunary sequence θ, and they identified connections among the two types of classes of sequences. Researchers Ercan et al. [5], Gumus [7], Tripathy and Et [12], Dowari and Triptahy [3,4] have all investigated further lacunary sequences. Recently, generalized difference lacunary weak convergence of sequences was investigated by Tamuli and Tripathy [14,15].

The idea of weak convergence, first proposed by Banach [1], is interesting but has several limitations. Numerous conclusions related to these ideas are generally only valid for separable space. Many others, like Mahanta [11] and Tripathy, have studied vector-valued sequence spaces in recent years.

2. Definition and Preliminaries

Let $\theta = (k_s)$ a sequence of positive integers, then it is called lacunary if $k_0 = 0, 0 < k_s < k_{s+1}$ and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s)$ and $q_s = k_s/k_{s-1}, \forall s \in N$.
According to Freedman et al., the space of lacunary strongly convergent sequence N_θ was defined as follows. [6]

$$N_\theta = \left\{x: \lim_{s\to\infty} \frac{1}{h_s} \sum_{i \in I_s} |x_i - L| = 0, \text{ for some }, L\right\}$$

Difference sequence was introduced by Kizmaz [8]. After that, as explained in [13], the generalized difference sequence spaces were discussed by Esi, Tripathy, and Sarma in the following ways:

Let $p, q \geq 0$ be fixed integers,

$$Z(\Delta_q^p) = \{x = (x_k) \in \omega: \Delta_q^p x = (\Delta_q^p x_k) \in Z\}$$

for $Z = \ell_\infty, c$ and c_0; where $\Delta_q^p x_k = \Delta_q^{p-1} x_k - \Delta_q^{p-1} x_{k+q}$ and $\Delta_q^0 x_k = x_k, \forall k \in \mathbb{N}$. The binomial representation of this generalized difference operator is shown below:

$$\Delta_q^p x_k = \sum_{v=0}^{p} (-1)^v \binom{p}{v} x_{k+qv}, \text{ for all } k \in \mathbb{N} \tag{1}$$

The spaces $\ell_\infty(\Delta), c(\Delta)$, and $c_0(\Delta)$, introduced and investigated by Kizmaz [8], are represented by these spaces for $p = 1$ and $q = 1$. The spaces $\ell_\infty(\Delta^p), c(\Delta^p)$ and $c_0(\Delta^p)$, which were introduced and examined by Et and Colak [2], are represented by these spaces for $q = 1$. The spaces $\ell_\infty(\Delta_q), c(\Delta_q)$ and $c_0(\Delta_q)$, which were introduced and examined by Tripathy and Esi [13], are represented by these spaces for $p = 1$.

The space of sequences $Z(\Delta_q^p)$ for $Z = \ell_\infty, c$ and c_0 are Banach spaces, for the given norm

$$\| x \|_w = \sum_{i=1}^{pq} |x_i| + \sup_k \left|\Delta_q^p x_k\right|, \text{ for } p \geq 1, q \geq 1$$

A function $\mathcal{H}: [0, \infty) \to [0, \infty)$ with $\mathcal{H}(0) = 0, \mathcal{H}(x) > 0$ for $x > 0$, and $\mathcal{H}(x) \to \infty$, as $x \to \infty$ is called an Orlicz function. It is continuous, non-decreasing, and convex.

The concept of the Orlicz function was applied by Lindenstrauss and Tzafriri [9] to create the given sequence space.

$$\ell_{\mathcal{H}} = \left\{(x_i) \in \omega: \sum_{i=1}^{\infty} \mathcal{H}\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

where the class of every sequence is denoted by ω.
For the given norm the sequence space $\ell_{\mathcal{H}}$

$$\| x \| = \inf\left\{\rho > 0: \sum_{i=1}^{\infty} \mathcal{H}\left(\frac{|x_i|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space, also known as an Orlicz sequence space. Tripathy and Esi [13], Parashar and Choudhury [10], Tripathy and Mahanta [11], and many researchers have investigated different types of Orlicz sequence spaces.

Definition 2.1. A sequence (x_i) is said to be weakly convergent in a norm linear space X, if there is an element $L \in X$ such that

$$\lim_{i \to \infty} f(x_i - L) = 0, \text{ for all } f \in X'$$

Here, X' denotes the continuous dual space of X.

Definition 2.2. A sequence (x_i) in a norm linear space X is lacunary weakly convergent to $L \in X$ if

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} f(x_i - L) = 0$$

for all $f \in X'$, where X' is the continuous dual of X.
In this paper, the term $\mathcal{D}_\theta^w$ refers to the lacunary weak convergent.

Definition 2.3. If $(\alpha_i x_i) \in E$ whenever $(x_i) \in E$, for all sequences of scalar (α_i) with $|\alpha_i| \leq 1, \forall i \in \mathbb{N}$, then the sequence space E is said to be solid

Definition 2. 4. The sequence space $E \subset \omega$ is called as monotone if it contains all of the pre-images of its step spaces.

Definition 2.5. The sequence space $E \subset \omega$ is known as symmetric if it satisfies the condition $(x_i) \in E$ implies $\left(x_{\pi(i)}\right)$ belongs to E, where π is a permutation of $\mathbb{N}$.

Definition 2.6. The $\Delta_2 -$ condition is satisfied by an Orlicz function $\mathcal{H}$ if there is a constant $T > 0$ such that, for each $z, \mathcal{H}(2z) \leq T\mathcal{H}(z)$, for $z \geq 0$.

3. Main Result

In this section we introduce the following classes of sequences and establish result invloving them.

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0 = \left\{ x = (x_k) \colon \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g}\right) = 0, \text{ for some } g > 0 \right\}$$

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_1 = \left\{ x = (x_k) \colon \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k - L)|}{g}\right) = 0, \text{ for some } L \text{ and } g > 0 \right\}$$

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\infty = \left\{ x = (x_k) \colon \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g}\right) < \infty, \text{ for some } g > 0 \right\}$$

We state, without proof, the following result.

Theorem 3. 1. The classes of sequences $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_1$ and $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\infty$ are linear spaces.

Theorem 3.2. For any Orlicz function $\mathcal{H}$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\infty$ is a normed linear space for the given norm

$\xi_\Delta(x) = \sum_{i=1}^p |f(x_i)| + \inf\left\{g > 0: \sup_s \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g}\right) \le 1, s = 1,2,3, \ldots\right\};$

where the infimum is taken over all $g > 0$.

Proof: Clearly, $\xi_\Delta(x) = \xi_\Delta(-x), x = \theta$ implies $\Delta x_k = 0$ and as such we have $\mathcal{H}(\theta) = 0$. Therefore $\xi_\Delta(\theta) = 0$. Conversely support that $\xi_\Delta(x) = 0$, then

$$\sum_{i=1}^{p} |f(x_i)| + \inf\left\{g > 0: \sup_s \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g}\right) \le 1, s = 1,2,3, \ldots\right\} = 0$$

$$\Rightarrow \sum_{i=1}^{p} |f(x_i)| = 0 \text{ and } \inf\left\{g > 0: \sup_s \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g}\right) \le 1, s = 1,2,3, \ldots\right\} = 0$$

From the first part we have

$$x_i = \bar{\theta}, \text{ for } i = 1,2,3, \ldots, m$$

where, $\bar{\theta}$ is the zero element. In accordance with this second section, there exists some $g_\varepsilon (0 < g_\varepsilon < \varepsilon)$ for a given $\varepsilon > 0$. such that

$$\sup_s \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g_\varepsilon}\right) \le 1$$

$$\Rightarrow \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g_\varepsilon}\right) \le 1$$

Thus,

$$\sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{\varepsilon}\right) \leq \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g_\varepsilon}\right) \leq 1$$

Suppose $\Delta x_{c_i} \neq \bar{\theta}$, for each i. Taking $\varepsilon \to 0$, we have $\frac{\left|f\left(\Delta x_{c_i}\right)\right|}{\varepsilon} \to \infty$. It follows that

$$\frac{1}{h_s}\sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{\varepsilon}\right) \to \infty$$

as $\varepsilon \to 0$, for $c_i \in I_s$. Hence, we arrive at a contradiction. Therefore, $\Delta x_{c_i} = \bar{\theta}$, for each $i \in N$. Thus $\Delta x_k = \bar{\theta}, \forall k \in N$.

Therefore, it follows from (1) and (2) that $x_k = \bar{\theta}, \forall k \in N$. Hence $x = \theta$. Next let $g_1, g_2 > 0$ such that

$$\sup_s \frac{1}{h_s}\sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g_1}\right) \leq 1$$

and

$$\sup_s \frac{1}{h_s}\sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g_2}\right) \leq 1$$

Let $g = g_1 + g_2$, then we have

$$\sup_s \frac{1}{h_s}\sum_{k\in I_s} \mathcal{H}\left(\frac{|\ f(\Delta(x_k + y_k)\ |}{g}\right) \leq 1$$

Let $\varphi \neq 0$, then

$$\xi_\Delta(\varphi x) = \sum_{i=1}^{p} |\varphi x_i| + \inf\left\{g > 0: \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta(\varphi x_k))|}{g}\right) \leq 1, s = 1,2,3, \ldots\right\}$$

$$\leq |\varphi| \xi_\Delta f(x)$$

This completes the theorem's proof.

Theorem 3.3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Orlicz functions satisfying $\Delta_2 -$ condition. Then
(i) $[\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta]_{\mathcal{G}} \subseteq [\mathcal{D}_\theta^w, \mathcal{H}_2.\mathcal{H}_1, \Delta]_{\mathcal{G}}$.
(ii) $[\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta]_{\mathcal{G}} \cap [\mathcal{D}_\theta^w, \mathcal{H}_2, \Delta]_{\mathcal{G}} \subseteq [\mathcal{D}_\theta^w, \mathcal{H}_1 + \mathcal{H}_2, \Delta]_{\mathcal{G}}$, where $\mathcal{G} = 0$, and ∞.

Proof We prove it in the case of $\mathcal{G} = 0$, we will apply same methods to the remaining cases.
(i) Let $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta]_0$. Then there exists $g > 0$ such that

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}_1\left(\frac{|f(\Delta x_k)|}{g}\right) = 0$$

Let $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that $\mathcal{H}_2(t) < \varepsilon$, for $0 \leq t < \delta$.
Let $y_k = \mathcal{H}_1\left(\frac{|f(\Delta x_k)|}{g}\right)$ and consider

$$\sum_{k \in I_s} \mathcal{H}_2(y_k) = \sum_1 \mathcal{H}_2(y_k) + \sum_2 \mathcal{H}_2(y_k)$$

where the summations are over $y_k > \delta$ in the second summation and over $y_k \leq \delta$ in the first.
Since,

$$\frac{1}{h_s} \sum_1 \mathcal{H}_2(y_k) < \mathcal{H}_2(2) \frac{1}{h_s} \sum_1 (y_k)$$

for $y_k > \delta$, we have

$$y_k < 1 + \frac{y_k}{\delta}$$

Given that $\mathcal{H}_2$ is convex and non-decreasing, it follows thatSince, $\mathcal{H}_2$ is nondecreasing and convex, it follows that

$$\mathcal{H}_2(y_k) < \frac{1}{2}\mathcal{H}_2(2) + \frac{1}{2}\mathcal{H}_2\left(\frac{2y_k}{\delta}\right)$$

Since, $\mathcal{H}_2$ satisfies $\delta_2 -$ conditions, we have

$$\mathcal{H}_2(y_k) = K\frac{y_k}{\delta}\mathcal{H}_2(2)$$

Hence,

$$\frac{1}{h_s}\sum_{2} \mathcal{H}_2(y_k) \leq \max(1, K\delta^{-1}\mathcal{H}_2(2))\frac{1}{h_s}\sum_{2} y_k$$

Taking limit as $s \to \infty$, from (3) and (4) we have

$$(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}_2.\mathcal{H}_1, \Delta]_0$$

Similar proof can be shown for the other cases.
(ii) The proof is obvious and omitted.

Result 3.1. The space $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\mathcal{G}$, where, in general, $\mathcal{G} = 0,1,\infty$ are not solid.

To show that the spaces $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_1, [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\infty$ are not solid, in general, we illustrate the folowing examples.

Example 1. Considering the function $f(x) = x, \forall x \in R$, and for $X = R$. Let us consider the sequence (x_k), defined by $x_k = 1/k, \forall k \in N$. Let $\mathcal{H}(x) = x^r, r \geq 1$ and the lacunary sequence $\theta = (2^s)$. Then $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_1$ and $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\infty$. Let $(\gamma_k) = ((-1)^k)$, then $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_1$ and $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\infty$.

We consider the following example to show that $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0$ is not solid in general.

Example 2. Taking $X = R$ and the function $f(x) = x, \forall x \in R$. Let us now consider the sequence (x_k), which is defined as $x_k = 1, \forall k \in N$. Assume that $\mathcal{H}(x) = x^r, r = 2$, and that the lacunary sequence is $\theta = (2^s)$. Let $(\gamma_k) = ((-1)^k), \forall k \in N$. Then, $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0$.
Thus, the set $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0$ is not solid.

Result 3.2. The spaces $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\mathcal{G}$, where $\mathcal{G} = 0,1,\infty$ are not symmetric in general.
The following example is given to support the previous result.

Example 3. Let $X = R$ and the function $f(x) = x, \forall x \in R$ be considered. Let $\mathcal{H}(x) = x$, and a lacunary sequence $\theta = (2^s)$. Considering the sequence (x_k), we can define it as $x_k = k, \forall k \in N$ is in $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0$. After rearranging the sequence (x_k) as follows, (y_k) will be considered as

$$y_k = (x_1, x_4, x_5, x_8, x_9., ..)$$

Then $(y_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\mathcal{G}$, where $\mathcal{G} = 0,1,\infty$ are not symmetric in general. Hence the spaces $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_\mathcal{G}$, where $\mathcal{G} = 0,1,\infty$ are not symmetric in general.

Proposition 3.1. Let a lacunary sequence $\theta = (k_s)$ with $\liminf_s \mu_s > 1$, then for any Orlicz function $\mathcal{H}$, $[\Omega, \mathcal{H}, \Delta\,]_0 \subseteq [\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0$, where

$$[\Omega, \mathcal{H}, \Delta]_0 = \left\{x = (x_k): \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathcal{H}\left(\frac{|f(\Delta x_k)|}{g}\right) = 0, \text{ for some } g > 0\right\}.$$

Proposition 3.2. Let a lacunary sequence $\theta = (k_s)$ with $\limsup_s \mu_s < \infty$, then for any Orlicz function $\mathcal{H}$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0 \subseteq [\Omega, \mathcal{H}, \Delta]_0$.

Proposition 3.3. Let a lacunary sequence $\theta = (k_s)$ with $1 < \liminf_s \mu_s \leq \limsup_s \mu_s < \infty$, then for any Orlicz function $\mathcal{H}$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta]_0 = [\Omega, \mathcal{H}, \Delta]_0$.

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On Orbits of Sequences of Non-Newtonian Numbers

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Abstract: In this paper, we have investigated the different types of stability of orbits of sequences under a bounded linear operator T, and also we have developed a theory on orbits of sequences.

Keywords: Orbits, stability, orbit space, non-Newtonian numbers

AMS Subject Classification No: 08A05; 11N05; 40A05; 54H20;

1. Introduction

The idea of non-Newtonian numbers was presented by Grossman and Katz [1] in 1972 as an alternative to classical calculus, often known as Newtonian calculus. Çakmak and Basar [2] followed next, defining the field $\mathbb{R}^{(N)}$ of non-Newtonian real numbers and introducing the notions of norm, non-Newtonian metric, and several well-known inequalities following them. Cakmak and Basar [3] classified all classes of non-Newtonian numbers that are all sequences, absolutely p- summable, bounded, convergent, and null. ω^N ℓ_p^N, ℓ_∞^N , c^N, and c_0^N, and are the symbols that represent these classes, respectively. It was demonstrated by Gungor [4] that the sequence of spaces ℓ_∞^N and l_p^N have non-Newtonian strictly convexity and non-Newtonian uniform convexity, also defined some geometric properties of the non-Newtonian sequence spaces ℓ_p(N).

A bijective function α with a domain of R and a range that is a subset of R is called a generator. Consider the following: $\mathbb{R}^{(N)} = \{\alpha(x): x \in R\}$. $\mathbb{R}^{(N+)}$ represents the α - positive real numbers, which are the numbers x in $\mathbb{R}^{(N)}$ for which $\dot{0} < x$, and α-negative real numbers, denoted by $\mathbb{R}^{(N)-}$, are the numbers x for which $\dot{0} > x$. We denote $\alpha(n) = \dot{n}$ for each $n \in \mathbb{Z}$. The arithmetic with the operations described as follows and whose domain is $\mathbb{R}^{(N)}$ is called α-arithmetic.

Considering $x, y \in \mathbb{R}$ and for any generator α,

α-addition	$x \dot{+} y = \alpha\,\{\alpha^{-1}(x) + \alpha^{-1}(y)\}$
α-subtraction	$x \dot{-} y = \alpha\,\{\alpha^{-1}(x) - \alpha^{-1}(x)\}$
α-multiplication	$x \dot{\times} y = \alpha\,\{\alpha^{-1}(x) \times \alpha^{-1}(y)\}$
α-division	$x \dot{/} y = \alpha\,\{\alpha^{-1}(x) / \alpha^{-1}(y)\},\ \alpha^{-1}(y) \neq 0$
α-order	$x \dot{\leq} y \Leftrightarrow \alpha^{-1}(x) \leq \alpha^{-1}(y)$

The set $\mathbb{R}^{(N)}$, with the above operations, forms a complete order field.

The preliminary work on lacunary sequences was done by Freedman et al. [5]. With consideration for a general lacunary sequence θ, they examined strongly lacunary convergent sequences and strongly Cesaro summable sequences, and discovered correlations between the classes of the two types of sequences. In addition, Ercan et al. [6], Gumus [7] explored lacunary sequences. Colak et al. [8], Tripathy and Et [9], Dowari and Tripathy [10]. Karakaya [11] developed some conclusions about the space $\ell_{p,\theta}$ and addressed some inclusion link between ces(p) and $\ell_{p,\theta}$.

Let X be a Banach space over the non-Newtonian field $\mathbb{R}^{(N)}$, and $\|\cdot\|_N: X \to \mathbb{R}^{(N)+} \cup \{\dot{0}\}$ be a function satisfying the non-Newtonian norm axioms: For all $x, y \in X$, and $\eta \in \mathbb{R}^{(N)}$,
(1) $\|x\|_N = \dot{0} \Leftrightarrow x = \dot{0}$
(2) $\|\eta \dot{\times} x\|_N = \eta \dot{\times} \|x\|_N$
(3) $\|x \dot{+} y\|_N \dot{\leq} \|x\|_N \dot{+} \|y\|_N$.

Then $(X, \|x\|_N)$ is said to be a non-Newtonian norm space.

Let X be a Banach space over the non-Newtonian field $\mathbb{R}^{(N)}$, and $<,>_N: X \dot{\times} X \to \mathbb{R}^{(N)}$ be a function satisfying the non-Newtonian inner product axioms:

For all $x, y \in X$, and $\eta \in \mathbb{R}^{(N)}$,

(1) $< x, y >_N \geq \dot{0}$,

(2) $< x, y >_N = \dot{0}$ iff $x = y$,

(3) $< \eta \dot{\times} x, y >_N = \eta \dot{\times} < x, y >_N$,

(4) $< x \dot{+} z, y >_N \dot{\leq} < x, y >_N \dot{+} < z, y >_N$.

Then $(X, <,>_N)$ is said to be a non-Newtonian norm space.

2. **Definition and Preliminaries**:

A sequence of positive integers $\theta = (k_r)$ is said to be b a lacunary sequence if $k_0 = 0, 0 < k_r < k_{(r+1)}$ and $h_r = k_{(r+1)} - k_r \to \infty$, as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r)$ and $q_r = \frac{k_r}{k_{r-1}}$.

Definition 2.1. Let X be a Banach space over $\mathbb{R}^N$, and T be a linear bounded operator. Then the orbit at x is defined by $[x] = \{x, T(x), T^2(x), \dots, T^n(x), \dots\}$ and $V = \{[x]: x \in X\}$ be the set of all orbits of the sequence space X.

Definition 2.2. An orbit $\{x_k\}$ is called Lyapunov stable if for each $\varepsilon \dot{>} \dot{0}$, there exists a $\delta \dot{>} \dot{0}$, for any initial point x such that $\|x \dot{-} y\|_N \dot{<} \delta$ implies

$$\|T^n(x) \dot{-} T^n(y)\|_N \dot{<} \varepsilon, \text{ for all } n \geq 0.$$

Definition 2.3. An orbit $\{x_k\}$ is called lacunary-Lyapunov stable if for each $\varepsilon \dot{>} \dot{0}$, there exists a $\delta \dot{>} \dot{0}$, for any initial point y in X such that $\|x \dot{-} y\|_N \dot{<} \delta$ implies

$$\|T^n(x) \dot{-} T^n(y)\|_{N,\theta} \dot{<} \varepsilon$$

$$\text{i.e., } \frac{1}{h_r} \sum_{k \in I_r} |T^n x_k - T^n y_k|_N \dot{<} \varepsilon, \text{ for all } n \geq 0.$$

Definition 2.4. An orbit $\{x_k\}$ is called asymptotic stable if it is Lyapunov stable and if $\|T^n(x) \dot{-} T^n(x_0)\|_N \to \dot{0}$, as $n \to \infty$.

Definition 2.5. An orbit $\{x_k\}$ is called lacunary-asymptotic stable if it is Lyapunov stable and if

$$\|T^n(x)\dot{-}T^n(x_0)\|_{N,\theta} \to \dot{0}, \text{ as } n \to \infty$$
$$i.e, \frac{1}{h_r}\sum_{k\in I_r} |T^n x_k \dot{-} T^n y_k|_N \to \dot{0}, \text{ as } n \to \infty.$$

Definition 2.6. An orbit $\{x_k\}$ is called exponential stable if there exists contents $C \dot{>} \dot{0}$ and $\dot{0} \dot{<} \lambda \dot{<} \dot{1}$ such that $\|T^n(x) - T^n(x_0)\|_N \leq C\lambda^n \|x - x_0\|_N$, for all $n \in \mathbb{N}$.

Definition 2.7. An orbit $\{x_k\}$ is called lacunary-exponential stable if there exist constants $C \dot{>} \dot{0}$ and $\dot{0}\dot{\lambda} \dot{<} \dot{1}$ such that

$$\|T^n(x)\dot{-}T^n(x_0)\|_{N,\theta} \leq C\lambda^n \|x \dot{-} x_0\|_{N,\theta}, \text{ for all } n \geq 0$$
$$\text{i.e., } \frac{1}{h_r}\sum_{k\in I_r} |T^n x_k \dot{-} T^n y_k|_N \leq C\lambda^n \frac{1}{h_r}\sum_{k\in I_r} |x_k \dot{-} y_k|_N, \text{ for all } n \geq 0.$$

3. Main Results based on orbits of sequences under bounded linear operator

Theorem 3.1. Let X be a Banach space over $\mathbb{R}^N$ and $V = \{[x]: x \in X\}$ be the set of all orbits of the sequence space X. Then the set V is normed linear space with respect to the norm defined by

$$\|[x]\|_N = \sup_{k\geq 0} \|T^k(x)\|_N \text{ ---------- } (1)$$

Theorem 3.2. Let X be a Banach space over $\mathbb{R}^N$, T be a linear bounded operator and $V = \{[x]: x \in X\}$ be the set of all orbits of the sequence space X. The following statements are held: then for each $x \in X$

(i) The orbit $[x]$ is neither open nor closed in general,

(ii) The orbit $[x]$ is connected,

(iii) The orbit $[x]$ is not compact.

Proof: (*i*) First, we prove that $[x]$ is not open, we consider $x = (\dot{1}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots)$, and

$$T^n(\dot{1}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots) = (\dot{0}, \dot{0}, \dots \dot{0}, \dot{1}, \dot{0}, \dot{0}, \dots), (\text{ first n terms are zero })$$

such that $\|T^n(x)\|_N = \dot{1}$, for all $n \in \mathbb{N}$.
Then, for $\varepsilon = \frac{1}{2}$, and $T^k(x)$ be any element in orbit $[x]$, then there is an open set U_ε contains infinitely many elements [for separable sequence space] except $T^k(x)$ and none of them belongs to the orbits $[x]$. Hence, U_ε is not subset of $[x]$. Therefore, the orbit $[x]$ is not open.

To prove $[x]$ is not closed, we consider $X = (l_p)^N$ and T be a right-shift operator on X defined by
$T^n(x_1, x_2, \dots., x_n, \dots) = (\dot{0}, \dots. \dot{0}, x_1, x_2, \dots., x_n, \dots)$(first n terms are zero) for all $n \geq 0$.

Consider $x = (\dot{1}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots)$, Then

$$T^n(\dot{1}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots) = (\dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dot{1}, \dot{0}, \dot{0}, \dots), (\text{ first n terms are zero })$$

Now,

$$T^n(\dot{1}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots) \to (\dot{0}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots), \text{ as } n \to \infty$$

Hence, the orbit $[x]$ does not contains $(\dot{0}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots)$ i.e., non-Newtonian zero element. Therefore, the orbit $[x]$ is not closed.

(*ii*) Suppose the orbit $[x]$ is not connected, then there are two non-empty open sets U and V such that

$$[x] = U \cup V, \text{ with } U \cap V = \phi$$
$$\bar{U} \cup V = \phi$$
$$U \cup \bar{V} = \phi.$$

There exists a natural number $k_0 \in \mathbb{N}$ such that

$$T^{k_0}(x) \in U \text{ but } T^{k_0+i}(x) \in V, \text{ for all } i \in \mathbb{N}$$

Therefore, U contains only finite number of elements, so it is not open set. This contradicts our assumption. Hence $[x]$ is connected.
(iii) The orbit $[x]$ is not closed, this follows from the proof of (i). Therefore $[x]$ is not compact.

Theorem 3.3. Let X be a Banach space over $\mathbb{R}^N$, T be a bounded linear operator with $\|T\|_N < 1$ and $V = \{[x]: x \in X\}$ be the space of orbits of the sequence space X. Then for any initial point $y \in X$, the following statements are equivalent:
(i) The orbit is lacunary Lyapunov stable at any point x
(ii) The orbit is lacunary asymptotic stable at any point x
(iii) The orbit is lacunary exponentially stable at any point x.

Proof: (ii) $\Longrightarrow$ (i) is obvious. We prove that (i) $\Longrightarrow$ (ii).

First, we assume $[x]$ is lacunary Lyapunov-stable i.e., for each $\varepsilon \dot{>} \dot{0}$, there exists a $\delta \dot{>} \dot{0}$, and an initial point y in X such that

$$\|x \dot{-} y\|_{N,\theta} \dot{<} \delta \Longrightarrow \|T^n(x) \dot{-} T^n(y)\|_{N,\theta} \dot{<} \varepsilon$$

Now, for all $n \geq 0$, we have

$$\|T^n(x) \dot{-} T^n(y)\|_{N,\theta} \dot{\leq} \frac{1}{h_r} \sum_{k \in I_r} |T^n x_k \dot{-} T^n y_k|_N$$

$$\dot{\leq} \|T^n\|_N \dot{\times} \tfrac{1}{h_r} \textstyle\sum_{k \in I_r} |x_k \dot{-} y_k|_N$$

$$\dot{\leq} \|T\|_N^n \dot{\times} \|x \dot{-} y\|_{N,\theta}$$

$$\dot{\leq} \|T\|_N^n \dot{\times} \delta$$

As $\|T\|_N \dot{<} \dot{1}$, then $\|T\|_N^n \to \dot{0}$, as $n \to \infty$

$$\text{i.e., } \|T^n(x)\dot{-}T^n(y)\|_{N,\theta} \to \dot{0}, \text{ as } n \to \infty$$

Therefore, (i) ⟺ (ii).
$(iii) \Longrightarrow (ii)$ is obvious. Now, we prove (ii) ⟹ (iii):

We assume $[x]$ is lacunary asymptotic-stable. Then for each $\epsilon > 0$, there exists a constant $C = \dot{1}$ and $\dot{0} \dot{<}\dot{<} \dot{1}$, take $\|T\|_N = \lambda$ and an initial point y in X such that for $\delta = \frac{\epsilon}{C}$, we obtain

$$\|x\dot{-}y\|_{N,\theta} \dot{<} \delta \Longrightarrow \|T^n(x)\dot{-}T^n(y)\|_{N,\theta} \dot{\leq} \frac{1}{h_r}\sum_{k\in I_r} |T^n x_k \dot{-} T^n y_k|_N$$

$$\dot{\leq} \|T^n\|_N \dot{\times} \frac{1}{h_r}\sum_{k\in I_r} |x_k \dot{-} y_k|_N$$

$$\dot{\leq} C \dot{\times} \|T\|_N^n \dot{\times} \|x\dot{-}y\|_{N,\theta}$$

i.e., $\|T^n(x)\dot{-}T^n(y)\|_{N,\theta} \dot{\leq} C \dot{\times} \lambda^n \dot{\times} \|x\dot{-}y\|_{N,\theta}$, for all $n \geq 0$.

Therefore, $[x]$ is lacunary exponential-stable. Therefore, (ii) ⟺ (iii).
Hence, (i) ⟺ (iii).

Corollary 3.4. Let X be a Banach space over $\mathbb{R}^N, T$ be a bounded linear operator with $\|T\|_N \dot{<} \dot{1}$ and $V = \{[x]: x \in X\}$ be the space of orbits of the sequence space X. Then for any initial point $y \in X$, the following statements are equivalent:
(i) The orbit is Lyapunov stable at any point x
(ii) The orbit is asymptotic stable at any point x
(iii) The orbit is exponentially stable at any point x.

Corollary 3.5. Let X be a Banach space over $\mathbb{R}^N, T$ be a bounded linear operator with $\|T\|_N = \dot{1}$ and $V = \{[x]: x \in X\}$ be the space of orbits of the sequence space X. Then for any initial point $y \in X$, then the following statements are held:
(i) The orbit is Lyapunov stable at any point x
(ii) The orbit is not asymptotic stable at any point x
(iii) The orbit is not exponentially stable at any point x.

Proof: (i) Let $[x]$ be an orbit. Then for each $\varepsilon \dot{>} \dot{0}$, there exists a $\delta = (\varepsilon) \dot{>} \dot{0}$ and an initial point y in X such that $\|x - y\|_N \dot{<} \delta$ implies

$$\|T^n(x) \dot{-} T^n(y)\|_{N,\theta} \ \dot{\leq} \frac{1}{h_r} \sum_{k \in I_r} \ |T^n x_k \dot{-} T^n y_k|_N$$

$$\dot{\leq} \|T^n\|_N \dot{\times} \tfrac{1}{h_r} \textstyle\sum_{k \in I_r} \ |x_k \dot{-} y_k|_N$$

$$\dot{\leq} \|x \dot{-} y\|_{N,\theta}$$

$$\dot{<} \varepsilon, \text{ for all, } n \geq 0$$

(ii) Let $\mathrm{X} = (l_\infty)^N$ be a normed space with sup-norm and T be a right shift operator on $(l_\infty)^N$ i.e.,
$T^n(x_1, x_2, \dots., x_n, \dots) = \big(\dot{0}, \dots \dots, \dot{0}, x_1, x_2, \dots., x_n, \dots\big)$ (first n terms are zero), for all $n \in \mathbb{N}$.

Consider $(\dot{1}, \dot{0}, \dot{0}, \dot{0}, \dots, \dot{0}, \dots)$ and $y = (\dot{0}, -1, \dot{0}, \dot{0}, \dots, \dot{0}, \dots)$ with

$$\|x\|_N = \dot{1}, \text{ and } \|y\|_N = \dot{1}$$

Now,

$$\|x \dot{-} y\|_N = \dot{1}$$

$$\Longrightarrow \|T^n(x) \dot{-} T^n(y)\|_N = \|(\dot{0}, \dot{0}, \dots, \dot{0}, \dot{1}, \dot{1}, \dot{0}, \dot{0}, \dots)\|_N$$

$$= \sup_k |x_k|_N = \dot{1}$$

Therefore $\|T^n(x)\dot{-}T^n(y)\|_N \nrightarrow \dot{0}$, as $n \to \infty$.
(iii) This follows immediately from the above part (ii).

Corollary 3.6. Let X be a Banach space over $\mathbb{R}^N, T$ be a bounded linear operator with $\|T\| \dot{>} \dot{1}$ and $V = \{[x]: x \in X\}$ be the space of orbits of the sequence space X. Then for any initial point $y \in X$, then the following statements are held:
(i) The orbit is not Lyapunov stable at any point x,
(ii) The orbit is not asymptotic stable at any point x,
(iii) The orbit is not exponentially stable at any point x.

Proof: (i) Let $[x]$ be an orbit. Then for each $\varepsilon \dot{>} \dot{0}$, there exists a $\delta \dot{>} \dot{0}$ and an initial point y in X such that

$$\|x\dot{-}y\|_{N,\theta} \dot{<} \delta \Longrightarrow \|T^n(x)\dot{-}T^n(y)\|_{N,\theta} \dot{\leq} \frac{1}{h_r}\sum_{k\in I_r} |T^n x_k \dot{-} T^n y_k|_N$$

$$\dot{\leq} \|T^n\|_N \dot{\times} \frac{1}{h_r}\sum_{k\in I_r} |x_k \dot{-} y_k|_N$$

$$\dot{\leq} \|T\|_N^n \dot{\times} \|x\dot{-}y\|_{N,\theta}.$$

As $\|T\| \dot{>} \dot{1}$, then $\|T\|_N^n \to \infty$, as $n \to \infty$

i.e., $\|T^n(x)\dot{-}T^n(y_N)\|_N \to \infty$, as $n \to \infty$

Therefore, $[x]$ is not Lyapunov-stable. Since $[x]$ is not lyapunov-stable, then it can't be asymptotic and exponential-stable.

Theorem 3.7. Let X be a Banach space over $\mathbb{R}^N$ and $V = \{[x]: x \in X\}$ be the space of orbits of the sequence space X. Then
(i) There is a bijective linear function $F: X \to V$ i.e., there is a one-one correspondence from X to V,
(ii) $(x_k) \to x \Leftrightarrow [x_k] \to [x]$
(iii) (x_k) is Cauchy sequence in $X \Leftrightarrow [x_k]$ is Cauchy sequence in V.

Proof: (i) We define $F: X \to V$ by F(x) = [x].
Then it is cleared that F is bijective because for each $x \in X$, there exists al unique orbit [x] in V.
Now, let $x, y \in X$ and $\alpha \in \mathbb{R}^N$. Then

$$F(\alpha x + y) = [\alpha x + y] = \{\alpha x + y, T(\alpha x + y), T^2(\alpha x + y), \dots\}$$

$$= \alpha\{x, Tx, T^2(x), \dots\} + \{y, Ty, T^2(y), \dots\}$$

$$= \alpha[x] + [y]$$

$$= \alpha F(x) + F(y)$$

Therefore, F is linear bijective function.

(ii) Let $(x_k) \to x$ in X. Then for each $\varepsilon > \dot{0}$, there exists a $N \in \mathbb{N}$ such that

$$\|x_n \dot{-} x\| < \frac{\epsilon}{\|T\|_N^\beta}, \text{ for all } n \geq N$$

where $\|T\|_N^\beta \geq \|T\|_N^k$, for all $k \geq 0$.
We have,

$$\|[x_n] \dot{-} [x]\|_N = \|(x_n, T(x_n), T^2(x_n), \dots\dots) - (x, T(x), T^2(x), \dots\dots)\|_N$$

$$= \|(x_n \dot{-} x, T(x_n) \dot{-} T(x), T^2(x_n) \dot{-} T^2(x), \dots\dots)\|_N$$

$$= \sup_{k \geq 0} \|T^k(x) \dot{-} T(x)\|_N$$

Now,

$$\|T^k(x_k)\dot{-}T(x)\|_N \leq \|T^k\|_N \|(x_k\dot{-}x)\|_N$$

$$\leq \|T\|_N^k \|(x_k\dot{-}x)\|_N$$

$$\leq \frac{\varepsilon}{\|T\|_N^\beta} \|T\|_N^k$$

$$\leq \frac{\varepsilon}{\|T\|_N^\beta} \|T\|_N^\beta = \varepsilon$$

i.e., $\sup_{k\geq 0} \|T^k(x)\dot{-}T(x)\|_N \dot{<} \varepsilon$ implies $\|[x_n]\dot{-}[x]\|_N \dot{<} \varepsilon$, for all $n \geq N$.

Conversely, we assume for each $\epsilon \dot{>} \dot{0}$, there exists a $N \in \mathbb{N}$ such that

$$\|[x_n]\dot{-}[x]\|_N \dot{<} \varepsilon, \text{ for all } n \geq N$$
$$\Longrightarrow \sup_{k\geq 0} \|T^k(x_k\dot{-}x)\|_N \dot{<} \varepsilon$$
$$\Longrightarrow \|T^k(x_k\dot{-}x)\|_N \dot{<} \varepsilon, \text{ for all } k \geq 0$$

In particular $\mathrm{k} = 0, T^0 = I$, identity map, so we have

$$\Longrightarrow \|(x_k\dot{-}x)\|_N \dot{<} \varepsilon$$

Hence, $(x_k) \to x$ in X.
(iii) This can be proved from the following proof of (i).

Theorem 3.8. The orbit space V is a Banach space with respect to the norm defined by $Eq(1)$.

Proof: Let $[x_n]$ is Cauchy sequence in V. Then, for each $\varepsilon \dot{>} \dot{0}$, there exists a natural number N such that

$$\|[x_n] \dot{-} [x_m]\|_N \dot{<} \varepsilon, \text{ for all } n, m \geq N. ----------(2)$$

$$\Rightarrow \sup_{k \geq 0} \|T^k(x_n \dot{-} x_m)\|_N \dot{<} \varepsilon, \text{ for all } n, m \geq N$$

$$\Rightarrow \|T^k(x_n \dot{-} x_m)\|_N \dot{<} \epsilon, \text{ for all } n, m \geq N \text{ and for all } k \geq 0.$$

In particular k $= 0, T^0 = I$, Identity map, so we have

$$\Rightarrow \|(x_n \dot{-} x_m)\|_N \dot{<} \varepsilon, \text{ for all } n, m \geq N$$

Since (x_m) be a Cauchy sequence in X, so it is convergent to some element x in X.

Then by Theorem 3.7. (ii), we have

$$[x_m] \to [x]$$

Now, from Eq(2), we have

$$\|[x_n] \dot{-} [x]\|_N \dot{<} \varepsilon, \text{ for all } n, m \geq N$$

Therefore, every Cauchy sequence $[x_n]$ converges in V. Consequently, V is a Banach space.

Theorem 3.9. Let X be a Banach space over $\mathbb{R}^N$ and $V = \{[x]: x \in X\}$ be the space of orbits of the sequence space X. Let Y be a subset of X and W be the corresponding subset of V. Then
(i) If Y is a closed subspace of X, then W is a closed subspace of V,
(ii) If Y is a T-invariant subspace of X, then W is a T-invariant subspace of V,
(iii) If $Y_1 \subset Y_2 \subset X$, then the corresponding subspaces W_1, and W_2 such that $W_1 \subset W_2 \subset V$
(iv) If Y is a dense subspace of the separable space X, then W is a dense subspace of V, and V is also a separable space.

Proof: (*i)* Suppose Y is a closed subspace of X. First, show that W is a subspace of V. For this, consider $x, y \in Y$ and α, β in $\mathbb{R}^N$ such that

$$\alpha x + \beta y \in Y \Longrightarrow [\alpha x + \beta y] \in W$$

. Further, $x, y \in Y$, we have $[x], [y] \in W$. Now,

$$\alpha[x] + \beta[y] = \alpha\{x, T(x), T^2(x), \dots\} + \beta\{y, T(y), T^2(y), \dots\}$$

$$= \{\alpha x, T(\alpha x), T^2(\alpha x), \dots.\} + \{\beta y, T(\beta y), T^2(\beta y), \dots.\}$$

$$= \{\alpha x + \beta y, T(\alpha x + \beta y), T^2(\alpha x + \beta y), \dots\} = [\alpha x + \beta y]$$

Therefore, $[x], [y] \in W$ implies $\alpha[x] + \beta[y] \in W$.
Now, for closedness, suppose $[z] \in \bar{W}$, we show $[z] \in W$. Since $[z] \in \bar{W}$, then $z \in \bar{Y}$, by our assumption, $Y = \bar{Y}$ implies $z \in Y$, we have $[z] \in W$.
The proof of the statements (*ii*), (*iii*) and (*iv*) are easy.

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Some Properties of Ideal Convergent Sequence Spaces of Bi-Complex Numbers

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Abstract: In this article, we study the properties of ideal convergence of a sequence of bi-complex numbers. We shall discuss some of the basics of bi-complex numbers. We have introduced and explored the classes of sequences of $\mathbb{I}$-convergent, $\mathbb{I}$-null, $\mathbb{I}$-Cauchy, $\mathbb{I}$-bounded, $\mathbb{I}_p$-summable sequences of bi-complex numbers and studied their properties. Throughout the article, we use the notation $\mathbb{C}_2$(or $\mathbb{BC}$) as the set of bi-complex numbers with respect to the imaginary units $i_1$and i_2 and $w(\mathbb{BC})$as the class of all sequences of bi-complex numbers.

1 Introduction

The notion of the usual convergence of real sequences was extended to statistical convergence by Fast [8] and Schoenberg [15] independently. Substantial advancements ensued after the pioneering work of Šalát [13], Šalát et al. [14], Tripathy and Hazarika [18], Bera and Tripathy ([2], [3], [4]), and many others. In our study, certain classes of $\mathbb{I}$-convergent sequences of bi-complex numbers have been studied with a functional analytic viewpoint where some properties of ideal convergent sequence spaces of bi-complex numbers are analyzed.

Let X be a non-empty set. A non-empty collection $\mathbb{I}$ of subsets of X is called an ideal, if it satisfies the hereditary Property: If $A \in \mathbb{I}$ and $B \subseteq A$, then $B \in \mathbb{I}$, and the additive Property: If $A, B \in \mathbb{I}$, then $A \cup B \in \mathbb{I}$.

A non-trivial ideal $\mathbb{I}$ in a set X is an ideal that satisfies: $\mathbb{I} \neq 2^X$. A non-empty ideal $\mathbb{I}$ on a set X is said to be admissible, if it contains every singleton subset of X. A non-empty collection F of subsets of X is a filter, if it is closed under finite intersections and supersets, and does not include the empty set. For every ideal $\mathbb{I}$, $\mathbb{F}(\mathbb{I}) = \{K \subseteq \mathbb{N} : \mathbb{N} \setminus K \in \mathbb{I}\}$ is the corresponding filter of $\mathbb{I}$.

Example 1.1. The following are some examples of ideals:

1. The class $\mathbb{I}_f$ of all finite subsets of $2^{\mathbb{N}}$ is a non-trivial admissible ideal of $\mathbb{N}$.

2. Let $\mathbb{I}_\delta = \{A \in 2^{\mathbb{N}} : \delta(A) = 0\}$. Then $\mathbb{I}_\delta$ is a non-trivial admissible ideal of $\mathbb{N}$, where δ is the statistical density of sequences.

3. Similarly, let $\mathbb{I}_d = \{A \in 2^{\mathbb{N}} : d(A) = 0\}$. Then $\mathbb{I}_d$ is an ideal of $\mathbb{N}$, where d is the logarithmic density of sequences.

Ideals and filters are interconnected due to their complementary nature. Specifically, given a filter, an ideal can often be constructed, and vice versa. One may refer to [9] for the details on ideals.

2 Definitions and preliminaries:

Bi-complex numbers have been studied for quite a long time, and a lot of work has been done on them. The work probably began with the work of the Italian school of Segre [16], Spampinato [17], and dragoni [7]. Following this, Price [11], Alpay et al. [1], Değirmen and Sağır [6], Kumar and Tripathy [10], and many others contributed to the study of bi-complex numbers.

Segre defined the bi-complex number ξ in the following manner:

$$\begin{aligned}\xi &= z_1 + i_2 z_2 \\ &= (x_1 + i_1 x_2) + i_2(x_3 + i_1 x_4) \\ &= x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\end{aligned}$$

where $z_1, z_2 \in \mathbb{C}_1$ and the two imaginary units i_1 and i_2 are such that $i_1^2 = i_2^2 = -1$; $i_1 i_2 = i_2 i_1$.

The set of bi-complex numbers is denoted by $\mathbb{C}_2$(or $\mathbb{BC}$)and defined as

$$\mathbb{C}_2 \ = \{z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1(i_1)\},$$

where $\mathbb{C}_1(i_1) = \{x_1 + i_1 x_2 : x_1, x_2 \in \mathbb{C}_0\}$, $\mathbb{C}_0$ is the set of real numbers.

In $\mathbb{BC}$, they are $0, 1, \frac{1+i_1 i_2}{2}$ and $\frac{1-i_1 i_2}{2}$. $\frac{1+i_1 i_2}{2}$, and $\frac{1-i_1 i_2}{2}$ are denoted by e_1 and e_2 and they satisfy: $e_1 + e_2 = 1$, $e_1 e_2 = 0$. Furthermore, every bi-complex number $\xi = z_1 + i_2 z_2 \in \mathbb{BC}$, can be expressed as $\xi = \mu_1 e_1 + \mu_2 e_2$,

where $\mu_1 = z_1 - i_1 z_2$ and $\mu_2 = z_1 + i_1 z_2$ and $\mathbb{BC}$ can be represented as

$$\mathbb{BC} = X_1 e_1 + X_2 e_2,$$

where $X_1 = \{z_1 - i_1 z_2 : z_1, z_2 \in \mathbb{C}_1\}$ and $X_2 = \{z_1 + i_1 z_2 : z_1, z_2 \in \mathbb{C}_1\}$.

The Euclidean norm $\|\cdot\|$ on $\mathbb{C}_2$ is defined as

$$\| \xi \| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

Let $\xi, \eta \in \mathbb{BC}$, then $\| \xi \cdot \eta \| \leq \sqrt{2} \| \xi \| \cdot \| \eta \|$.

One may refer to ([1], [11]) for the details on $\mathbb{BC}$.

We denote the set of all sequences of bi-complex numbers as $w(\mathbb{BC})$. The algebraic operations addition $\oplus$, scalar multiplication $\odot$ and multiplication $\otimes$ defined on $w(\mathbb{BC})$ as follows, respectively:

$\oplus : w(\mathbb{BC}) \times w(\mathbb{BC}) \to w(\mathbb{BC}), (\xi, \eta) \to \xi \oplus \eta = (\xi_k + \eta_k)$,
$\odot : \mathbb{C}_0 \times w(\mathbb{BC}) \to w(\mathbb{BC}), (a, \xi) \to a \odot \xi = (a\xi_k)$,
$\otimes : w(\mathbb{BC}) \times w(\mathbb{BC}) \to w(\mathbb{BC}), (\xi, \eta) \to \xi \otimes \eta = (\xi_k \eta_k)$,
where $\xi = (\xi_k)$, $\eta = (\eta_k) \in w(\mathbb{BC})$ and $a \in \mathbb{C}_0$.

Definition 2.1. [5] A sequence $(\xi_k) \in w(\mathbb{BC})$ is called $\mathbb{I}$-convergent to $\zeta \in \mathbb{BC}$, if for every $\varepsilon > 0$, such that

$$\{k \in \mathbb{N} : \| \xi_k - \zeta \| \geq \varepsilon\} \in \mathbb{I},$$

and it is written as $\mathbb{I} - \lim \xi_k = \zeta$.

Definition 2.2. [5] A sequence $(\xi_k) \in w(\mathbb{BC})$ is called $\mathbb{I}$-null, if $\zeta = 0$ and it is written as $\mathbb{I}-\lim \xi_k = 0$.

Definition 2.3. [5] A sequence $(\xi_k) \in w(\mathbb{BC})$ is called $\mathbb{I}$ -Cauchy, if for every $\varepsilon > 0$, there exists a number m (depending on ε), such that

$$\{k \in \mathbb{N} : \| \xi_k - \xi_m \| \geq \varepsilon\} \in \mathbb{I},$$

and written as $I - \lim \xi_k = \zeta$.

Definition 2.4. [5] A sequence $(\xi_k) \in w(\mathbb{BC})$ is called $\mathbb{I}$-bounded if there exists $M > 0$ such that

$$\{k \in \mathbb{N} : \| \xi_k \| > M \} \in \mathbb{I}.$$

Definition 2.5. [5] Let $(\xi_k), (\eta_k) \in w(\mathbb{BC})$ be two sequences. We say that $\xi_k = \eta_k$, for almost all k relative to $\mathbb{I}(a.a.k.r.\mathbb{I})$, if

$$\{k \in \mathbb{N} : \xi_k \neq \eta_k\} \in \mathbb{I}.$$

Here we define the sets $\mathbb{I}_c(\mathbb{BC}), \mathbb{I}_\theta(\mathbb{BC}), \mathbb{I}_\infty(\mathbb{BC})$ and $\mathbb{I}_p(\mathbb{BC})$ of $\mathbb{I}$-convergent, $\mathbb{I}$-null, $\mathbb{I}$-bounded, $\mathbb{I}_p$-summable.

$\mathbb{I}_c(\mathbb{BC}) \coloneqq \{(\xi_k) \in w(\mathbb{BC}) : \{k \in \mathbb{N} : \|\xi_k - \zeta\| \geq \varepsilon\} \in \mathbb{I}\}$

$\mathbb{I}_\theta(\mathbb{BC}) \coloneqq \{(\xi_k) \in w(\mathbb{BC}) : \{k \in \mathbb{N} : \|\xi_k\| \geq \varepsilon\} \in \mathbb{I}\}$

$\mathbb{I}_{Ca}(\mathbb{BC}) \coloneqq \left\{(\xi_k) \in w(\mathbb{BC}) : \exists m = m(\varepsilon) \text{ such that } \{k \in \mathbb{N} : \|\xi_k - \zeta\| \geq \varepsilon\}\right\}$

$\mathbb{I}_\infty(\mathbb{BC}) \coloneqq \{(\xi_k) \in w(\mathbb{BC}) : \exists M > 0 \text{ such that } \{k \in \mathbb{N} : \|\xi_k\| > M\} \in \mathbb{I}\}$

$$\mathbb{I}_p(\mathbb{BC}) \coloneqq \{(\xi_k) \in w(\mathbb{BC}) : \sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\|^p < \infty \text{ for } 0 < \mathrm{p} < 1 \ \& \left(\sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\|^p\right)^{\frac{1}{p}} < \infty, \text{for } 1 \leq \mathrm{p} < \infty \text{ for some } \{k_1 < k_2 < k_2 < \cdots\} \in F(\mathbb{I})\}$$

Lemma 2.1. [12] The algebraic operations addition $\oplus$, scalar multiplication $\odot$ and multiplication $\otimes$ defined on $w(\mathbb{BC})$ as follows, respectively:

$\oplus: \; w(\mathbb{BC}) \times w(\mathbb{BC}) \to w(\mathbb{BC}), (\xi, \eta) \to \xi \oplus \eta = (\xi_k + \eta_k),$
$\odot: \; \mathbb{C}_0 \times w(\mathbb{BC}) \to w(\mathbb{BC}), (a, \xi) \to a \odot \xi = (a\xi_k),$
$\otimes: \; w(\mathbb{BC}) \times w(\mathbb{BC}) \to w(\mathbb{BC}), (\xi, \eta) \to \xi \otimes = (\xi_k \eta_k),$
where $\xi = (\xi_k)$, $\eta = (\eta_k) \in w(\mathbb{BC})$ and $a \in \mathbb{C}_0$.

Lemma 2.2. [12] The set of all sequences of bi-complex numbers $w(\mathbb{BC})$ is a sequence space.

Lemma 2.3. [12] Let p and q be real numbers with $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\xi_k, \eta_k \in \mathbb{BC}$ for $k \in \mathbb{N}$, then

$$\sum_{i=1}^{\infty} \|\xi_k \eta_k\|^p \leq \sqrt{2} \left(\sum_{i=1}^{\infty} \|\xi_k\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} \|\eta_k\|^q \right)^{\frac{1}{q}}.$$

Lemma 2.4. [12] (Bi-complex Minkowski's Inequality) Let p be real numbers with $1 < p < \infty$ and $\xi_k, \eta_k \in \mathbb{BC}$ for $k \in \mathbb{N}$, then

$$\left(\sum_{k=1}^{n} \|\xi_k + \eta_k\|^p \right)^{\frac{1}{p}} \leq \left[\left(\sum_{k=1}^{n} \|\xi_k\|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} \|\eta_k\|^p \right)^{\frac{1}{p}} \right]$$

3.1. Main Results:

Theorem 3.1. The set $\mathbb{I}_\infty(\mathbb{BC})$ is a sequence space.

Proof: Let, $\xi = (\xi_k), \eta = (\eta_k) \in \mathbb{I}_\infty(\mathbb{BC})$ and $a \in \mathbb{C}_0$ then $\sup_{k_i'} \left\|\xi_{k_i'}\right\| < \infty, \sup_{k_i'} \left\|\eta_{k_i''}\right\| < \infty,$ for some

$K_1 = \{k_1' < k_2' < k_3' < \cdots\} \in \mathbb{F}(\mathbb{I})$ and $K_2 = \{k_1'' < k_2'' < k_3'' < \cdots\} \in \mathbb{F}(\mathbb{I})$.

Let, $K = \{k_1 < k_2 < k_3 < \cdots\} = K_1 \cap K_2 \in \mathbb{F}(\mathbb{I})$.

Now $\sup_{k_i \in K}\|\xi_k + b\eta_k\| \leq \sup_{k_i \in K}\left\|\xi_{k_i}\right\| + \sup_{k_i \in K}\left\|\eta_{k_i}\right\|$

$$\leq \sup_{k_i \in K}\left\|\xi_{k_i}\right\| + \sup_{k_i \in K}\left\|\eta_{k_i}\right\| < \infty$$

Therefore, $\xi \oplus \eta \in \mathbb{I}_\infty(\mathbb{BC})$.

For $a \in \mathbb{C}_0, \sup_{k_i \in K}\left\|a\xi_{k_i}\right\| \leq |a| \sup_{k_i \in K}\left\|\xi_{k_i}\right\| < \infty$.

Therefore, $a \odot \xi \in \mathbb{I}_\infty(\mathbb{BC})$.

And so $\mathbb{I}_\infty(\mathbb{BC})$ is a subspace of the space $w(\mathbb{BC})$.

Hence, the set $\mathbb{I}_\infty(\mathbb{BC})$ is a sequence space.

Theorem 3.2. The class of sequences $\left(\mathbb{I}_\infty(\mathbb{BC}), d_{\mathbb{I}_\infty(\mathbb{BC})}\right)$ is a complete metric space with the metric $d_{I_\infty(\mathbb{BC})}$ defined by

$$d_{I_\infty(\mathbb{BC})}: \mathbb{I}_\infty(\mathbb{BC}) \times \mathbb{I}_\infty(\mathbb{BC}) \longrightarrow [0,\infty), (\xi,\eta) \longrightarrow d_{I_\infty(\mathbb{BC})}(\xi,\eta) = \sup_{k_i \in K}\left\|\xi_{k_i} - \eta_{k_i}\right\| \quad \text{- - (1)}$$

for some $K = \{k_1 < k_2 < k_3 < \cdots\} = K_1 \cap K_2 \in F(\mathbb{I})$ where $\xi = (\xi_k), \eta = (\eta_k) \in \mathbb{I}_\infty(\mathbb{BC})$.

Proof: 1st we proof the metric axioms.

i) $d_{I_\infty(\mathbb{BC})}(\xi,\eta) = 0$

$$\Leftrightarrow \sup_{k_i \in K}\left\|\xi_{k_i} - \eta_{k_i}\right\| = 0$$

$$\Leftrightarrow \xi_{k_i} = \eta_{k_i} \ \forall i \in \mathbb{N}$$

$$\Leftrightarrow \xi_k = \eta_k \text{ for } a.a.k.r.\mathbb{I}.$$

ii) $d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi,\eta) = \sup\limits_{k_i\in K}\left\|\xi_{k_i} - \eta_{k_i}\right\| = \sup\limits_{k_i\in K}\left\|\eta_{k_i} - \xi_{k_i}\right\| = d_{\mathbb{I}_\infty(\mathbb{BC})}(\eta,\xi)$.

iii) Let $\mu \in \mathbb{I}_\infty(\mathbb{BC})$ and $K = K_1 \cap K_2 \cap K_3 \in \mathbb{F}(\mathbb{I})$.

$$d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi,\eta) = \sup_{k_i\in K}\left\|\xi_{k_i} - \eta_{k_i}\right\| \leq \sup_{k_i\in K}\left\|\xi_{k_i} - \mu_{k_i} + \mu_{k_i} - \eta_{k_i}\right\|$$

$$\leq \sup_{k_i\in K}\left\|\xi_{k_i} - \mu_{k_i}\right\| + \sup_{k_i\in K}\left\|\mu_{k_i} - \eta_{k_i}\right\| = d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi,\mu) + d_{\mathbb{I}_\infty(\mathbb{BC})}(\mu,\eta).$$

Therefore, $d_{\mathbb{I}_\infty(\mathbb{BC})}$ satisfies the metric axioms on the space $\mathbb{I}_\infty(\mathbb{BC})$.

Next, we show that $\mathbb{I}_\infty(\mathbb{BC})$ is complete.

Let (ξ_m) be an arbitrary Cauchy Sequence in $\mathbb{I}_\infty(\mathbb{BC})$, where $\xi_m = (\xi_k^m)_k$.

Then, $\exists n_o(\varepsilon) \in \mathbb{N}$, such that $d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi_m,\xi_r) = \sup\limits_{i\in\mathbb{N}}\left\|\xi_{k_i}^m - \xi_{k_i}^r\right\| < \varepsilon \ \forall m,r \geq n_o(\varepsilon)$.

Then, for fixed i, $\left\|\xi_{k_i}^m - \xi_{k_i}^r\right\| < \varepsilon \ \forall m,r \geq n_o$. - - - - - - - - - - - - - (2)

In this case for any fixed $i, (\xi_{k_i}^1, \xi_{k_i}^2, \xi_{k_i}^3, \dots, \xi_{k_i}^m, \dots)$ is a bi-complex Cauchy sequence. So it converges to a point say $\xi_k^* \in \mathbb{BC}$. Define the sequence $\xi^* = (\xi_k^*) = (\xi_1^*, \xi_2^*, \xi_3^*, \dots)$, with infinitely many limits $\xi_1^*, \xi_2^*, \xi_3^*, \dots$ and show $\xi^* \in \mathbb{I}_\infty(\mathbb{BC})$ and $\xi_m \longrightarrow \xi^*\ as\ m \longrightarrow \infty$.

Indeed in (2), by letting $r \longrightarrow \infty$ for any fixed k and using the continuity of Euclidean norm function $\|\cdot\|\ \forall m > n_o(\varepsilon)$, we get $\left\|\xi_{k_i}^m - \xi_{k_i}^*\right\| < \varepsilon$

And so $d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi_m, \xi^*) = \sup\limits_{i\in\mathbb{N}}\left\|\xi_{k_i}^m - \xi_{k_i}^*\right\| < \varepsilon$.

$(\xi_m) \subset \mathbb{I}_\infty(\mathbb{BC})$ converges to $\xi^* = (\xi_k^*) \in w(\mathbb{BC})$.

On the other hand, as $\xi_m = (\xi_k^m)_k \in \mathbb{I}_\infty(\mathbb{BC})$ for each $m \in \mathbb{N}, \exists t_m \in (0,\infty)$ such that $\left\|\xi_{k_i}^m\right\| < t_m$, for all $i \in \mathbb{N}$.

$\left\|\xi_{k_i}^*\right\| \leq \left\|\xi_{k_i}^* - \xi_{k_i}^m\right\| + \left\|\xi_{k_i}^m\right\| < \varepsilon + t_m$ holds for $k_i \in K \in \mathbb{F}(\mathbb{I})$ and $m \geq n_o(\varepsilon)$, which is independent of k. Therefore $\xi^* = (\xi_k^*) \in \mathbb{I}_\infty(\mathbb{BC})$. Hence, $\mathbb{I}_\infty(\mathbb{BC})$ is complete.

Corollary 3.1. The sequence space $\mathbb{I}_\infty(\mathbb{BC})$ is a Banach space with the norm $\|\cdot\|_{I_\infty(\mathbb{BC})}$ defined by

$$\|\xi\|_{\mathbb{I}_\infty(\mathbb{BC})} = \sup_{i\in\mathbb{N}}\left\|\xi_{k_i}\right\|, \xi = (\xi_k) \in \mathbb{I}_\infty(\mathbb{BC}). \text{- - - - - - - - -} (3)$$

Proof: Since Theorem 3.2. confirms that $\mathbb{I}_\infty(\mathbb{BC})$is a complete metric space with the metric $d_{\,\mathbb{I}_\infty(\mathbb{BC})}$ induced by the norm $\|\xi\|_{\mathbb{I}_\infty(\mathbb{BC})}$ as defined by (3), the proof is evident**.**

Theorem 3.3. The sets $\mathbb{I}_c(\mathbb{BC}), \mathbb{I}_\theta(\mathbb{BC})$ and $\mathbb{I}_p(\mathbb{BC})$, for $0 < p < \infty$ are sequence spaces.

Proof: (i) Let $\xi = (\xi_k), \eta = (\eta_k) \in \mathbb{I}_c(\mathbb{BC})$.

Then there exists $\zeta_1^*, \zeta_2^* \in \mathbb{BC}$ such that $I - \lim\limits_{k\to\infty} \xi_k = \zeta_1^*$ and $\mathbb{I} - \lim\limits_{k\to\infty} \eta_k = \zeta_2^*$ and so for every $\varepsilon > 0 \,\exists\, K_1, K_2 \in \mathbb{F}(\mathbb{I})$ such that $K_1 = \{k_1' < k_2' < k_3' < \cdots\} = \{k_i' \in \mathbb{N} : \| \xi_{k_i'} - \zeta_1^* \| < \frac{\varepsilon}{2}\} \in \mathbb{F}(\mathbb{I})$ and $K_2 = \{k_1'' < k_2'' < k_3'' < \cdots\} = \{k_i'' \in \mathbb{N} : \| \eta_{k_i''} - \zeta_2^* \| < \frac{\varepsilon}{2}\} \in \mathbb{F}(\mathbb{I})$.

Let $K = \{k_1 < k_2 < k_3 < \cdots\} = K_1 \cap K_2 \in \mathbb{F}(\mathbb{I})$.

Then, $\left\|(\xi_{k_i} + \eta_{k_i}) - (\zeta_1^* + \zeta_2^*)\right\| \le \left\|(\xi_{k_i} - \zeta_i^*)\right\| + \left\|(\eta_{k_i} - \zeta_2^*)\right\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \;\; \forall k_i \in K.$

Which means $\mathbb{I} - \lim\limits_{k\to\infty} (\xi_k - \eta_k) = \zeta_1^* + \zeta_2^* = \mathbb{I} - \lim\limits_{k\to\infty} \xi_k + \mathbb{I} - \lim\limits_{k\to\infty} \eta_k$.

Therefore, $\xi \oplus \eta \in \mathbb{I}_c(\mathbb{BC})$.

(ii) Let $\xi = (\xi_k) \in I_c(\mathbb{BC})$ and $a \in \mathbb{C}_0 \setminus \{0\}$.

Then there exists $\zeta^* \in \mathbb{BC}$ such that $\mathbb{I} - \lim\limits_{k\to\infty} \xi_k = \zeta^*$ and so for every $\varepsilon > 0 \,\exists\, K, \in \mathbb{F}(\mathbb{I})$ such that $K = \{k_1 < k_2 < k_3 < \cdots\} = \left\{k_i \in \mathbb{N} : \| \xi_{k_i} - \zeta^* \| < \frac{\varepsilon}{|a|}\right\} \in \mathbb{F}(\mathbb{I})$.

Then, $\| a\xi_{k_i} - a\zeta^* \| = \| a(\xi_{k_i} - \zeta^*) \| = |a| \, \| \xi_{k_i} - \zeta^* \| \le |a| \frac{\varepsilon}{|a|} = \varepsilon, \forall k_i \in K.$

Which means, $\mathbb{I} - \lim\limits_{k\to\infty} a\xi_k = a\zeta^* = a\,(\mathbb{I} - \lim\limits_{k\to\infty} \xi_k)$,

Therefore, $a \odot \xi \in \mathbb{I}_c(\mathbb{BC})$.

For $a = 0$, the proof is obvious.

Hence, $\mathbb{I}_c(\mathbb{BC})$ is the sequence space.

For $\mathbb{I}_\theta(\mathbb{BC})$, it is easy to prove that it is the sequence space by taking $\zeta_1^* = \zeta_2^* = \zeta^* = 0$ in the above.

For $\mathbb{I}_p(\mathbb{BC}), 0 < p < \infty$, Let $\xi = (\xi_k), \eta = (\eta_k) \in \mathbb{I}_p(\mathbb{BC})$. Then there exists $K_1 = \{k_1' < k_2' < k_3' < \cdots\} \in \mathbb{F}(\mathbb{I})$ and $K_2 = \{k_1'' < k_2'' < k_3'' < \cdots\} \in \mathbb{F}(\mathbb{I})$ such that $\sum_{i=1}^{\infty} \left\|\xi_{k_i'}\right\| < \infty$ and $\sum_{i=1}^{\infty} \left\|\xi_{k_i^{"}}\right\| < \infty$, respectively.

Let, $K = \{k_1 < k_2 < k_3 < \cdots\} = K_1 \cap K_2 \in \mathbb{F}(\mathbb{I})$. Then the above inequalities hold for all $k_i \in K$.

(i) Now for $0 < p \leq 1$

$$\sum_{i=1}^{\infty} \left\|\xi_{k_i} + \eta_{k_i}\right\|^p \leq \sum_{i=1}^{\infty} \left(\left\|\xi_{k_i}\right\|^p + \left\|\eta_{k_i}\right\|^p\right) = \sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\|^p + \sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\|^p < \infty, \forall k_i \in K.$$

For $1 < p < \infty$ (by Lemma 2.4),

$$\sum_{i=1}^{\infty} \left\|\xi_{k_i} + \eta_{k_i}\right\|^p \leq \left[\left(\sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \left\|\eta_{k_i}\right\|^p\right)^{\frac{1}{p}}\right]^p < \infty, \forall k_i \in K.$$

Therefore, $\xi \oplus \eta \in \mathbb{I}_p(\mathbb{BC})$.

(ii) Let $\xi = (\xi_k) \in \mathbb{I}_p(\mathbb{BC})$ and $a \in \mathbb{C}_0 \setminus \{0\}$. Then there exists $K = \{k_1 < k_2 < k_3 < \cdots\}$ such that $\sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\| < \infty \ \forall k_i \in K \in \mathbb{F}(\mathbb{I})$.

Then, we have $\sum_{i=1}^{\infty} \left\|a\xi_{k_i}\right\|^p = \sum_{i=1}^{\infty} |a|^p \left\|\xi_{k_i}\right\|^p = |a|^p \sum_{i=1}^{\infty} \left\|\xi_{k_i}\right\|^p < \infty$, $\forall k_i \in K \in \mathbb{F}(\mathbb{I})$.

Therefore, $a \odot \xi \in \mathbb{I}_p(\mathbb{BC})$. For $a = 0$, the proof is understood.

Hence, the set $\mathbb{I}_p(\mathbb{BC})$ is a sequence space.

Theorem 3.4. The class of sequences $\left(\mathbb{I}_c(\mathbb{BC}), d_{\mathbb{I}_\infty(\mathbb{BC})}\right)$ is a complete metric space with the metric $d_{\mathbb{I}_\infty(\mathbb{BC})}$ defined by (1).

Proof: Let (ξ_m) be an arbitrary Cauchy Sequence in $\mathbb{I}_\infty(\mathbb{BC})$, where $\xi_m = (\xi_k^m)_k$.

Let (ξ_m) be an arbitrary Cauchy Sequence in $\mathbb{I}_\infty(\mathbb{BC})$, where $\xi_m = (\xi_k^m)_k$.

Then $\exists n_o(\varepsilon) \in \mathbb{N}$, such that $d_{I_\infty(\mathbb{BC})}(\xi_m, \xi_r) = \sup_{i\in\mathbb{N}} \left\|\xi_{k_i}^m - \xi_{k_i}^r\right\| < \frac{\varepsilon}{3}\ \forall m, r \geq n_o(\varepsilon)$.

Then for fixed i, $\left\|\xi_{k_i}^m - \xi_{k_i}^r\right\| < \frac{\varepsilon}{3}\ \forall m, r \geq n_o(\varepsilon)$ - - - - - - - - - - - - - - - - - - (4)

In this case, for any fixed $i, (\xi_{k_i}^1, \xi_{k_i}^2, \xi_{k_i}^3, \dots, \xi_{k_i}^m, \dots)$ is a bi-complex Cauchy sequence. So it converges to a point say $\xi_k^* \in \mathbb{BC}$. Define the sequence $\xi^* = (\xi_k^*) = (\xi_1^*, \xi_2^*, \xi_3^*, \dots)$, with infinitely many limits $\xi_1^*, \xi_2^*, \xi_3^*, \dots$ and show $\xi^* \in \mathbb{I}_\infty(\mathbb{BC})$ and $\xi_m \longrightarrow \xi^*\ as\ m \longrightarrow \infty$.

In (4), by letting $r \longrightarrow \infty$, for any fixed k and using the continuity of Euclidean norm function $\|\cdot\|\ \forall m > n_o(\varepsilon)$, we get $\left\|\xi_{k_i}^m - \xi_{k_i}^*\right\| < \frac{\varepsilon}{3}$.

And so $d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi_m, \xi^*) = \sup_{i\in\mathbb{N}} \left\|\xi_{k_i}^m - \xi_{k_i}^*\right\| < \varepsilon\ \forall m, r \geq n_o(\varepsilon)$.

Therefore, the sequence $(\xi_m) \subset \mathbb{I}_c(\mathbb{BC})$ converges to $\xi^* = (\xi_k^*) \in w(\mathbb{BC})$.

On the other hand as $\left(\xi_k^{n_0}\right)_{k\in\mathbb{N}} \in \mathbb{I}_c(\mathbb{BC})$ is a bi-complex $\mathbb{I}$-Cauchy sequence, for every $\varepsilon > 0\ \exists l$ and $\left\|\xi_{k_i}^{n_0} - \xi_l^{n_0}\right\| < \frac{\varepsilon}{3}\ \forall k_i \in K \in \mathbb{F}(\mathbb{I})$.

Therefore, for every $\varepsilon > 0$,

$$\begin{aligned}\left\|\xi_{k_i}^* - \xi_l^*\right\| &= \left\|\xi_{k_i}^* - \xi_{k_i}^{n_0} + \xi_{k_i}^{n_0} - \xi_l^{n_0} + \xi_l^{n_0} + \xi_l^*\right\| \\ &\leq \left\|\xi_{k_i}^* - \xi_{k_i}^{n_0}\right\| + \left\|\xi_{k_i}^{n_0} - \xi_l^{n_0}\right\| + \left\|\xi_l^{n_0} - \xi_l^*\right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \forall k_i \in K \\ &\in F(\mathbb{I}).\end{aligned}$$

Therefore, $\xi^* = (\xi_k^*) \in \mathbb{I}_c(\mathbb{BC})$ and is a bi-complex $\mathbb{I}$ -Cauchy sequence. Hence $\mathbb{I}_c(\mathbb{BC})$ is complete.

We state the following Corollary without proof.

Corollary 3.2. The sequence space $\left(\mathbb{I}_\theta(\mathbb{BC}), d_{\mathbb{I}_\infty(\mathbb{BC})}\right)$ is a complete metric space with the metric $d_{\mathbb{I}_\infty(\mathbb{BC})}$ defined by (1).

Theorem 3.5. The sequence spaces $\mathbb{I}_c(\mathbb{BC})$ and $\mathbb{I}_\theta(\mathbb{BC})$ are Banach spaces with the norm $\|\cdot\|_{\mathbb{I}_\infty(\mathbb{BC})}$ defined by (3).

Proof: Since theorem 4 and Corollary 2 confirm that $\mathbb{I}_c(\mathbb{BC})$ and $\mathbb{I}_\theta(\mathbb{BC})$ are complete metric spaces with the metric $d_{\mathbb{I}_\infty(\mathbb{BC})}$ induced by the norm $\|\xi\|_{\mathbb{I}_\infty(\mathbb{BC})}$ as defined by (2), the proof is evident**.**

Theorem 3.6. The sequence spaces $(\mathbb{I}_p(\mathbb{BC}), d_{\mathbb{I}_p(\mathbb{BC})})$ are complete metric spaces for $0 < p < \infty$, where $d_{\mathbb{I}_p(\mathbb{BC})}$ is defined as follows:

$$d_{I_p(\mathbb{BC})}(\xi,\eta): \mathbb{I}_p(\mathbb{BC}) \times \mathbb{I}_p(\mathbb{BC}) \to [0,\infty),$$

$$(\xi,\eta) \to d_{\mathbb{I}_p(\mathbb{BC})}(\xi,\eta) = \begin{cases} \displaystyle\sum_{i=1}^{\infty} \left\|\xi_{k_i} - \eta_{k_i}\right\|^p, 0 < p \le 1 \\ \left(\displaystyle\sum_{i=1}^{\infty} \left\|\xi_{k_i} - \eta_{k_i}\right\|^p\right)^{\frac{1}{p}}, 1 < p < \infty \end{cases}$$

for some $K = \{k_1 < k_2 < k_3 < \cdots\} = K_1 \cap K_2 \in \mathbb{F}(\mathbb{I})$ where $\xi = (\xi_k), \eta = (\eta_k) \in \mathbb{I}_p(\mathbb{BC})$.

Proof: Let $1 < p < \infty$.

Now 1st we proof the metric axioms.

i) $d_{\mathbb{I}_p(\mathbb{BC})}(\xi,\eta) \ge 0$ as $\left\|\xi_{k_i} - \eta_{k_i}\right\| \ge 0\ \forall \xi,\eta \in \mathbb{I}_p(\mathbb{BC})$.

Now $d_{\mathbb{I}_p(\mathbb{BC})}(\xi,\eta) = 0 \Leftrightarrow \left(\sum_{i=1}^{\infty}\left\|\xi_{k_i} - \eta_{k_i}\right\|^p\right)^{\frac{1}{p}} = 0$

$$\Leftrightarrow \left\|\xi_{k_i} - \eta_{k_i}\right\|^p = 0, \forall i \in \mathbb{N}$$

$$\Leftrightarrow \left\|\xi_{k_i} - \eta_{k_i}\right\| = 0, \forall i \in \mathbb{N}$$

$$\Leftrightarrow \xi_{k_i} = \eta_{k_i}, \forall i \in \mathbb{N}$$

$$\Leftrightarrow (\xi_k) = (\eta_k) \text{ for } a.a.k.r.\mathbb{I}.$$

ii) $d_{\mathbb{I}_p(\mathbb{BC})}(\xi,\eta) = \left(\sum_{i=1}^{\infty}\|\xi_{k_i}-\eta_{k_i}\|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}\|\eta_{k_i}-\xi_{k_i}\|^p\right)^{\frac{1}{p}} = d_{\mathbb{I}_\infty(\mathbb{BC})}(\eta,\xi)$.

iii) Let $\mu = (\mu_k) \in \mathbb{I}_p(\mathbb{BC})$ and $K = K_1 \cap K_2 \cap K_3 \in \mathbb{F}(\mathbb{I})$.

$$d_{\mathbb{I}_p(\mathbb{BC})}(\xi,\eta) = \left(\sum_{i=1}^{\infty}\|\xi_{k_i}-\eta_{k_i}\|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{\infty}\|(\xi_{k_i}-\mu_{k_i}) + (\mu_{k_i}-\eta_{k_i})\|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=1}^{\infty}\|\xi_{k_i}-\mu_{k_i}\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}\|\mu_{k_i}-\eta_{k_i}\|^p\right)^{\frac{1}{p}}$$

$$= d_{\mathbb{I}_p(\mathbb{BC})}(\xi,\mu) + d_{\mathbb{I}_p(\mathbb{BC})}(\mu,\eta).$$

Therefore, $d_{\mathbb{I}_p(\mathbb{BC})}$ satisfies the metric axioms on the space $\mathbb{I}_p(\mathbb{BC})$ for $0 < p < \infty$.

Next, we show that $\mathbb{I}_p(\mathbb{BC})$ is complete.

Let (ξ_m) be an arbitrary Cauchy Sequence in $\mathbb{I}_p(\mathbb{BC})$, where $\xi_m = (\xi_k^m)_{k\in\mathbb{N}}$.

Then for every $\varepsilon > 0, \exists n_o(\varepsilon) \in \mathbb{N}$, such that

$$d_{\mathbb{I}_\infty(\mathbb{BC})}(\xi_m,\xi_r) = \left(\sum_{i=1}^{\infty}\|\xi_{k_i}-\eta_{k_i}\|^p\right)^{\frac{1}{p}} < \varepsilon\ \forall m,r \geq n_o(\varepsilon).$$

Then for fixed i, $\|\xi_{k_i}^m - \xi_{k_i}^r\| < \varepsilon\ \forall m,r \geq n_o(\varepsilon)$. - - - - - - - - - - - - - - - -(5)

In this case for any fixed $i, (\xi_{k_i}^1, \xi_{k_i}^2, \xi_{k_i}^3, \dots, \xi_{k_i}^m, \dots)$ is a bi-complex Cauchy sequence. So it converges to a point say $\xi_k^* \in \mathbb{BC}$. Define the sequence $\xi^* = (\xi_k^*) = (\xi_1^*, \xi_2^*, \xi_3^*, \dots)$, with infinitely many limits $\xi_1^*, \xi_2^*, \xi_3^*, \dots$ and show $\xi^* \in \mathbb{I}_p(\mathbb{BC})$ and $\xi_m \longrightarrow \xi^*\ as\ m \longrightarrow \infty$.

And in (5), by letting $r \longrightarrow \infty$ we have $\left(\sum_{i=1}^{n}\left\|\xi_{k_i}^{m}-\xi_{k_i}^{*}\right\|^{p}\right)^{\frac{1}{p}} < \varepsilon \; \forall n \in \mathbb{N}$

Now for n tends to infinity, $d_{I_p(\mathbb{BC})}(\xi_m, \xi^*) = \left(\sum_{i=1}^{n}\left\|\xi_{k_i}^{m}-\xi_{k_i}^{*}\right\|^{p}\right)^{\frac{1}{p}} \leq \varepsilon$.

$(\xi_m) \subset \mathbb{I}_p(\mathbb{BC})$ converges to $\xi^* = (\xi_k^*) \in w(\mathbb{BC})$.

On the other hand, as $\xi_m = (\xi_k^m)_k \in \mathbb{I}_p(\mathbb{BC})$. By bi-complex Minkowski's inequality and convergence of series $\sum_{i=1}^{n}\left\|\xi_{k_i}^{m}-\xi_{k_i}^{*}\right\|^{p}$,

$$\left(\sum_{i=1}^{n}\left\|\xi_{k_i}^{*}\right\|^{p}\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n}\left\|\xi_{k_i}^{m}-\xi_{k_i}^{*}\right\|^{p}\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=1}^{n}\left\|\xi_{k_i}^{m}\right\|^{p}\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}\left\|\xi_{k_i}^{*}-\xi_{k_i}^{m}\right\|^{p}\right)^{\frac{1}{p}} < \infty$$

holds for $k_i \in K \in F(\mathbb{I})$, which is independent of k. Therefore $\xi^* = (\xi_k^*) \in \mathbb{I}_\infty(\mathbb{BC})$. Hence, $\mathbb{I}_p(\mathbb{BC})$ is complete for $1 < p < \infty$.

Similarly, we can show that $d_{\mathbb{I}_p(\mathbb{BC})}(\xi_m, \xi^*) = \sum_{i=1}^{\infty}\left\|\xi_{k_i}-\eta_{k_i}\right\|^{p}$, is a metric and $\mathbb{I}_p(\mathbb{BC})$ is complete for $1 < p < \infty$.

Corollary 3.3. The sequence spaces $\mathbb{I}_p(\mathbb{BC})$ are Banach spaces with the norm $\|\cdot\|_{I_p(\mathbb{BC})}$ defined by

$$\|\xi\|_{\mathbb{I}_p(\mathbb{BC})} = \begin{cases} \sum_{i=1}^{\infty}\left\|\xi_{k_i}-\eta_{k_i}\right\|^{p}, 0 < p \leq 1 \\ \left(\sum_{i=1}^{\infty}\left\|\xi_{k_i}-\eta_{k_i}\right\|^{p}\right)^{\frac{1}{p}}, 1 < p < \infty \end{cases},$$

for some $K = \{k_1 < k_2 < k_3 < \cdots\} = K_1 \cap K_2 \in \mathbb{F}(\mathbb{I})$, where $\xi = (\xi_k) \in \mathbb{I}_p(\mathbb{BC})$.

Proof: The proof is clear from the above theorem 6.

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A New Generalization of Set Theory

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Abstract: In this article we are going to define a set theoretic approach which is the generalization of set theory, fuzzy set and multiset. We define some operation on the basis of the new generalization set, some application of this set with interesting examples. We investigate and established some valuable result on the set which shows this theoretic approach very much advanced then other sets.

Keywords: New Generalization Set; N-G membership, Union, Intersection, symmetric difference.

AMS Classification No: 03Exx, 03E15, 03C62.

1. Introduction:

In the classical set theory Cantor set has property that the appearance of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set which is a set of points lying on a single line segment that has a number of remarkable and deep properties. It was discovered in 1874 by Henry John Stephen Smith and introduced by German mathematician Cantor in 1883[2]. Later on fuzzy sets (aka uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Zadeh [7] in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structure called an L-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are used now in different areas, such as linguistics, decision-making and clustering are special cases of L-relations when L is the unit interval [0, 1].

We have Cantor set, fuzzy set but in some case they also fail to represent real objects for overcome such kind of problems Wayne Blizard [1] in 1970 established the multisets theory. Multiset or bag set is a modification of the concept of a set that, unlike a set, allows for multiple instances for each of its elements. The positive integer number of instances, given for each element is called the multiplicity of this element in the multiset. As a consequence, an infinite number of multisets exist, which contain only elements *x* and *y*, but vary by the multiplicity of their elements. Blizard traced multisets back to the very origin of numbers, arguing that "in ancient times, the number n was often represented by a collection of n strokes, tally marks, or units. These and similar collections of objects are multisets, because strokes, tally marks, or units are considered indistinguishable. This shows that people implicitly used multisets even before mathematics emerged. After discovering the Blizard's multiset theory classical set theory has become a particular case of multiset theory. Many mathematicians had studied on this set. Now a day we have multiset and fuzzy sset for multiset we had defined count function and for other type of set which is not well define we have membership function but we can't apply both the case at a times. But in real life we may have uncertinity and multiplicity. In this article we have solve this problems which is also may considerable for the generalization of multiset, fuzzy set and classical set theory.

2. Preliminary

In this section we will give some definition from previous research work for the establishment of this paper.

Definition 2.1. A domain X, is defined as the set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are from X such that no element occurs more than w times.

Throughout this paper, we denote a multiset drawn from the multiset space $[X]^w$ by M

Definition 2.2. An mset M drawn from the set X is represented by a count function M or C_M:$X\rightarrow N$, where N represents the set of nonnegative integers.

Here $C(x)$ is the number of occurrences of the element x in the mset M drawn from the set X = $\{x_1, x_2, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ...m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, i = 1, 2, ... n in the mset M. The elements which are not included in the mset M have zero count.

Remark 2.1. We introduce some new notions on multiset topological space on the basis of the count function. Whenever $C_M(x) = 1$ for every $x \in X$ multisets become structurally equivalent to the class of sets. So whatever results and definition we establish when restricted to this condition must be equivalent to some results in classical set theory.

Consider two msets M and N drawn from a set X. The following are the operations defined on the msets will be used in this article

For the details of multisets such as addition, union, intersection, subtraction, compliment, one may refer [4, 5]

The following are basic operations under collection of msets. Let $[X]^w$ be an mset space with $C_Z(x)$ as the multiplicities of $x \in X$ and $\{M_1, M_2, ...\}$be a collection of msets drawn from $[X]^w$. Then the following operations are possible under arbitrary collections of msets.

1. The Union is defined by
   $\cup_{i \in I} M_i = \{C_M(x)/x : C_M(x) = \max\{C_{M_i}(x) : i \in I\}$,for all $x \in X\}$.
2. The intersection is defined by
   $\cap_{i \in I} M_i = \{C_M(x)/x : C_M(x) = \min\{C_{M_i}(x) : i \in I\}$,for all $x \in X\}$.
3. The mset complement is defined by
   $M^c = Z \ominus M = \{C_M^c(x)/x : C_M^c(x) = C_Z(x) - C_M(x)$, for all $x \in X\}$.

Definition 2.3. Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$.

Definition 2.4. An mset M is said to be an empty set if for all $x \in X$, $C_M(x) = 0$.

Definition 2.5. Let X be a support set and $[X]^w$be the mset space defined over X. Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C_{Mc} = w - C_M(x)$ for all $x \in X$.

The following types of submsets of *M* can be defined from the mset space $[X]^w$ on the basis of multiplicity of elements.

Definition 2.6. A submset *N* of *M* is a whole submset of *M* with each element in *N* having full multiplicity as in *M* i.e., $C_N(x) = C_M(x)$ for every $x \in N$.

Definition 2.7. A submset *N* of *M* is a partial whole submset of *M* is a partial whole submset of *M* with at least one element in *N* having full multiplicity as in *M* i.e. $C_N(x)=C_N(x)$ for some *x* in *N*.

Definition 2.8. A submset N of M is a full submset of *M* if each element in *M* is an element in *N* with the same or lesser multiplicity as in *M*

i.e.$C_N(x) \leq C_M(x)$ for every $x \in N$.

Some basic information about fuzzy set.

Definition 2.9 A fuzzy set is a pair (*X*, *m*) where *X* is a set and m: $X \to [0, 1]$ a membership function. The reference set *X* is called universe of discourse, and for each $x \in X$ the value $m(x)$ is called the grade of membership of *x* in (*X*, *m*). The function $m = \mu_A$ is called the membership function of the fuzzy set *A* = (*X*, *m*).

For a finite set $X = \{x_1, x_2, x_3, \ \ , x_n\}$ the fuzzy set (*X*, *m*) is often denoted by

$$\{m(x_1)/x_1, m(x_2)/x_2, m(x_3)/x_3, ... \ , m(x_n)/x_n\}$$

Let $x \in X$ Then *x* is called

- not included in the fuzzy set (*X*, *m*) if $m(x) = 0$ (no member),
- fully included if $m(x) = 1$ (full member),
- partially included if $0 \leq m(x) \leq 1$ (fuzzy member).

Definition 2.10 A fuzzy set $A = (X, m)$ is empty ($A = \emptyset$) iff (if and only if)

for each $x \in X$ such that $\mu_A(x) = m(x) = 0$

Definition 2.11 Two fuzzy sets A and B are equal ($A = B$) iff

for each $x \in X$ such that $\mu_A(x) = \mu_B(x)$

Definition 2.12 A fuzzy set A is included in a fuzzy set B ($A \subseteq B$) iff

for each $x \in X$ such that $\mu_A(x) \leq \mu_B(x)$.

Definition 2.13 The complement of a fuzzy set A is denoted by $\neg A$ (sometimes denoted as A^c) is defined by the following membership function:

for each $x \in X$ such that $\mu_{A^c}(x) = 1 - \mu_A(x)$.

Cardinality of fuzzy set:

For a fuzzy set A with finite supp(A) (i.e. a 'finite fuzzy set'), its cardinality or scalar cardinality or sigma-count is given by

Card(A) = sc(A) = |A| = $\sum_{x \in X} \mu_A(x)$.

In case that X itself a finite set, the relative cardinality is given by

$RealCard(A) = \|A\| = sc(A)/|X| = |A|/|X|$

Disjoint fuzzy sets:

Two fuzzy set A and B is said to be disjoint fuzzy sets iff the following mathematical condition holds

$\forall x \in X$: $\mu_A(x) = 0 \vee \mu_B(x) = 0$.

Which is equivalent to
$\nexists x \in X: \mu_A(x) > 0 \wedge \mu_B(x) > 0$

and also equivalent to

$$\forall x \in X: \min\{\mu_A(x),\ \mu_B(x)\} = 0$$

For disjoint fuzzy sets A, B any intersection will give $\emptyset$, and any union will give the same result.

For more about fuzzy set theory we may refer to [7,8, 9].

3. Main result

In this section we define some new definition and established some results on New GS set

The New Generalization set in shortly denoted by New-GS set.

Define The New Membership function Γ: $X \rightarrow Q^{+} \cup \{0\}$

Definition 3.1. A domain *X* is defined as the set of elements from which New Generalization set is constructed. The New Generalization set $[X]^w$ is the set of all New Generalization set whose elements are from *X* such that no element occurs more than *w* times.

Definition 3.2. A New Generalization set *N* drawn from the set *X* is represented by a New Membership function Γ: $X \rightarrow Q^{+} \cup \{0\}$, where Q^{+} represents the set of nonnegative rational.

Here $\Gamma(x)$ is the number of occurrences of the element *x* in the New Generalization set *N* drawn from the set $X = \{x_1, x_2, ..., x_n\}$ as $N = \{\frac{\Gamma_1}{x_1}, \frac{\Gamma_2}{x_2}, ... \frac{\Gamma_n}{x_n}\}$ where Γ_i is the number of occurrences of the element x_i, i = 1, 2, ...*n* in the New Generalization set *N*. The elements which are not included in the New Generalization set *N* have zero count.

Example 3.1 Let $[X]^{10}$ ={x, y, z, t} are the elements the New-GS set is N= $\{\frac{3_5^2}{x}, \frac{4_7^2}{y}, \frac{4_5^1}{z}, \frac{8_9^2}{t}\}$, where the New Membership of the element x is 3_5^2 means the appearance 3 times and $\frac{2}{5}$ part of another one element which is fuzziness of the element.

The following types of sub New Generalization sets of N can be defined from the domain of New Generalization set $[X]^w$ on the basis of New Membership of elements.

Definition 3.3. A sub New-GS set M of N is a whole sub New-GS of N with each element in M having full mew membership as in N i.e., $\Gamma_N(x) = \Gamma_M(x)$ for every $x \in N$.

Definition 3.4. A sub New-GS set N of M is said to be empty New-GS set if the New Membership function is zero for all x in N.

The empty New-GS set is denoted by $Ne_\emptyset$ where $Ne_\emptyset$ = { } and $\Gamma_{Ne_\emptyset}$= 0, $\forall x \in X$.

Definition 3.5. A sub New-GS set N_1 of N is a full sub New-GS set of N if each element in N_1 is an element in N with the same or lesser New memberhip as in N

i.e, $\Gamma_{N_1}(x) \leq \Gamma_N(x)$.

Definition 3.6 Let $N_1$and N_2 be two New-GS set the union of two New-GS set is defined by $N = N_1 \cup N_2$ and Γ_N= $\Gamma_{N_1 \cup N_2}$=max$\{\Gamma_{N_1}, \Gamma_{N_2}\}$.

Definition 3.7 Let $N_1$and N_2 be two New-GS set the intersection of two New-GS set is defined by $N = N_1 \cap N_2$ and Γ_N= $\Gamma_{N_1 \cap N_2}$=min$\{\Gamma_{N_1}, \Gamma_{N_2}\}$.

Definition 3.8 The complement of a New GS set A is denoted by $\neg A$ (sometimes denoted as A^c) is defined by the following New GS membership function:

for each $x \in X$ such that $\Gamma_{A^c}(x) = w - \Gamma_A(x)$.

Cardinality of New Generalization set:

For a New GS set A the cardinality of A is the total number of element in A in case of finite set the cardinality will be finite in case of infinite set the cardinality will have different notion for countable infinite and uncountable New GS set.

In case of finite set with weight *w* (maximum occurrence of any element not more than *w*) the cardinality of *A* is given by

Card(A) = |A| = $\sum_{x\in X} \Gamma_A(x)$.

In case of *A* countable infinite New Generalization set the cardinality of *A* is define by Card(A) = |A| = $\sum_{x\in X} \Gamma_A(x) = NG_{n_0}$.

In case of *A* uncountable New Generalization set the cardinality of *A* is define by Card(A) = |A| = $\sum_{x\in X} \Gamma_A(x) = NG_{\mathcal{C}_0}$.

Disjoint of New Generalization set:

Two New GS set *A* and *B* is said to be disjoint New GS sets iff the following mathematical condition holds

$$\forall x \in X\text{: min } \{\Gamma_A(x),\ \Gamma_B(x)\} = 0$$

Which is equivalent to

$\forall x \in X: \Gamma_A(x) = 0 \lor \Gamma_B(x) = 0$ .or

$\nexists x \in X: \Gamma_A(x) > 0 \land \Gamma_B(x) > 0$

For disjoint New GS sets A, B any intersection will give $\Gamma_{Ne_\emptyset}$.

We have established some result and supportable example

Theorem 3.1 Every multiset is a New-GS set but not necessarily conversely.

Proof: from the definition of multiset let $M = \{\frac{m_1}{x_1}, \frac{m_2}{x_2}, , \ldots , \frac{m_n}{x_n}\}$ be a multiset where m_i be the multiplicity of each x_i and each m_i is a positive integer now this set can also be a New-GS set where the m_i be the New membership treat as Γ_i of each x_i , where the fractional part of the new membership is zero.

In case of multiset every fractional part of new membership always zero.

Hence every multiset is a New-GS set.

For converse part we need an example 2.1

Where example 2.1 is a New-GS set but not a multiset.

Hence prove the theorem.

Theorem 3.2 Every Scrip set is a New-GS set but not necessarily conversely.

Proof: Since every scrip set is a multiset and every multiset is a New-GS set.

So every Scrip set is a New-GS set.

Using above theorem we can say that the converse is not true.

Theorem 3.3 Every fuzzy set is a New-GS set but not necessarily conversely.

Proof: Let $X = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$ be a domain set

Then A = $\{(x_1, \mu_1), (x_2, \mu_2), (x_3, \mu_3), \ldots\ldots, (x_n, \mu_n)\}$ is a fuzzy set where each μ_i is the membership value of x_i.

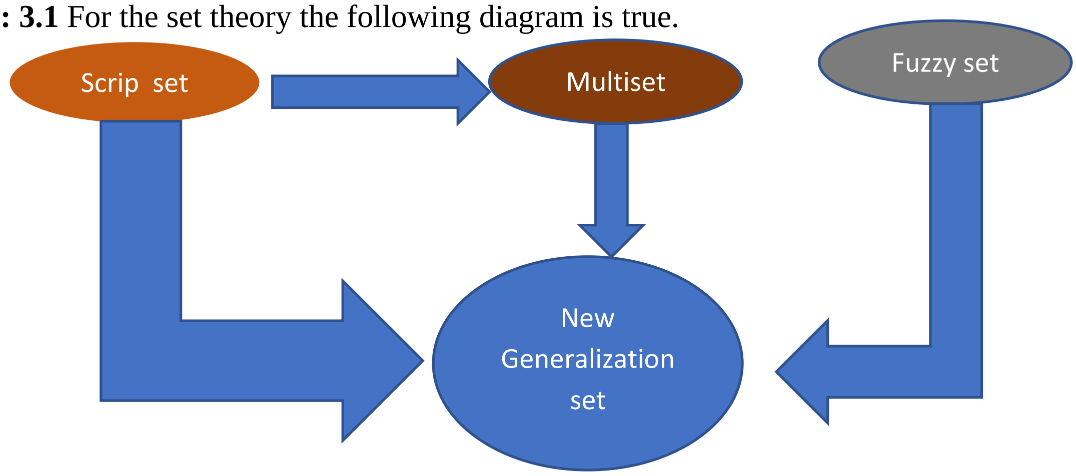
To prove A is also a New-GS set

Here μ_i can be consider as a New membership function where each integer part of the New membership function is zero.

Hence prove the Necessary part of the theorem.

For converse part use example 2.1 this shows that the converse of the theorem is not true.

Lemma: 3.1 For the set theory the following diagram is true.



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