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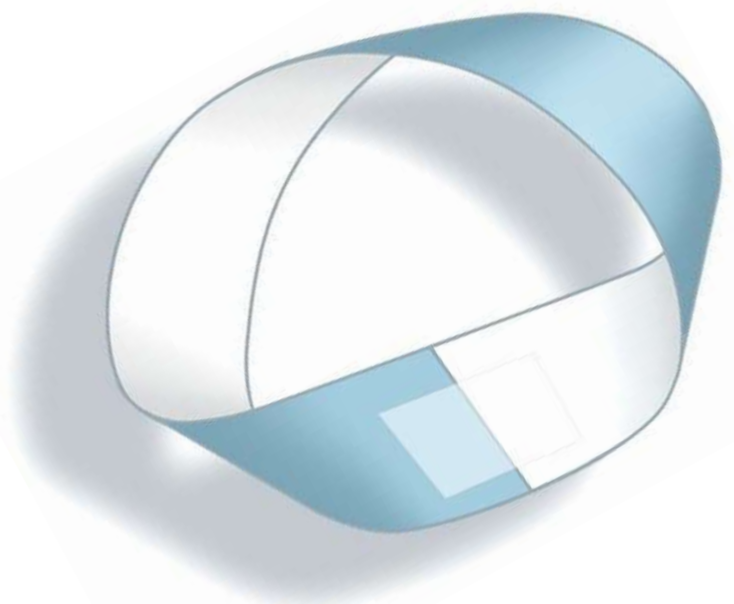

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Contents

Prof. Zdzislaw Pawlak And Rough Sets in Tripura, India

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Abstract.

It is an honour to contribute my short article to this special commemorating the life and work of Professor Zdzislaw Pawlak. In this article, I would like to discuss our encounters with the field of Rough Set Theory and the memory of Prof. Z. Pawlak in Tripura, a small state in the eastern part of India.

1. Introduction

It is my great proud that I was associated with the “ROUGH SET YEAR IN INDIA 2009” [3], where it was included International Conference on Rough Sets, Fuzzy Sets and Soft Computing, November 5-7, 2009, organized at Tripura University, Tripura, India in collaboration with IRSS & ISFUMIP.

A number of conferences (International and National) on the Rough Sets and its associate concepts and several ceremonies on the memory of Prof. Z. Pawlak were organized by **Fuzzy & Rough Sets Association** and the Department of Mathematics, Tripura University**,** Tripura, India jointly.

2. The International Conference on Rough Sets, Fuzzy Sets and Soft Computing:

The International Conference on Rough Sets, Fuzzy Sets and Soft Computing, organized at Tripura University, November 5-7, 2009 in collaboration with IRSS & ISFUMIP [1]. The main aim of the conference was particularly to expose young researchers to the latest trends in fuzzy and rough systems through deliberations by well-known scientists. The committee has chosen over 49 papers to be included into the conference materials published by Serial Publications, New Delhi. It is also important to acknowledge that there were 141 participants from both India and abroad.

3. Fuzzy & Rough Sets Association (FRSA), Tripura , India

It is our Pleasure to inform you that the inauguration ceremony of **'Fuzzy & Rough Sets Association (FRSA)** was held on **21st January, 2009** at the Dept of Mathematics, Tripura University at 2.30 PM. Prof A. Saha, Hon'ble Vice Chancellor, Tripura Univ. inaugurated the Fuzzy & Rough Set Association.

Inaugural Address by the Hon'ble Vice – Chancellor *Logo of FRSA*

The following Advisory Board and First Executive Committee of FRSA was formed:
a) An Advisory Board (11 members) from different countries
Honorar Co-Chairs - Prof. J. Peters, Canada, Prof. S.K. Pal, India

Other members

i) Prof. S. Dominik, Poland
ii) Prof. S. Ramanna, Canada
iii) Prof. M. K. Chakraborty, CU, India
iv) Prof. S. Ramanna, Canada
v) Prof. S. Dominik, Poland
vi) Prof. E. Turunen, Italy
vii) Prof. E. E. Kerre, Belgium
viii) Prof. S. Jafari, Denmark
ix) Prof. A. K. Srivastava, BHU, India

b) **First Executive Committee (7 members)**

i. President	:	Prof. R. N. Bhaumik
ii. Secretary	:	Prof. Anjan Mukherjee
iii. Treasurer	:	Dr. S. Bhattacharya (Halder)
iv. Editor	:	Dr. Subrata Bhowmik
v. Library-in-charge	:	Dr. Mrinal Kanti Bhowmik
vi. Members	:	1. Dr. Debasish Bhattacharya
Members	:	2. Dr. Dulal Dey

4. 4th Death Anniversary of Prof. Z. Pawlak (April 7, 2010)

In the Inaugural session of the Fourth Death Anniversary of Prof. Z. Pawlak held at the court Hall, Tripura University, Prof. A. Mukherjee, Head, Dept. of Mathematics and Secretary. FRSA, gave welcome address, Hon'ble Vice Chancellor Prof, A. Saha, presided over the Inaugural session, Prof. R.N. Bhaumik, President, FRSA gave the Keynote address, The Dean, Faculty of Science was Guest of Honour. In the Technical session 8 papers were presented by the teachers and research scholars.

5. 5th Death Anniversary of Prof. Z. Pawlak (April 7, 2011)

FRSA organized the 5th Death Anniversary of Prof. Z. Pawlak on 7th November, 2011.

6. National Seminar on Rough set, Fuzzy set and Soft Computing, November 11- 12, 2011

The seminar was organized by the Dept. of Mathematics, Tripura University in collaboration with Fuzzy and Rough Sets Association. 50 participants were present.

7. 85th Birth Anniversary of Prof. Z. Pawlak (Nov 11, 2011)

On November 11, 2011, FRSA paid homage to Prof. Z. Palak on his 85th Birth Anniversary.

8. Workshop on Rough set & its Applications, Nov.16-17, 2012

This Workshop, held at the Dept. of Mathematics, Tripura University, was organized by FRSA to observe the **Rough Set Day** and to pay tribute to Prof. Z. Pawlak in presence of Prof. Dominik Slezak.

In the Inaugural session of the workshop, we observed the *Rough Set Day* and paid tribute to Prof. Z. Pawlak. Prof. Dominik delivered the Keynote address of the said workshop. 60 participants from different institutes took part in the workshop. Some pictures of the Workshop with Prof. D. Slezak.

(1)

(2)

(3)

(4)

9. The first News Bulletin of FRSA

In this occasion, the first News Bulletin of Fuzzy & Rough Sets Association was released by Prof. D. Slezak. In the 4th picture above, Dr. D. Slezak is releasing the News Bulletin -1.

10. 2nd Int. conf. on Rough Sets, Fuzzy Sets and Soft Computing, January. 17-19, 2013

2nd Int. conf. on Rough Sets, Fuzzy Sets and Soft Computing, January17-19 ,2013, in collaboration with FRSA. 90 participants were present and 26 papers were published by Narosa Publishing House, New Delhi [2].

11. 7th Death Anniversary of Prof. Z. Pawlak (April 10,2013)

FRSA organized the 7th Death Anniversary of Prof. Z. **Pawlak** on 10.04.2013 at the Dept. of Mathematics, Tripura University. In the Inaugural session, Prof. A Mukherjee, Head, Dept. of Mathematics & Secretary, FRSA, gave welcome address. Honourble Prof. Anjan Ghosh, VC, Tripura University was Chief Guest. Prof. S. Sinha, Dean of Science was special Guest. Prof. R. N. Bhaumik, President, FRSA, discussed on Life & work of Prof. Z. Pawlak. 5 papers were presented by the research Scholars.

12. News Bulletin of FRSA, Volume- II

In this volume, 21 abstracts of published papers on Fuzzy Sets and Rough Sets were included and the following are cited.

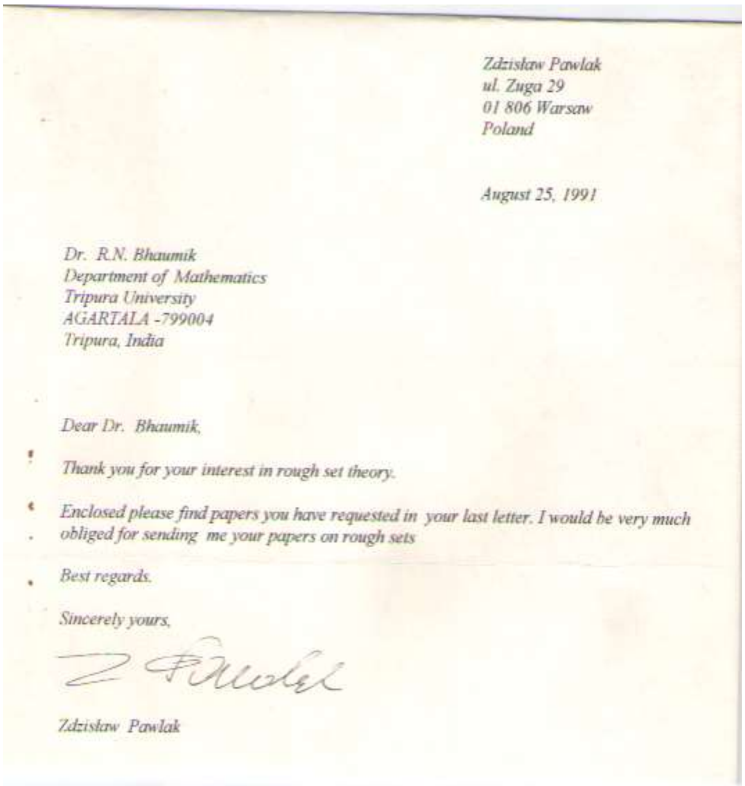
a) One photograph (shown in next page) of Prof. Z. Pawlak with Prof. R. N. Bhaumik, Prof. M.K. Chakraborty with others, during 2002 AFSS, Int. Conf. on Fuzzy Systems, held on Feb. 3-6,2002 at Calcutta.

(b) One letter of Prof. Z. Pawlak, written to Prof. R. N. Bhaumik in 1991 is attached in next page.

13. National Seminar on Rough Sets, Fuzzy Sets and their Applications, May 06, 2016

The National Seminar was organized by the Dept. of Mathematics, Tripura University in collaboration with **FRSA** on the **10th Death anniversary of Prof. Z. Pawlak.** In the Inaugural session, Dr. S. Bhattacharya (Halder), Head, Dept. of Mathematics and Treasurer, FRSA, gave the welcome address, Hon'ble Vice Chancellor, Tripura University, delivered talk on the topic. Prof. R.N. Bhaumik, President, FRSA and Prof. A. Mukherjee and Secretary, FRSA discussed how Fuzzy Sets and Rough Sets play an important role. After tea break 19 papers were presented by the Teachers and scholars of different institutes of India. The proceeding of seminar will be published as a book edited by Prof. B. C. Tripathy. Prof. R. N. Bhaumik and Prof. Anjan Mukherjee.

14. Rough Set Day will be observed **on 13th Nov., 2016 by FRSA.**



Zdzisław Pawlak
ul. Zuga 29
01 806 Warsaw
Poland

August 25, 1991

Dr. R.N. Bhaumik
Department of Mathematics
Tripura University
AGARTALA -799004
Tripura, India

Dear Dr. Bhaumik,

Thank you for your interest in rough set theory.

Enclosed please find papers you have requested in your last letter. I would be very much obliged for sending me your papers on rough sets

Best regards.

Sincerely yours,

Zdzisław Pawlak

8

References

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[2] S. Bhattacharya, Subrata Bhowmik, Proc. of the 2nd Int.conf. on Rough Sets, Fuzzy Sets and Soft Computing, January 17-19, 2013, in collaboration with FRSA, was published by Narosa Publishing House, New Delhi, 284 pages, 2015.

[3] Manish Joshi, Rabi N. Bhaumik, Pawan Lingras, Nitin Patil, Ambujan Salgaonkar, and Dominik Ślęzak, Rough Set Year in India 2009.

Pythagorean Neutrosophic Vague Soft Sets and Its Application on Decision Making Problem

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Abstract:

In this paper we study the concept of pythagorean neutrosophic vague soft sets (*PNVS*sets). Some definitions and operations have been proposed. It is a combination of soft set and pythagorean neutrosophic vague set. Lastly an application has been shown with the above concepts in decision making problem. For further study, it may be applied to real world problems with realistic data and extend proposed algorithm to other decision making problem with vagueness and uncertainty. Here we need less calculation and few steps to get our result

Keywords: Neutrosophic Set, PythagoreanNeutrosophic Set, Pythagorean Neutrosophic Soft Set, Vague set, Pythagorean Neutrosophic Vague Soft Set, Decision Making Problem.

1.Introduction: Yager [16] introduced the new concept known as Pythagorean fuzzy sets. Pythagorean fuzzy sets has limitation that their square sum is less than or equal to Smarandache proposed neutrosophic logic and neutrosophic sets (*NSs*) in 1999 [11]. A *NS* is a set in which elements of the universe has respective degrees of truth, indeterminacy and falsity. They lie in the nonstandard unit interval of $]0^-, 1^+[$. The uncertainty presented here (i.e, indeterminacy factor) is independent of the truth and falsity values. In 2019 Jansi at.el.[3] studied the concept of Pythagorean neutrosophic set with T and F are neutrosophic components and also define the correlation measure of Pythagorean neutrosophic set with$T$ and F are

dependent neutrosophic components [*PNS*] and prove some of its properties.
In 1999 Molodtsov [8] introduce the concept soft set which was completely a new approach for deal with vagueness and uncertainties. Maji [7] introduced neutrosophic soft set by the concept of neutrosophic set and soft set. This paper is an attempt to introduce the concept of pythagoreanneutrosophic vague soft sets (*PNVS*sets). Some definitions and operations have been proposed. It is a combination of soft set and pythagorean neutrosophic vague set. Lastly an application has been shown with the above concepts in decision making problem.In *PNS* sets, membership, non-membership and indeterminacy degrees are gratifying the condition $0 \leq (\mu_A(x))^2 + (v_A(x))^2 + (\delta_A(x))^2 \leq 2$ instead of $\mu_A(x) + v_A(x) + \delta_A(x) > 2$ as in neutrosophic sets. The theory of vague set was first proposed by Gau and Buehrer [2] as an extension of fuzzy set theory and vague sets are regarded as a special case of content-dependent fuzzy sets. Neutrosophic vague set was defined by S. Allehezaleh [1] in 2015. We further study [10,12,15,17].

The organization of this paper is as follows: in section 2 we briefly present some basic definitions and resultsIn section 3, we introduce the concept of pythagorean neutrosophic vague soft sets (*PNV*Ssets). Some definitions and results are established. In section 4, an application has been shown in decision making problem.

2, Preliminaries
In this section, we recall some basic notions for future work.

Definition2.1:[2] A vague set A in the universe of discourse U is a pair(t_A, f_A) where t_A, f_A:$U \to [0, 1]$ such that $t_A + f_A \leq 1$ for all $u \in U$. The function t_A and f_A are called the true membership function and the false membership function respectively. The interval $[t_A, 1 - f_A]$ is called the value of u in A and is denoted by$V_A = [t_A, 1 - f_A]$.

Definition2.2: [2] Let X be a non-empty set. Let A and B be two vague sets in the form$A = \{< x, t_A, 1 - f_A >: x \in X\}$, $B = \{< x, t_B, 1 - f_B >: x \in X\}$. Then
(i) $A \subseteq B \ if \ and \ only \ if \ t_A \leq t_B$and $1 - f_A \leq 1 - f_B$.
(ii) $A \cup B = \{< x, \max(t_A(x), t_B(x)), \max(1 - f_A(x), 1 - f_B(x)) >: x \in X\}$
(iii) $A \cap B = \{< x, \min(t_A(x), t_B(x)), \min(1 - f_A(x), 1 - f_B(x)) >: x \in X\}$
(iv) $A^c = \{< x, f_A, 1 - t_A >: x \in X\}$.

Definition2.3:[10,11] A neutrosophic set A on the universe of discourse U is defined as $A=\{\langle x,\mu_A(x),\gamma_A(x),\delta_A(x)\rangle : x\in U\}$, where $\mu_A,\gamma_A,\delta_A : U\to]^-0,1^+[$ are functions such that the condition: $\forall x\in U,\quad {}^-0\le \mu_A(x)+\gamma_A(x)+\delta_A(x)\le 3^+$ is satisfied.
Here $\mu_A(x),\gamma_A(x),\delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership respectively of the element $x\in U$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. But in real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0,1^+[$. Hence, we consider the neutrosophic set which takes the value from the subset of $[0,1]$.

Definition2.4: [1] A neutrosophic vague set A_{NV} on the universe of discourse U written as A_{NV} = {<x; $\hat{T}A_{NV}(x)$; $\hat{I}A_{NV}(x)$; $\hat{F}A_{NV}(x)$>; $x\in U$} whose truth-membership, indeterminacy-membership, and falsity-membership functions is defined as $\hat{T}A_{NV}(x) = [T^-, T^+]$, $\hat{I}A_{NV}(x) = [I^-, I^+]$ and $\hat{F}A_{NV}(x) = [F^-, F^+]$, where (1) $T^+ = 1 - F^-$, (2) $F^+ = 1 - T^-$ and (3) ${}^-0 \le T^- + I^- + F^- \le 2^+$.

Definition2.5:[8] Let U be an initial universal set and let E be a set of parameters. Let P(U) denotes the power set of all subsets of U and let $A\subseteq E$. A collection of pairs (f, A) is called a soft set over U, where f is a mapping given by $f : A \to P(U)$.

Definition 2.6. [1] A neutrosophic vague set A_{NV}(NVS in short) on the universe of discourse X written as A_{NV} = {<x; $\hat{T}A_{NV}(x)$; $\hat{I}A_{NV}(x)$; $\hat{F}A_{NV}(x)$>; $x\in X$} whose truth-membership, indeterminacy-membership, and falsity-membership functions is defined as $\hat{T}A_{NV}(x) = [T^-, T^+]$, $\hat{I}A_{NV}(x) = [I^-, I^+]$ and $\hat{F}A_{NV}(x) = [F^-, F^+]$, where (1) $T^+ = 1 - F^-$, (2) $F^+ = 1 - T^-$ and (3) ${}^-0 \le T^- + I^- + F^- \le 2^+$.

Definition 2.7. [1] If Ψ_{NV} is a NVS of the universe U, where $\forall u_i\in U$, $\hat{T}\Psi_{NV}(x) = [1, 1]$, $\hat{I}\Psi_{NV}(x) = [0, 0]$, $\hat{F}\Psi_{NV}(x) = [0, 0]$, then Ψ_{NV} is called a unit NVS, where $1 \le i \le n$. If Φ_{NV} is a NVS of the universe U, where $\forall u_i\in U$, $\hat{T}\Psi_{NV}(x) = [0, 0]$, $\hat{I}\Psi_{NV}(x) = [1, 1]$, $\hat{F}\Psi_{NV}(x) = [1, 1]$, then Φ_{NV} is called a zero NVS, where $1 \le i \le n$.

Definition 2.8.[1] Let A_{NV} and B_{NV} be two NVSs of the universe U. If $\forall u_i \in U$, (1) $\hat{T}A_{NV}(u_i) = \hat{T}B_{NV}(u_i)$, (2) $\hat{I}A_{NV}(u_i) = \hat{I}B_{NV}(u_i)$ and (3) $\hat{F}A_{NV}(u_i) = \hat{F}B_{NV}(u_i)$, then the NVS A_{NV} is equal to B_{NV}, denoted by $A_{NV} = B_{NV}$, where $1 \leq i \leq n$.

Definition 2.9. [1] Let A_{NV} and B_{NV} be two NVSs of the universe U. If $\forall u_i \in U$, (1) $\hat{T}A_{NV}(u_i) \leq \hat{T}B_{NV}(u_i)$, (2)$\hat{I}A_{NV}(u_i) \geq \hat{I}B_{NV}(u_i)$ and (3)$\hat{F}A_{NV}(u_i) \geq \hat{F}B_{NV}(u_i)$, then the NVS A_{NV} is included by B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$, where $1 \leq i \leq n$.

Definition 2.10. [1] The complement of a NVS A_{NV} is denoted by A^c and is defined by
$\widehat{T^c}A_{NV}(x) = [1 - T^+, 1 - T^-]$,
$\widehat{I^c}A_{NV}(x) = [1 - I^+, 1 - I^-]$, and
$\widehat{F^c}A_{NV}(x) = [1 - F^+, 1 - F^-]$.

Definition 2.11.[1] The union of two NVSs A_{NV} and B_{NV} is a NVS C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A_{NV} and B_{NV} given by
$T_{C_{NV}}(x) = [\max(T^-_{A_{NV_x}}, T^-_{B_{NV_x}}), \max(T^+_{A_{NV_x}}, T^+_{B_{NV_x}})]$
$I_{C_{NV}}(x) = [\min(I^-_{A_{NV_x}}, I^-_{B_{NV_x}}), \min(I^+_{A_{NV_x}}, I^+_{B_{NV_x}})]$ and
$F_{C_{NV}}(x) = [\min(F^-_{A_{NV_x}}, F^-_{B_{NV_x}}), \min(F^+_{A_{NV_x}}, F^+_{B_{NV_x}})]$

Definition 2.12. [1] The intersection of two NVSs A_{NV} and B_{NV} is a *NVS* C_{NV}, written as $H_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A_{NV} and B_{NV} given by
$T_{H_{NV}}(x) = [\min(T^-_{A_{NV_x}}, T^-_{B_{NV_x}}), \min(T^+_{A_{NV_x}}, T^+_{B_{NV_x}})]$
$I_{H_{NV}}(x) = [\max(I^-_{A_{NV_x}}, I^-_{B_{NV_x}}), \max(I^+_{A_{NV_x}}, I^+_{B_{NV_x}})]$ and
$F_{H_{NV}}(x) = [\max(F^-_{A_{NV_x}}, F^-_{B_{NV_x}}), \max(F^+_{A_{NV_x}}, F^+_{B_{NV_x}})]$

Definition 2.13[16] Let X be a nonempty set and I the unite interval [0, 1]. A pythagorean fuzzy set is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)): x \in X\}$, where the function $\mu_A: X \to [0,1]$ and $\gamma_A: X \to [0,1]$ denote the respectively degree of membership and degree of non-membership of each element $x \in X$ to the set A and $0 \leq (\mu_A(x))^2 + (\gamma_A(x))^2 \leq 1$ for each $x \in X$. Supposing, $0 \leq (\mu_A(x))^2 + (\gamma_A(x))^2 \leq 1$, then the degree of indeterminency of $x \in X$ to A is denoted by $\pi_A(x) = \sqrt{(\mu_A(x))^2 + (\gamma_A(x))^2}$&$\pi_A(x) \in [0, 1]$.

Definition 2.14[3] Let X be a nonempty set (Universe). A pythagorean neutrosophic set with truth, falsity an dependent neutrosophic components [PNS] an a non-empty set X is an object of the form A = {(x, $\mu_A(x)$, $\nu_A(x)$, $\delta_A(x)$): x∈ X} where $\mu_A(x)$, $\nu_A(x)$, $\delta_A(x) \in [0, 1]$, $0 \leq (\mu_A(x))^2 + (\nu_A(x))^2 + (\delta_A(x))^2 \leq 2$ for all $x \in X$. Where $\mu_A(x)$ is the degree of membership, $\nu_A(x)$ degree of indeterminacy and, $\delta_A(x)$ degree of non-membership. Here $\mu_A(x)$ and $\delta_A(x)$ are dependent component and $\nu_A(x)$ is independent component.

Definition 2.15[3] Let X be a nonempty set and I be the unit interval [0, 1]. A pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] A and B of the form

A = {(x, $\mu_A(x)$, $\nu_A(x)$, $\delta_A(x)$): x∈ X} and B = {(x, $\mu_B(x)$, $\nu_B(x)$, $\delta_B(x)$): x∈ X} then

1) A^c = {(x, $\delta_A(x)$, $\nu_A(x)$, $\mu_A(x)$): x∈ X}

2) $A \cup B$ = {(x, max $\{\mu_A(x), \mu_B(x)\}$, max $\{\nu_A(x), \nu_B(x)\}$, min $\{\delta_A(x), \delta_B(x)\}$): x∈ X}

3) $A \cap B$ = {(x, min $\{\mu_A(x), \mu_B(x)\}$, min $\{\nu_A(x), \nu_B(x)\}$, max $\{\delta_A(x), \delta_B(x)\}$): x∈ X}

3. Pythagorean Neutrosophic Vague Set.

Definition 3.1 Let X be a nonempty set. A pythagorean neutrosophic vague set with T and F are dependent neutrosophic components [$PNVS$]

A_{PNV} = {(x, $T_{A_{PNV}}(x)$, $I_{A_{PNV}}(x)$, $F_{A_{PNV}}(x)$): x∈ X} where truth membership, indeterminacy membership and falsity membership function is defined as

$T_{A_{PNV}}(x)$= [T^+, T^-], $I_{A_{PNV}}(x)$= [I^+, I^-] and $F_{A_{PNV}}(x)$= [F^+, F^-]

Where 1)T^+=1-F^-, 2) F^+=1-T^- and 3) 0≤ $(T^-)^2$+$(I^-)^2$+$(F^-)^2 \leq 2$.

Example 3.2. Let X={u_1, u_2, u_3} be a set of universe. Then the PNV set A_{PNV} is as follows A_{PNV} = $\{\frac{u_1}{[0.3,0.5],[0.5,0.5],[0.5,0.7]}, \frac{u_2}{[0.3,0.7],[0.4,0.6],[0.3,0.7]}, \frac{u_3}{[0.4,0.7],[0.4,0.6],[0.3,0.6]}\}$ satisfies (1), (2) and (3) of definition 3.1

(a) 0≤ $(0.3)^2$+$(0.5)^2$+$(0.5)^2 = 0.09 + 0.25 + 0.25 = 0.59 \leq 2$.

(*b*) 0$\leq (0.3)^2$+$(0.4)^2$+$(0.3)^2 = 0.09 + 0.16 + 0.09 = 0.34 \leq 2.$
(c) 0$\leq (0.4)^2$+$(0.4)^2$+$(0.3)^2 = 0.16 + 0.16 + 0.09 = 0.41 \leq 2.$
Note: In particular, PNV set A_{PNV} may be as follows
A_{PNV}= = $\{\frac{u_1}{[0,1],[0,1],[0,1]}, \frac{u_2}{[0,1],[0,1],[0,1]}, \frac{u_3}{[0,1],[0,1],[0,1]}\}$
Then we have the condition $0 \leq (T^+)^2$+$(I^+)^2$+$(F^+)^2 \leq 3$

Definition 3.3 Let *APNV* and *BPNV* be two *PNV* sets of the universal set *U*. If $\forall u_i \in U$

1) $T_{A_{PNV}}(u_i) = T_{B_{PNV}}(u_i)$
2) $I_{A_{PNV}}(u_i) = I_{B_{PNV}}(u_i)$ and
3) $F_{A_{PNV}}(u_i) = F_{B_{PNV}}(u_i)$

Then the *PNV* sets APNVis equals to *PNV* set *BPNV*, denoted by *APNV* =*BPNV*, where 1$\leq i \leq n$

Definition 3.4 Let *APNV* and *BPNV* be two *PNV* sets of the universal set *U*. If $\forall u_i \in U$

1) $T_{A_{PNV}}(u_i) \leq T_{B_{PNV}}(u_i)$
2) $I_{A_{PNV}}(u_i) \geq I_{B_{PNV}}(u_i)$
3) $F_{A_{PNV}}(u_i) \geq F_{B_{PNV}}(u_i)$

Then the *PNV* sets APNV is included in *BPNV*; denoted by *APNV*⊆*BPNV*, where 1$\leq i \leq n$.

Definition 3.5 The compliment of a *PNV* set *APNV* is denoted by $A^c_{PNV}$and is defined by

$T_{A^c_{PNV}}(x)$= $[1 - T^+,\ 1 - T^-]$, $I_{A^c_{PNV}}(x)$= $[1 - I^-,\ 1 - I^+]$ and $F_{A^c_{PNV}}(x)$= $[1 - F^-,$ $1 - F^+]$

Example 3.6 Consider the example 3.2
ThenA^c_{PNV} = $\{\frac{u_1}{[0.5,0.7],[0.5,0.5],[0.3,0.5]}, \frac{u_2}{[0.7,0.3],[0.4,0.6],[0.3,0.7]}, \frac{u_3}{[0.3,0.6],[0.4,0.6],[0.4,0.7]}\}$
Note: The example 3.6 satisfies the definition 3.5 with the conditions
0$\leq (T^-)^2$+$(I^-)^2$+$(F^-)^2 \leq 3.$
0$\leq (T^+)^2$+$(I^+)^2$+$(F^+)^2 \leq 3.$

Definition 3.7 Pythagorean Neutrosophic Vague Soft Set.
Let U be a universal set. E be a set of parameters and $A \subseteq E$. Let PNVset(U) denotes the set of all Pythagorean neutrosophic vague set of U. Then the pair (f, A) is called Pythagorean neutrosophic vague soft set ($PNVS$ set in short) over U. Here f is a mapping $f: A \to PNV\ set(u)$. The collection of all Pythagorean neutrosophic vague soft sets over U is denoted by$PNVS\ set(U)$.

Example 3.8 Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$. Then Pythagorean neutrosophic vague soft sets A_1 and A_2 over U are as follows-
$A_1 = [(e_1, \{(u_1, [0.3, 0.5], [0.5, 0.5], [0.5, 0.7]\ ), (u_2, [0.2, 0.6], [0.6, 0.7], [0.4, 0.8]\ ),$
$(u_3, [0.4, 0.6], [0.3, 0.4], [0.4, 0.6]\ )\ \}),$
$(e_2, \{(u_1, [0.5, 0.6], [0.3, 0.4], [0.4, 0.5],), (u_2, [0.3, 0.4], [0.6, 0.8], [0.6, 0.7]),$
$(u_3, [0.5, 0.6], [0.7, 0.8], [0.4, 0.5]\ )\ \}].$

$A_2 = [(e_1, \{(u_1, [0.4, 0.5], [0.3, 0.4], [0.5, 0.6],), (u_2, [0.3, 0.7], [0.5, 0.6], [0.3, 0.7]),$
$(u_3, [0.5, 0.7], [0.2, 0.3], [0.3, 0.5],)\ \}),$
$(e_2, \{(u_1, [0.6, 0.7], [0.2, 0.4], [0.3, 0.4],), (u_2, [0.4, 0.5], [0.5, 0.7], [0.5, 0.6]),$
$(u_3, [0.6, 0.7], [0.5, 0.7], [0.3, 0.4]\ )\ \}\ ].$

Definition 3.9: An empty Pythagorean neutrosophic vague soft set $\emptyset$ in U is defined as $\emptyset = \{(e, \{(u, [0, 0], [0, 0], [1, 1])\ \}: e \in E \text{and } u \in U\}.$

Definition 3.10: An absolute Pythagorean neutrosophic vague soft set I in U is defined as
$I = \{(e, \{(u, [1, 1], [1, 1], [0, 0]\ )\ \}: e \in E \text{ and } u \in U\}.$
Definition 3.11 C^i={e, (u, $T_{C^i_{PNVS}}$, $I_{C^i_{PNVS}}$, $F_{C^i_{PNVS}}$): u$\in U, e \in E\}$ where i=1, 2 be the pythagorean neutrosophic vague soft set over U. Then $C^1$is pythagorean neutrosophic vague softsubset ofC^2 is defined by $C^1 \subseteq C^2$ if
$T_{C^1_{PNVS}} \leq T_{C^2_{PNVS}}$, $I_{C^1_{PNVS}} \geq I_{C^2_{PNVS}}$, $F_{C^1_{PNVS}} \geq F_{C^2_{PNVS}}$

Example 3.12 Consider the example 3.8 Here,$A_1 \subseteq A_2$ as per our definition 3.12.
Definition 3.13 Let A be a pythagorean neutrosophic vague soft set over U. Then the compliment of A is defined by A^c is defined by
A^c={e, (u, $T_{A^c_{PNVS}}$, $I_{A^c_{PNVS}}$, $F_{A^c_{PNVS}}$): u$\in U, e \in E\}$

$T_{A^c_{PNVS}}$(u) = [(1-$T^+(u)$), (1-$T^-(u)$)]
$I_{A^c_{PNVS}}$(u) = [(1-$I^+(u)$), (1-$I^-(u)$)]
$F_{A^c_{PNVS}}$(u) = [(1-$F^+(u)$), (1-$F^-(u)$)]

Example 3.14 Let U={u_1, u_2}and E ={e_1, e_2}then the pythagorean neutrosophic vague soft set A is
A = [(e_1, {(u_1, [0.1, 0.3], [0.2, 0.4], [0.7, 0.9])}, {(u_2, [0.6, 0.8], [0.3, 0.5], [0.2, 0.4])}, (e_2, {(u_1, [0.7, 0.9], [0.2, 0.5], [0.1, 0.3])}, {(u_2, [0.8, 0.9], [0.5, 0.6], [0.1, 0.2])}] Then the compliment of A is defined by A^c is as follows
A^c = [(e_1, {(u_1, [0.7, 0.9], [0.6, 0.8], [0.1, 0.3])}, {(u_2, [0.2, 0.4], [0.5, 0.7], [0.6, 0.8])}, (e_2, {(u_1, [0.1, 0.3], [0.5, 0.8], [0.7, 0.9])}, {(u_2, [0.1, 0.2], [0.4, 0.5], [0.8, 0.9])}]

Definition 3.15
A^i={e, (u, $T_{A^i_{PNVS}}$, $I_{A^i_{PNVS}}$, $F_{A^i_{PNVS}}$): u$\in U, e \in E$} where i=1, 2 be the two pythagorean neutrosophic vague soft set over U, then the union and intersection of A^1 and A^2 of two pythagorean neutrosophic vague soft set are defined as follows:

(a) $A^1 \cup A^2 = A^3$
$$= \{e, (u, T_{A^3_{PNVS}}, I_{A^3_{PNVS}}, F_{A^3_{PNVS}})\}$$
where,
$T_{A^3_{PNVS}}(u)$ = $[(T^-_{A^1_{PNVS}}(u)) \vee (T^-_{A^2_{PNVS}}(u)), (T^+_{A^1_{PNVS}}(u)) \vee (T^+_{A^2_{PNVS}}(u))]$
$I_{A^3_{PNVS}}(u)$ = $[(I^-_{A^1_{PNVS}}(u)) \wedge (I^-_{A^2_{PNVS}}(u)), (I^+_{A^1_{PNVS}}(u)) \wedge (I^+_{A^2_{PNVS}}(u))]$
$F_{A^3_{PNVS}}(u)$ = $[(F^-_{A^1_{PNVS}}(u)) \wedge (F^-_{A^2_{PNVS}}(u)), (F^+_{A^1_{PNVS}}(u)) \wedge (F^+_{A^2_{PNVS}}(u))]$

(b) $A^1 \cap A^2 = A^4$
$$= \{e, (u, T_{A^4_{PNVS}}, I_{A^4_{PNVS}}, F_{A^4_{PNVS}})\}$$
where,
$T_{A^4_{PNVS}}(u)$ = $[(T^-_{A^1_{PNVS}}(u)) \wedge (T^-_{A^2_{PNVS}}(u)), (T^+_{A^1_{PNVS}}(u)) \wedge (T^+_{A^2_{PNVS}}(u))]$
$I_{A^4_{PNVS}}(u)$ = $[(I^-_{A^1_{PNVS}}(u)) \vee (I^-_{A^2_{PNVS}}(u)), (I^+_{A^1_{PNVS}}(u)) \vee (I^+_{A^2_{PNVS}}(u))]$
$F_{A^4_{PNVS}}(u)$ = $[(F^-_{A^1_{PNVS}}(u)) \vee (F^-_{A^2_{PNVS}}(u)), (F^+_{A^1_{PNVS}}(u)) \vee (F^+_{A^2_{PNVS}}(u))]$

Definition 3.16 Let A={e, (u, $T_{A_{PNV}}(u), I_{A_{PNV}}(u),\ F_{A_{PNV}}(u)):):$ u∈ $U, e \in E\}$ be a pythagorean neutrosophic vague soft set over U. Then aggregation pythagorean neutrosophic vague softoperator denoted by A_{agg} is denoted as

A_{agg}={$\frac{[\delta_A^+, \delta_A^-]}{u} : u \in U\}$

where $[\delta_A^+,\ \delta_A^-] = \frac{1}{2|E\times U|}[\sum_{e\in E}([1,1] - I_e(u)[T_e - F_e(u)]$

where $I_e(u) = [I_e^+(u) - I_e^-(u)]$,

$$T_e(u) = [T_e^+(u) - T_e^-(u)]$$
$$F_e(u) = [F_e^+(u) - F_e^-(u)]$$

$|E \times U|$is the cardinality of $E \times U$.

4. Application of pythagorean neutrosophic vague soft set.

In our daily life we face problems in decision making such as education, economy, management, politics and technology. The results for education to choose the best college education. In the selection of college teaching education, the evaluation of teacher education is carried out according to various standards of experts.

There are various studies, primarily conducted that have investigated the reasons why parents select a college, which they think best suit their college students needs and parental aspirations for their college student. We identify a factor regarded as parental decision making: Academic Factor - divided into three identified elements namely Campus Environment, Academic Quality, and Career Opportunities. Our goal is to select the optimal one out of a number of alternatives based on the assessment of experts against the criteria.

A parent's committee intends to choose popular college education. Here the committee intends to choose three colleges U={u_1, u_2 , u_3}. The score of the college education evaluated by the experts is represented by E={e_1 = Popular Environments, e_2 = Academic quality, e_3 = Career Opportunity} Algorithm

1. First, we construct the pythagorean neutrosophic vague soft set on U.
2. We compute the pythagorean neutrosophic vague soft set aggregation operator.
3. Average of each interval and find $|A_{agg}|$. (The numerical value)
4. Find the optimum value on U. Assume that the set of colleges U={u_1, u_2 , u_3} which may be characterized by a set of parameters E ={e_1, e_2, e_3}

(a) The parents committee constract a pythagorian neutrosophic vague soft set A over U as

A = [{(e_1, (u_1, [0.8, 0.9], [0.5, 0.7], [0.1, 0.2]), (u_2, [0.5, 0.7], [0.4, 0.6], [0.3, 0.5]), (u_3, [0.7, 0.9], [0.2, 0.4], [0.1, 0.3])}, {e_2, (u_1, [0.5, 0.7], [0.4, 0.6], [0.3, 0.5]), (u_2, [0.7, 0.9], [0.4, 0.6], [0.1, 0.3]), (u_3, [0.6, 0.8], [0.8, 0.9], [0.2, 0.4])}, {e_3, (u_1, [0.7, 0.9], [0.2, 0.4], [0.1, 0.3]), (u_2, [0.6, 0.8], [0.4, 0.6], [0.2, 0.4]), (u_3, [0.5, 0.7], [0.5, 0.7], [0.3, 0.5])}]

(b) Then we find the pythagorean neutrosophic vague soft set aggregation operator A_{agg} of A as follows:

For u_1

$\frac{1}{18}$[[1, 1]-[0.5, 0.7]([0.8, 0.9]-[0.1, 0.2])+[1, 1]-[0.4, 0.6]([0.5, 0.7]-[0.3, 0.5])+[1, 1]-[0.2, 0.4]([0.7, 0.9]-[0.1, 0.3])]

For u_2

$\frac{1}{18}$[[1, 1]-[0.4, 0.6]([0.5, 0.7]-[0.3, 0.5])+[1, 1]-[0.4, 0.6]([0.7, 0.9]-[0.1, 0.3])+[1, 1]-[0.4, 0.6]([0.6, 0.8]-[0.2, 0.4])]

For u_3

$\frac{1}{18}$[[1, 1]-[0.2, 0.4]([0.7, 0.3]-[0.1, 0.3])+[1, 1]-[0.8, 0.9]([0.6, 0.8]-[0.2, 0.4])+[1, 1]-[0.5, 0.7]([0.5, 0.7]-[0.3, 0.5])]

(c) We take the average of each interval i.e [1, 1] $\&(u) = [T^-(u) - T^+(u)] I(u) = [I^-(u) - I^+(u)], F(u) = [F^-(u) - F^+(u)]$

(d) Then $|A_{agg}| = \frac{0.1277}{u_1}, \frac{0.1333}{u_2}, \frac{0.1311}{u_3}$

(e) Finally the parent's committee choose the college u_2, since $|A_{agg}|$ has the maximum degree 0.1333 among the colleges. . Here we need less calculation and few steps to get our result. Validity of this method is better than that of previous work.

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Declarations

Conflict of interest Authors declare that they have no conflict of interest.

Ethics approval Data has been collected from reliable sources. We follow all the ethical rules.

5. Conclusion

In this paper, we introduce the pythagorean neutrosophic vague soft set. It is a combination of soft set and the pythagorean neutrosophic vague set. We develop a decision making method based on pythagorean neutrosophic vague soft set. A numerical example has been given. Some new operations on pythagorean neutrosophic vague soft set have been designed. For further study, it may be applied to real world problems with realistic data and extend proposed algorithm to other decision making problem with vagueness and uncertainty. Here we need less calculation and few steps to get our result

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On Ideal Convergence of Sequences of Functions in Neutrosophic Normed Spaces

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Abstract:

In this paper, we would introduce sequence of convergence with the help of neutrosophic norm in neutrosophic normed spaces. We wish to introduce the notion of point wise convergence and uniform convergence of sequences of functions with the help of neutrosophic norm in neutrosophic normed spaces. Some basic properties and characterization theorems of these concepts would be investigated in neutrosophic normed spaces. Our purpose is also to introduce I - convergence, point wise and uniform I - convergence and I - Cauchy sequence of sequences of functions in neutrosophic normed space and to investigate the relationships among the concepts such as I - convergence, statistical convergence and the usual convergence of sequences of functions in neutrosophic normed spaces.

Keywords: I - point wise convergence, I - uniform convergence, sequence of functions, neutrosophic normed space.

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1. Introduction:

Human beings are always dealing with many real - life problems due to uncertainties which cannot always be explained by classic methods. To handle such situations, Zadeh [26] introduced the concept of fuzzy set theory which emerged as one of the most active areas of research in many branches of mathematics and engineering. But it is not sufficient to explain the indeterminacy states because it has only membership (truth) function. Thereafter, Atanassov [1] introduced the notion of intuitionistic fuzzy sets theory which deals with three states, such as truth, falsity, and indeterminacy. However, these states are dependent on each other. In order to solve real life problems on decision making under uncertainty, Smarandache [22] introduced the notion of neutrosophic set where each element had three associated defining functions, namely the membership function (T), the non - membership function (F) and the indeterminacy function

(I) defined on the universe of discourse X. These three functions are completely independent. Further investigation had been made by Smarandache [23] on the applications of the neutrosophic theory.

Lots of researchers [4, 6, 17, 25] contributed themselves to apply the notion of fuzzy set theory successfully in studying sequence spaces with classical metrics. Intuitionistic fuzzy set theory was used in all areas where fuzzy set theory was studied. Park [18] defined intuitionistic fuzzy metric space which is a generalization of fuzzy metric space. Using the idea of intuitionistic fuzzy sets, Park [18] defined the notion of intuitionistic fuzzy metric spaces by the help of the continuous t - norms and the continuous t - conorms as a generalization of fuzzy metric space due to George and Veeramani [6]. Saadati [20] investigated on intuitionistic fuzzy normed spaces. On the other hand, the notion of statistical convergence for real number sequences was first introduced by Fast [5]. Using the concept of an ideal, Kostyrko et al. [14] introduced the notion of I - convergence which is a generalization of ordinary convergence and statistical convergence. The I - convergence provides a general framework to study the properties of various types of convergence. Karakus¸ et al. [10] defined statistical convergence in intuitionistic fuzzy normed spaces (IFNS for short). Karakaya et al. [7] defined and studied I - convergence of sequences of functions in IFNS. Moreover, Karakaya et al. [8, 9] investigated λ - statistical convergence and lacunary statistical convergence of sequences of functions in IFNS respectively. More research works on I - convergence and statistical convergence can be found in [13, 15].

Bera and Mahapatra [2] introduced neutrosophic soft linear spaces (NSLSs). Thereafter, neutrosophic soft normed linear spaces (NSNLS) has been defined by Bera and Mahapatra [3]. In [3], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied. Kirişci and Şimşek [12] introduced the notion of neutrosophic normed space (NNS) and defined statistical convergence with respect to NNS. Muralikrishna and Kumar [16] investigated on neutrosophic approach to normed linear space. Tripathy and Hazarika [24] introduced paranorm I - convergent sequence spaces. After getting motivations of these works, we shall introduce the notion on I - convergence of sequences of functions in NSS. We shall investigate some of their basic properties and relations with other convergence of sequences of functions. The paper reveals as follows. The next section briefly focuses some known definitions and results which are related for investigation. In section 3, we introduce the notions of different types of convergence of sequences of functions and investigate some basic properties and results in neutrosophic normed spaces. Section 4 indicates the conclusion of the work.

2. Preliminaries:

In this section, some known results and definitions would be procured for ready reference.

Definition 2.1. [21] A binary operation $o : [0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous t - norm (TN) if it satisfies the following conditions:

(i) o is associative and commutative,

(ii) o is continuous,
(iii) $a\, o 1 = a$ for all $a \in [0,1]$,
(iv) $a\, o\, c \leq b\, o\, d$ whenever $a \leq b$ and $c \leq d$ for each $a, b, c, d \in [0,1]$.

For example, $a\, o\, b = a.b$ is a continuous t - norm.

Definition 2.2. [21] A binary operation $\bullet : [0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous t - conorm (TC) if it satisfies the following conditions:

(i) $\bullet$ is associative and commutative,
(ii) $\bullet$ is continuous,
(iii) $a \bullet 1 = a$ for all $a \in [0,1]$,
(iv) $a \bullet c \leq b \bullet d$ whenever $a \leq b$ and $c \leq d$ for each $a, b, c, d \in [0,1]$.

For example, $a \bullet b = \min\{a + b, 1\}$is a continuous t - conorm.

Definition 2.3. [**19**] Let o be a continuous t - norm, $\bullet$ be a continuous t - conorm and X be a linear space over the field IF ($\mathbb{R}$ or $\mathbb{C}$). If μ and ν are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions, the five-tuple $(X, \mu, \nu, o, \bullet)$ is said to be an intuitionstic fuzzy normed spaces (IFNS) and (μ, ν) is called an intuitionistic fuzzy norm. For every $x, y \in X \; and \; s, t > 0$,

(i) $\mu\,(x,t) + \nu\,(x,t) \leq 1$,
(ii) $\mu\,(x,t) > 0$,
(iii)$\mu\,(x,t) = 1 \Leftrightarrow x = 0$,
(iv)$\mu\,(ax,t) = \mu\,(x, \frac{t}{|a|}) \; for \; each \; a \neq 0$,
(v) $\mu\,(x,\ t) o\, \mu\,(y,s) \leq \mu\,(x+y,\ t+s)$,
(vi)$\mu\,(x,\ .) : (0,\infty) \rightarrow [0,1] \; is \; continuous$,
(vii) $\lim_{t\to\infty} \mu\,(x,t) = 1 \; and \; \lim_{t\to 0} \mu\,(x,t) = 0$,
(viii) $\nu\,(x,t) < 1$,
(ix)$\nu\,(x,t) = 0 \Leftrightarrow x = 0$,
(x) $\nu\,(ax,t) = \nu\,(x, \frac{t}{|a|}) \; for \; each \; a \neq 0$,
(xi)$\nu\,(x,t) \bullet \nu\,(y,s) \geq \nu\,(x+y, t+s)$,
(xii) $\nu\,(x,\ .) : (0,\infty) \rightarrow [0,1] is \; continuous.$
(xiii) $\lim_{t\to\infty} \nu\,(x,\ t) = 1 \; and \; \lim_{t\to\infty} \nu\,(x,\ t) = 0.$

Definition 2.4. (One may refer to Kirisci and Simsek [12]) If X is a non - empty set, then a family of set $I \subset P(X)$ is called an ideal in X if and only if

(i) For each $A, B \in I$, we have $A \cup B \in I$;
(ii) For each $A \in I$ and $B \subset A$, we have $B \in I$.

Definition 2.5. (One may refer to Kirisci and Simsek [12])] Let X be a non - empty set. A non - empty family of sets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

(i) $\theta \notin \mathcal{F}$;

(ii) For each $A, B \in F$ we have $A \cap B \in \mathcal{F}$;
(iii) For each $A \in \mathcal{F}$ and $A \subset B$ we have $B \in \mathcal{F}$.

Definition 2.6. (One may refer to Kirisci and Simsek [12])] An ideal I is called non - trivial if $I \neq \theta$ and $X \notin I$. $I \subset 2^X$ is a non - trivial ideal if and only if $F = F(I) = \{X \setminus A : A \in I\}$ is a filter on X. Also, a non - trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons i.e., if it contains $\{\{x\} : x \in X\}$, i.e., $I \supset \{\{x\}: x \in X\}$.

Definition 2.7. [11] Take F as a vector space, $N = \{< u, G(u), B(u), Y(u) >: u \in F\}$be a normed space (NS) such that $N: F \times \mathbb{R}^+ \to [0,1]$. Let o and ● show the continuous TN and continuous TC, respectively. If the following conditions are hold, then the four - tuple $V = (F,\ N, \text{o}, \bullet)$ is called neutrosophic normed space (NNS). For all $u, v, \in F$ and $\lambda, \mu > 0$ and for each $\sigma \neq 0$.

(i) $0 \leq G(\mu,\ \lambda) \leq 1, 0 \leq \beta(\mu,\ \lambda) \leq 1,\ 0 \leq Y(\mu,\ \lambda) \leq 1, \forall\, \lambda \in \mathbb{R}^+,$
(ii) $G(\mu, \lambda) + \beta(\mu, \lambda) + Y(\mu, \lambda) \leq 3, (for\ \lambda \in \mathbb{R}^+),$
(iii) $G(\mu, \lambda) = 1\ \ (for\ \lambda > 0)\ iff\ u = 0,$
(iv) $G(u, v, \lambda) = G(v, u, \lambda)(for\ \lambda > 0),$
(v) $G(u, v, \lambda)\text{o}\ G(v, u, \lambda) \leq G(u, y, \lambda + u\ (\forall\, \lambda, \mu > 0),$
(vi) $G(u, v, .): [0, \infty) \to [0,1]\ is\ continuous$
(vii) $\lim_{\lambda \to \infty} G(u, v, \lambda) = 1\ (\forall\, \lambda > 0)$
(viii) $B(u, v, \lambda) = 0\ (for\ \lambda > 0)\ iff\ u = v,$
(ix) $B(u, v, \lambda) = B(v, u, \lambda)(for\ \lambda > 0),$
(x) $B(u, v, \lambda) \bullet B(v, y, \mu) \geq B(u, y, \lambda + \mu)(\forall\, \lambda, \mu > 0),$
(xi) $B(u, v, .): [0, \infty) \to [0,1] is\ continuous,$
(xii) $\log_{\lambda \to \infty} B(u, v, \lambda) = 0\ (\forall\, \lambda > 0),$
(xiii) $Y(u, v, \lambda) = 0\ (for\ \lambda > 0) iff\ u = v,$
(xiv) $Y(u, v, \lambda) = Y(v, u, \lambda)(\forall\, \lambda > 0),$
(xv) $Y(u, v, \lambda) \bullet Y(v, y, \mu) \geq Y(u, y, \lambda + \mu)(\forall\, \lambda, \mu > 0),$
(xvi) $B(u, v, .): [0, \infty) \to [0,1] is\ continuous,$
(xvii) $\lim_{\lambda \to \infty} Y(u, v, \lambda) = 1\ (\forall\, \lambda > 0)$
(xviii) $If\ \lambda \leq 0, then\ G(u, v, \lambda) = 0, B(u, v, \lambda) = 1\ and\ Y(u, v, \lambda) = 1.$

Then $N = (G, B, Y)\ is\ called\ Neutrosophic\ Norm\ (NN).$

3. *I* - convergence of sequences of functions in NNS:

In this section, we would introduce the notion of I - convergence of sequences of functions and investigate their properties in NNS.

Definition 3.1. Let $(F, N, o, \bullet)$ be a NNS, $N = (G, B, Y)$ be neutrosophic norm and (x_n) be a sequence in F. Sequence (x_n) is said to be convergent to $L \in$ F with respect to the neutrosophic norm (G, B, Y) if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer n_0 such that

$$G\ (x_n - L, t) > 1 - \varepsilon, B(x_n - L, t) < \varepsilon \ and \ Y(x_n - L, t) < \varepsilon \text{ whenever } n > n_0$$

In this case we write (G, B, Y) $-$ limx_n = $L\ as\ n \to \infty$.

Definition 3.2. Let$(F, N, o, \bullet)\ and\ (G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a sequence of functions. The sequence (f_n) is said to be pointwise neutrosophic convergent on F to a function f with respect to (*G*, *B*, *Y*) if for each $x \in F$, the sequence $(f_n(x))$ is convergent to $f(x)$ with respect to (G', B', Y')

The sequence (f_n) is said to be uniformly neutrosophic convergent on F to a function f with respect to (*G*, *B*, *Y*) if for $0 < r < 1$, t> 0, there exists a positive integer $n_0 = n_0(r, t)$ such that $\forall x \in X$ and $\forall$ n $> n_0$,

$$G(f_n(x) - f(x), t) > 1 - r, B(f_n(x) - f(x), t) < r, Y(f_n(x) - f(x), t) < r.$$

Definition 3.3. Let $I \subset P(N)$ be a nontrivial ideal and $(F, N, o, \bullet)$ be a NNS. A sequence $x = (x_n)$ of elements in F is said to I - convergent to $L \in F$ with respect to the neutrosophic norm (G, B, Y) if for each $\varepsilon > 0$ and $t > 0$, the set

$$\{n \in \mathbb{N} : G\ (x_n - L, t) \leq 1 - \varepsilon\ or\ B(x_n - L, t) \geq \varepsilon, Y(x_n - L, t) \geq \varepsilon\} \in I.$$

In this case the element L is called I - limit of the sequence (x_n) with respect to the neutrosophic norm (*G*, *B*, *Y*) and we write $I_{(G,\ B,\ Y)} - lim x_n = L$.

Definition 3.4. Let $I \subset P(N)$ be a nontrival ideal. Also let $(F, N, o, \bullet)\ and\ (G, N', o, \bullet)$ be NNS and N = (*G*, *B*, *Y*) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a sequence of functions. If for each $x \in F$ and $\forall \varepsilon > 0, t > 0$.

$\{n \in \mathbb{N} : G'(f_n(x) - f(x), t) \leq 1 - \varepsilon\ or\ B'(f_n\ (x) - f\ (x), t) \geq \varepsilon\ or\ Y'(f_n\ (x) - f\ (x), t) \geq \varepsilon\} \in I$

then we say that the sequence (f_n) is pointwise I - convergent with respect to neutrosophic norm (G, B, Y) and we write it $I_{(G,\ B,\ Y)} - f_n \to f$.

Lemma 3.5. Let $(F,\ N,\ o,\ \bullet)\ and\ (G,\ N',\ o,\ \bullet)$ be NNS and N = (*G*, *B*, *Y*) and $N' = (G',\ B',\ Y')$ be respective NN. Let $f_n : (F,\ N,\ o, \bullet) \to (G,\ N',\ o,\ \bullet)$ be a sequence of functions. Then for every $\varepsilon > 0$, for each $x \in X$ and $t > 0$, the following statements are equivalent:

(i) $I_{(G,\ B,\ Y)} - f_n \to f$.

(ii) $\{n \in \mathbb{N} : G'(f_n(x) - f(x), t \leq 1 - \varepsilon\} \in I$, $\{n \in \mathbb{N} : B'(f_n(x) - f(x), t) \geq \varepsilon\} \in I$ and $\{n \in \mathbb{N} : Y'(f_n(x) - f(x), t) \geq \varepsilon\} \in I$.

(iii) $\{n \in \mathbb{N} : G'(f_n(x) - f(x), t) > 1 - \varepsilon\} \in \mathcal{F}(I)$, $\{n \in \mathbb{N} : B'(f_n(x) - f(x), t) < \varepsilon\} \in \mathcal{F}(I)$ and $\{n \in \mathbb{N} : Y'(f_n(x) - f(x), t) < \varepsilon\} \in \mathcal{F}(I)$.

(iv) $\{n \in \mathbb{N} : G'(f_n(x) - f(x), t) > 1 - \varepsilon\} \in \mathcal{F}(I)$, $\{n \in \mathbb{N} : B'(fn(x) - f(x), t) < \varepsilon\} \in \mathcal{F}(I)$ and $\{n \in \mathbb{N} : Y'(fn(x) - f(x), t) < \varepsilon\} \in \mathcal{F}(I)$.

(v) $I - lim\, G'(f_n(x) - f(x), t) = 1$, $I - lim\, B'(fn(x) - f(x), t) = 0$ and $I - limY'(f_n(x) - f(x), t) = 0$

Proof. The proof is standard verification.

Theorem 3.6. Let (f_n) and (g_n) be two sequences of functions in a NNS $(F, N, o, \bullet)$ $with\ norm\ N = (G, B, Y)$. If $I_{(G,\ B,\ Y)} - f_n = f$ and$I_{(G,\ B,\ Y)} - g_n = g$, then $I_{(G,\ B,\ Y)} - (\alpha f_n + \beta g_n) = \alpha f + \beta g$ where $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$.
Proof. The proof is clear for $\alpha = 0$ and $\beta = 0$. Now let $\alpha \neq 0$ and $\beta \neq 0$. Since $I_{(G,\ B,\ Y)} - f_n \to f$ and $I_{(G,\ B,\ Y)} - g_n \to g$, for each $x \in X$.

$$A_1 = \left\{ \begin{array}{c} n \in N: G'\left(f_n - f(x), \dfrac{t}{2\lceil\alpha\rceil}\right) \leq 1 - \varepsilon \\ or\ B'\ (f_n(x) - f(x), \dfrac{t}{2|\alpha|}) \geq \varepsilon \\ or\ Y'\ (f_n(x) - f(x), \dfrac{t}{2|\alpha|}) \geq \varepsilon \end{array} \right\} \in I$$

$$A_2 = \left\{ \begin{array}{l} n \in \mathbb{N}: G'(g_x(x) - g(x), \dfrac{t}{2\lceil\beta\rceil}) \\ or\ B'\ (g_n(x) - g(x), \dfrac{t}{2|\beta|}) \geq \varepsilon \\ or\ Y'\ (g_n(x) - g(x), \dfrac{t}{2|\beta|}) \geq \varepsilon \end{array} \right\} \in I$$

Define the set $A = (A_1 \cup A_2)$, so A belongs to I. It follows that A^c is a non-empty set in $\mathcal{F}(I)$. We shall show that for each $x \in X$

$$\begin{aligned} A^c \subset \{n \in \mathbb{N} :\ & G'\ ((\alpha\, f_n + \beta g_n)\ (x) - (\alpha f + \beta g)\ (x), t)\ >\ 1 - \varepsilon\ , n \in \mathbb{N} \\ & : B'\ ((\alpha f_n + \beta g_n)\ (x) - (\alpha f + \beta g)\ (x), t) < \varepsilon\ and\ n \in \mathbb{N} \\ & : Y'\ ((\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x), t) < \varepsilon\} \end{aligned}$$

Let $m \in A^c$, in this case $G'(f_m(x) - f(x), \frac{t}{2\lceil\alpha\rceil}) > 1 - \varepsilon$,

$B'(f_m(x) - f(x), \frac{t}{2\lceil\alpha\rceil}) < \varepsilon$,

and $Y'(f_m(x) - f(x), \frac{t}{2\lceil\alpha\rceil}) < \varepsilon$.

$G'(g_m(x) - g(x), \frac{t}{2\lceil\beta\rceil}) > 1 - \varepsilon$,

$B'(g_m(x) - g(x), \frac{t}{2\lceil\beta\rceil}) < \varepsilon$,

and $Y'(g_m(x) - g(x), \frac{t}{2\lceil\beta\rceil}) < \varepsilon$.

We have

$$\begin{aligned}
&G'((\alpha f_m + \beta g_m)(x) - (\alpha f(x) + \beta g(x)), t) \\
&\geq G'(\alpha f_m(x) - \alpha f(x), \tfrac{t}{2})\, o\, G'(\beta g_m(x) - \beta g(x), \tfrac{t}{2}) \\
&= G'\left(f_m(x) - f(x), \frac{t}{2\lceil\alpha\rceil}\right) o\, G'(g_m(x) - g(x), \frac{t}{2\lceil\beta\rceil}) \\
&> (1-\varepsilon) o (1-\varepsilon) \\
&= (1-\varepsilon)
\end{aligned}$$

$$\begin{aligned}
&B'((\alpha f_m + \beta g_m)(x) - (\alpha f(x) + \beta g(x)), t) \\
&\leq B'(\alpha f_m(x) - \alpha f(x), \tfrac{t}{2})\, o\, B'(\beta g_m(x) - \beta g(x), \tfrac{t}{2}) \\
&= B'\left(f_m(x) - f(x), \tfrac{t}{2\lceil\alpha\rceil}\right) o\, B'(g_m(x) - g(x), \tfrac{t}{2\lceil\beta\rceil}) \\
&< \varepsilon \bullet \varepsilon \\
&= \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&Y'((\alpha f_m + \beta g_m)(x) - (\alpha f(x) + \beta g(x)), t) \\
&\leq Y'\left(\alpha f_m(x) - \alpha f(x), \tfrac{t}{2}\right) o\, Y'(\beta g_m(x) - \beta g(x), \tfrac{t}{2}) \\
&= Y'\left(f_m(x) - f(x), \tfrac{t}{2\lceil\alpha\rceil}\right) o\, Y'(g_m(x) - g(x), \tfrac{t}{2\lceil\beta\rceil}) \\
&< \varepsilon \bullet \varepsilon \\
&= \varepsilon
\end{aligned}$$

This implies that

$$\begin{aligned}
A^c \subset \{n \in \mathbb{N} :\ & G'((\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x), t) > 1 - \varepsilon, n \in \mathbb{N} \\
& : B'((\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x), \\
& t) < \varepsilon \ and\ n \in \mathbb{N} : Y'((\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x), t) < \varepsilon\}
\end{aligned}$$

Since, $\mathcal{F}(I)$ is filter, it follows that the later set belongs to $\mathcal{F}(I)$. According to Lemma 3.5.,

$$I_{(G,\ B,\ Y)} - (\alpha f_n + \beta g_n) \to \alpha f + \beta g.$$

Definition 3.7. Let $(F,\ N, o, \bullet)$ and $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n\ :\ (F, N, o, \bullet)\ \rightarrow\ (G, N', o, \bullet)$ be a sequence of functions. The sequence (f_n) is a pointwise I - Cauchy sequence in NNS provided that for every $\varepsilon > 0$ and $t > 0$ there exists a number $N\ =\ N(\varepsilon, t, x)$ such that

$$\left\{\begin{array}{c} \mathrm{n} \in \mathbb{N}:\ \mathrm{G}'(f_n(\mathrm{x}) - f_N(\mathrm{x}), \mathrm{t}) \leq 1 - \varepsilon \\ \text{or } B'(f_n(\mathrm{x}) -\ f_N(\mathrm{x}), \mathrm{t}) \geq\ \varepsilon \text{ for each x} \in \mathrm{X} \\ \text{or } Y'(f_n(\mathrm{x}) -\ f_N(\mathrm{x}), \mathrm{t}) \geq\ \varepsilon \text{ for each x} \in \mathrm{X} \end{array}\right\} \in I$$

Theorem 3.8. Let $(F,\ N,\ o, \bullet)$ and $(G, N',\ o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n\ :\ (F,\ N,\ o, \bullet)\ \rightarrow\ (G, N', o, \bullet)$ be a sequence of functions. If (f_n) is a pointwise $I -$ convergent with respect to neutrosophic norm (G, B, Y), then (f_n) is a pointwise $I -$ Cauchy sequence with respect to neutrosophic norm (G, B, Y).

Proof. Suppose that $I_{(G,\ B,\ Y)} - f_n \rightarrow f$ and let $\varepsilon > 0, t > 0$. For a given $\varepsilon > 0$, choose $s > 0$ such that $(1-\varepsilon)\ o\ (1-\varepsilon)\ >\ 1\ -\ s$ and $\varepsilon \bullet \varepsilon\ <\ s$. Then for each $x \in X$,

$$A_x(\varepsilon, t)\ = \left\{\begin{array}{l} n \in \mathbb{N} : G'\left(f_n - f(x), \quad \dfrac{t}{2}\right) \leq 1 - \varepsilon \\ \quad or\ B'\left(f_n(x) - f(x), \quad \dfrac{t}{2}\right) \geq \varepsilon \\ \quad or\ Y'\left(f_n(x) - f(x), \quad \dfrac{t}{2}\right) \geq \varepsilon \end{array}\right\} \in I$$

which implies that

$$A_x^c(\varepsilon, t) = \left\{\begin{array}{l} n \in \mathbb{N}: G'\left(f_n(x) - f(x), \dfrac{t}{2}\right) > 1 - \varepsilon \\ \quad or\ B'\left(f_n(x) - f(x), \quad \dfrac{t}{2}\right) \geq \varepsilon \\ \quad or\ Y'\left(f_n(x) - f(x), \quad \dfrac{t}{2}\right) \geq \varepsilon \end{array}\right\} \in \mathcal{F}(I)$$

Let us choose $N \in A_x^c\ (\varepsilon, t)$. Then

$$G'\left(f_N(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon\ , B'\left(f_N(x) - f(x), \frac{t}{2}\right) < \varepsilon\ and\ Y'\left(f_n(x) - f(x), \frac{t}{2}\right) < \varepsilon$$

We want to show that there exists a number $N = N(x, \varepsilon, t)$ such that

$$\left\{\begin{matrix} n \in \mathbb{N}: G'(f_n(x) - f_N(x), t) \leq 1 - s \\ or\ B'(f_n(x) - f_N(x), t) \geq s\ for\ each\ x \in X \\ or\ Y'(f_n(x) - f_N(x), t) \geq s\ for\ each\ x \in X \end{matrix}\right\} \in I$$

For this, define for each $x \in X$

$$B_x(\varepsilon, t) = \left\{\begin{matrix} n \in \mathbb{N}: G'(f_n(x) - f_N(x), t) \leq 1 - s \\ or\ B'(f_n(x) - f_N(x), t) \geq s \\ or\ Y'(f_n(x) - f_N(x), t) \geq s \end{matrix}\right\}$$

We have to show that

$$B_x(\varepsilon, t) \subset A_x(\varepsilon, t)$$

Suppose that

$$B_x(\varepsilon, t) \nsubseteq A_x(\varepsilon, t)$$

In this case $B_x(\varepsilon, t)$ has at least one different element which $A_x(\varepsilon, t)$doesn't has. Let $n \in B_x\ (\varepsilon, t)\backslash A_x\ (\varepsilon, t)$. Then we have

$$G'(f_n(x) - f_N(x), t) \leq\ 1 - s$$
$$\text{and } G'\left(f_n(x) - f(x), \tfrac{t}{2}\right) > 1 - \varepsilon$$

in particularly $G'\left(f_N(x) - f(x), \frac{t}{2}\right) \geq 1 - \varepsilon$. In this case

$$\begin{aligned} 1 - s &\geq G'(f_n(x) - f_N(x), t) \\ &\geq G'\left(f_n(x) - f(x), \tfrac{t}{2}\right) o\ G'\left(f_N(x) - f(x), \tfrac{t}{2}\right) \\ &\geq (1 - \varepsilon) o (1 - \varepsilon) > 1 - s \end{aligned}$$

which is not possible. On the other hand

$$B'(f_n(x) - f_N(x), t) \geq\ s$$
$$and\ B'\ (f_n\ (x) - f\ (x), t/2)\ <\ \varepsilon,$$

in particularly $B'\ (f_N\ (x) - f\ (x), t/2) < \varepsilon$. In this case

$$s \leq B'\ (f_n\ (x) - f_N\ (x), t)$$
$$\leq\ B'\ (f_n\ (x) - f\ (x), \frac{t}{2}) \bullet B'\ (f_N\ (x) - f\ (x), \frac{t}{2})$$
$$<\ \varepsilon \bullet \varepsilon\ <\ s$$

Similarly

$$Y'(f_n(x) - f_N(x), t) \geq\ s$$
$$and\ Y'\ (f_n\ (x) - f\ (x), t/2)\ <\ \varepsilon,$$

in particularly, $Y'(f_N(x) - f(x), t/2) < \varepsilon$. In this case

$$\begin{aligned} s &\leq Y'(f_n(x) - f_N(x), t) \\ &\leq Y'(f_n(x) - f(x), \frac{t}{2}) \bullet Y'(f_N(x) - f(x), \frac{t}{2}) \\ &< \varepsilon \bullet \varepsilon < s \end{aligned}$$

which is not possible. Hence $B_x(\varepsilon, t) \subset A_x(\varepsilon, t)$. Therefore, since $A_x(\varepsilon, t) \in I, B_x(\varepsilon, t) \in I$, (f_n) is a pointwise I - Cauchy sequence with respect to neutrosophic norm (G, B, Y) on X.
In studying on sequences of functions, uniform convergence is another important concept. Now we introduce uniformly I - convergence of sequences of functions in a NNS. Let us start with the following definition.

Definition 3.9. Let $(F, N, o, \bullet)$ and $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. and $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a sequence of functions. If for every $x \in X$ and $\forall \varepsilon > 0, t > 0$

$$\left\{ \begin{matrix} n \in \mathbb{N}: G'(f_n(x) - f_N(x), t) \leq 1 - \varepsilon \\ or\ B'(f_n(x) - f(x), t) \geq \varepsilon \\ or\ Y'(f_n(x) - f(x), t) \geq \varepsilon \end{matrix} \right\} \in I$$

then we say that the sequence (f_n) is uniformly I - convergent with respect to neutrosophic norm (G, B, Y) and we denote $I_{(G,\ B,\ Y)} - f_n \rightrightarrows f$.

Corollary 3.10. Let $I = \{A \subset \mathbb{N} : A$ is a finite set$\}$. Then I is an admissible ideal in $\mathbb{N}$. Hence, in the Definition 3.4. and the Definition 3.9., I - convergence coincides usual convergence of sequences of functions with respect to neutrosophic norm.

Corollary 3.11. Let $I = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$. Hence, in the Definition 3.4. and the Definition 3.9., I - convergence coincides with statistical convergence of sequences of function with respect to neutrosophic norm.

Remark 3.12. Let $(F, N, o, \bullet)$ and $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a sequence of functions. If (f_n) is uniformly neutrosophic convergent on X to a function f with respect to (G, B, Y), then $I_{(G,\ B,\ Y)} - f_n \rightrightarrows f$. But the converse of this is not true.

Lemma 3.13. Let $(F, N, o, \bullet)$ and $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$be a sequence of functions. Then for every $\varepsilon > 0$, for every $x \in X$ and $t > 0$, the following statements are equivalent:

(i) $I_{(G,\ B,\ Y)} - f_n \rightrightarrows f$.

(ii) $\{n \in \mathbb{N} : G'(f_n(x) - f(x), t) \leq 1 - \varepsilon\} \in I\}$, $\{n \in \mathbb{N} : B'(f_n(x) - f(x), t) \geq \varepsilon\} \in I\}$ *and* $\{n \in \mathbb{N} : Y'(f_n(x) - f(x), t) \geq \varepsilon\} \in I\}$

(iii) $\{n \in \mathbb{N} : G'(f_n(x) - f(x), t\} > 1 - \varepsilon\}, \{B'(f_n(x) - f(x), t) < \varepsilon\}$ and $\{Y'(f_n(x) - f(x), t) < \varepsilon\} \in F(I)$.

(iv) $\{n \in \mathbb{N} : G'(f_n(x) - f(x), t) > 1 - \varepsilon\} \in F(I), \{n \in \mathbb{N} : \{B'(f_n(x) - f(x), t) < \varepsilon\} \in F(I)$ and $\{n \in \mathbb{N} : Y'(f_n(x) - f(x), t) < \varepsilon\} \in F(I)\}$.

(v) $I - \lim G'(f_n(x) - f(x), t) = 1, I - \lim B'(f_n(x) - f(x), t) = 0$ and $I - \lim Y'(f_n(x) - f(x), t) = 0$

Definition 3.14. Let $(F, N, o, \bullet)$ *and* $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a sequence of functions. The sequence (f_n) is a uniform I - Cauchy sequence in neutrosophic normed space provided that for every $\varepsilon > 0$ *and* $t > 0$ there exists a number $N = N(\varepsilon, t)$ such that

$$\{n \in \mathbb{N} : G'(f_n(x) - f_N(x), t) \leq 1 - \varepsilon$$
$$or\ B'(f_n(x) - f_N(x), t) \geq \varepsilon$$
$$or\ Y'(f_n(x) - f_N(x), t) \geq \varepsilon\ for\ every\ x \in X\} \in I$$

Theorem 3.15. Let $(F, N, o, \bullet)$ *and* $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a sequence of functions. If (f_n) is a uniformly I - convergent sequence with respect to neutrosophic norm (G, B, Y), then f_n is uniformly I - convergent Cauchy sequence with respect to neutrosophic norm (G', B', Y').

Proof. The proof is similar to the Theorem 3.8. We omit it.

Theorem 3.16. Let $(F, N, o, \bullet)$ *and* $(G, N', o, \bullet)$ be NNS and N = (G, B, Y) and $N' = (G', B', Y')$ be respective NN. Let $f_n : (F, N, o, \bullet) \to (G, N', o, \bullet)$ be a neutrosophic continuous mapping on X. If $(G, B, Y) - f_n \to f$, then $f : X \to Y$ is a neutrosophic continuous mapping on X.

Proof. Let $x_0 \in X$ be an arbitrary point. By the neutrosophic continuity of f_n's, there exist $\delta > 0$ *such that* $G'(x - x_0, t) > 1 - \delta$, $B'(x - x_0, t) < \delta$ *and* $Y'(x - x_0, t) < \delta \Rightarrow G'(f_n(x_0) - f_n(x), t) > 1 - \varepsilon$, $B'(f_n(x_0) - f_n(x), t) < \varepsilon$ *and* $Y'(f_n(x_0) - f_n(x), t) < \varepsilon$ for every $n \in \mathbb{N}$ and $t > 0$. Let $x \in B(x_0, \delta, t)$ (open ball with center x_0 and radius δ in neutrosophic normed space be fixed. Since $I_{(G,\ B,\ Y)} - f_n \to f$ on X, the sets

$$\mathrm{A} = \left\{ \begin{array}{c} n \in \mathbb{N} : G'\left(f_n - f(x), \frac{t}{2}\right) \leq 1 - \varepsilon \\ or\ B'\left(f_n(x) - f(x), \frac{t}{2}\right) \geq \varepsilon \\ or\ Y'\left(f_n(x) - f(x), \frac{t}{2}\right) \geq \varepsilon \end{array} \right\} \in I$$

and

$$A_x(\varepsilon, t) = \left\{ \begin{array}{c} n \in \mathbb{N} : G'\left(f_n(x_0) - f(x), \frac{t}{2}\right) \leq 1 - \varepsilon \\ or\ B'\left(f_n(x_0) - f(x_0), \frac{t}{2}\right) \geq \varepsilon\ for\ each\ x \in X \\ or\ Y'\left(f_n(x_0) - f(x_0), \frac{t}{2}\right) \geq \varepsilon\ for\ each\ x \in X \end{array} \right\} \in I$$

is in I so $A \cup B \in I$ and $A \cup B$ is different from $\mathbb{N}$ since I is non-trivial. Thus, there exists an $n_0 \in \mathbb{N}$ such that

$$G'(f_{n_0}(x) - f(x), \frac{t}{3}) > 1 - \varepsilon$$
$$B'(f_{n_0}(x) - f(x), \frac{t}{3}) < \varepsilon$$
$$Y'(f_{n_0}(x) - f(x), \frac{t}{3}) < \varepsilon$$

and

$$G'(f_{n_0}(x) - f(x), \frac{t}{3}) > 1 - \varepsilon$$
$$B'(f_{n_0}(x) - f(x), \frac{t}{3}) < \varepsilon$$
$$Y'(f_{n_0}(x) - f(x), \frac{t}{3}) < \varepsilon$$

Now, we will show that f is neutrosophic continuous at x_0. We have

$$\begin{aligned} G'(f(x) - f(x_0), t) &= G'\left(f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0), t\right) \\ &\geq G'^{\left(f(x) - f_{n_0}(x), \frac{t}{3}\right)} o \mathrm{G}'\left(f_{n_0}(\mathrm{x}) - f_{n_0}(x_0), \frac{t}{3}\right) o \mathrm{G}'\left(f_{n_0}(x_0) - f(x_0), \frac{t}{3}\right) \\ &> (1 - \varepsilon)\ o\ (1 - \varepsilon)\ o\ (1 - \varepsilon) \\ &= 1 - \varepsilon \end{aligned}$$

and

$$B'(f(x) - f(x_0), t) = B'\big(f(x) - (x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0), t\big)$$
$$\leq B'\left(f(x) - f_{n_0}(x), \frac{t}{3}\right) \bullet B'(f_{n_0}(\mathrm{x}) - f_{n_0}(x_0), \frac{t}{3}) \bullet B'(f_{n_0}(\mathrm{x}_0) - f(x_0), \frac{t}{3})$$
$$< (1-\varepsilon) \bullet (1-\varepsilon) \bullet (1-\varepsilon) = \varepsilon.$$

Similarly, $Y'(f(x) - f(x_0), t) < \varepsilon$.

Hence the proof is completed.

4. Conclusion:

In this article, we have introduced the notions of sequence of convergence, point wise convergence and uniform convergence of sequences of functions with the help of neutrosophic norm in neutrosophic normed spaces. We have investigated several basic properties and characterization theorems of these newly defined concepts in neutrosophic normed spaces. We have defined *I* -convergence, point wise and uniform *I* - convergence and *I* - Cauchy sequence of sequences of functions with the help of neutrosophic norm in neutrosophic normed spaces. We have studied their basic properties and the relationship among the concepts such as I - convergence, statistical convergence and the usual convergence of sequences of functions in neutrosophic normed spaces.

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A Priority-based Queuing Control System Using Fuzzy Logic: Its application in the field of Railway Network

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Abstract: This paper aims at developing a fuzzy control system for queuing control (especially in traffic system) based on customer priority. Here we propose one model in which the decision regarding which customer (vehicle) is to be served next is taken by calculating the priority measure of each customer (vehicle) on the basis of certain parameters. Here we have considered the rail traffic system as the field of study. It may also be helpful in various other traffic control problems but parameters may be different.

Keywords: Fuzzy control, waiting time, holding cost, priority measure, etc.

2000 Mathematics Subject classification: 93C42

1. Introduction:

In today's world, Queues are very common. We can find Queues at hospital outdoors, petrol pumps, ticket booking counters, airports, banks, machine service stations, communication channels, etc. Queues are formed when demand for a service is more than the service facility available. A proper and efficient management of queues are very much essential for minimizing customers' waiting times as well as the service cost incurred by the service providers and this is the motivation that led to the growth of queuing theory.

Queuing control deals with controlling the various parameters of a queuing system with the aim of minimizing the costs, reducing the waiting time as well as minimizing the customer's inconvenience. Traditional queuing control techniques uses conventional stochastic methods. Though these methods are often successful, but they have some serious computational limitations – often queuing systems do not have mathematical descriptions, or such descriptions are very complicated. Fuzzy logic has appeared to be a powerful tool to overcome such limitations. In Fuzzy control one do not need a mathematical model of the system under consideration, and it is suitable in cases of complex systems or ill-defined processes. Moreover, in some cases input values of various parameters of a system may have fuzziness, inaccuracy or incompleteness. Similarly, the control rules that derive output values may also be incomplete or inaccurate. In such cases, fuzzy logic allows decision making with estimated values under

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incomplete information. Lots of work has been done till date in the field of fuzzy queuing control, but less concentration has been given to the priority or emergency of the customer.
In this paper we propose a model for fuzzy queuing control which will decide the order in which the customers present in a queue at a certain time should be served. The controller will first calculate the priority of each customer in the queue on the basis of certain parameters and the final order of service is made on the basis of this priority measure. Priority is a fuzzy concept and the determination of priority will depend on some parameters which are fuzzy. So, fuzzy control is the best way to deal with it. We also take queuing control in the railway network to illustrate the proposed controller.

1. **Mathematical Preliminaries:**

2.1. Fuzzy Set [1,2]

Fuzzy logic was first proposed by Lotfi A. Zadeh of the University of California at Berkeley in a 1965 paper. He elaborated on his ideas in a 1973 paper that introduced the concept of "linguistic variables", which in this article equates to a variable defined as a fuzzy set.

***2.1.1. Definition*:** Let X be a set of elements x. A fuzzy set A is a collection of ordered pairs (x, μ(x)) for xεX. X is called the universe of discourse and $\mu_A(x)$: X→[0,1] is the membership function. The function $\mu_A(x)$ provides the degree of membership of x in A. When X is countable, the fuzzy set A is represented as

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots\ldots.. + \mu_A(x_n)/x_n$$

This is a common notation in the context of fuzzy sets. It simply states the elements x_i of X and the corresponding membership grades.

2.2. Fuzzy Control [2]

Fuzzy logic control system is an expert system that uses a collection of Fuzzy rules [Fuzzy Rule Base] that uses the membership functions to derive conclusions. The principal design parameters for a fuzzy logic controller are [Driankov D, Hellendoorn H, Reinfrank M (1996), An introduction to fuzzy control, Springer Verlag, Berlin Heidelberg Newyork]

1) Fuzzification Methods and meaning
2) Knowledge base
   a) discretization/normalization of the universe of discourse
   b) choice of inputs and outputs
   c) choice of membership functions
   d) derivation of fuzzy control rules
   e) consistency, completeness of fuzzy control rules.
3) Inference Engine
   a) definition of fuzzy implication
   b) inference mechanism
4) Defuzzification method.

3. Description of The Proposed Fuzzy Control System:

In today's world, queues are present everywhere. As a result, in order to provide quality service and make profit at the same time, proper and efficient control of queues is very important. At this point we would like to emphasize on the fact that while controlling queues, the objective should not only be to minimize the service and holding costs, but also to minimize the inconvenience of the customer, a concept which is totally fuzzy. To deal with this we propose a fuzzy queuing controller which will take into consideration all these parameters and determine the priority (a rough set concept) of each customer which will in turn give the order in which the customers currently present should be served.

The parameters on the basis of which the priority of each customer is decided are process dependent. Expert opinion may be taken to decide the parameters for a particular process.

However, in our model two parameters, viz. waiting time and customer category are used in every process. In general, we divide customers into four categories - Emergency, Very Important, Important and general.

We decide to fix a particular waiting time, say T_{max}, such that whenever a customer's waiting time exceeds T_{max} that customer should be the next to be served as and when the server is empty. Also, the category of the customer will play a vital role in deciding the priority.

3.1. The proposed model:

In this model, we use the category of customers while constucting the rule base. We agree that if an 'EMERGENCY' category or a 'VERY IMPORTANT' category customer is waiting in the queue, then it should be given first priority. If both 'EMERGENCY' and 'VERY IMPORTANT' category customers are present, we first allow the 'EMERGENCY' customer and then the 'VERY IMPORTANT' customer is allowed. If more than one customer of each of the above category is present we decide to serve them on the basis of their waiting time (more the waiting time, higher the priority).

(3.1.1.) The step-wise working procedure of the proposed queuing control system is as below:

Step 1: The system receives the necessary data from various sources.

Step 2: Checks whether any 'EMERGENCY' category customer(s) is(are) present or not. If such customer(s) is(are) present, then decide the priority as mentioned above and go to Step 1. If not, then go to Step 3.

Step 3: Checks whether the current waiting time of any customer is greater than T_{max} or not. If it is greater than T_{max} then give him the first priority as and when the server is empty and go to Step 1. If it is not, then go to Step 4.

Step 4: Checks whether any 'VERY IMPORTANT' category customer(s) is(are) present or not. If such customer(s) is(are) present, then decide the priority as mentioned above and go to Step 1. If not, then go to Step 5.

Step 5: Determines the priority of the other customers waiting for service, then decides the order in which they should be served and go to Step 1.

Step 1-4 is simple to execute and we are not going into details. We now explain the procedure of Step 5.

3.1.2. FUZZIFICATION:

As we know, in this step the crisp input values are fuzzified using membership functions for different fuzzy sets corresponding to the input parameters. The membership functions will vary from process to process and expert opinion will be taken to construct the most appropriate membership function for a particular process. We also define the fuzzy sets corresponding to the output and assign certain weights to them.

3.1.3. INFERENCE ENGINE:

The next step is the set up the fuzzy rule base, i.e. a set of IF-THEN rules applicable to the process. The fuzzy controller calculates the priority of a particular customer on the basis of this rule set. This rule set will be constructed by taking the opinion of an expert human operator. Taking the help of more than one expert finer tunings can be done on this rule base to get better results. In this model we will make different rule bases for different customer categories. We will see it when we take the example of rail traffic later.

3.1.4. DEFUZZIFICATION:

In this step the fuzzy set output is converted to real crisp value. The method used in this model is 'Centre of gravity' [2].

Crisp output = $\dfrac{(\sum(\text{Membership degree})\text{x}(\text{singleton output fuzzy set}))}{(\sum(\text{Membership degree}))}$

This crisp output will give the measure of priority of each customer. After calculating the priority measure for each customer in the queue, the one with the highest priority is selected to be served next. The entire procedure is repeated after a certain pre-assigned interval of time.

4. Application of the Proposed Controller in the Indian Railway Network:

In the Indian rail network it is often seen that more than one train is waiting to enter a platform or to pass through a particular track at the same instant. In this case an experienced operator decides which train should be passed first on the basis of different parameters and executes the action. The entire process is done manually. Our aim is to automate the system using a fuzzy logic controller. Here, we use the two fuzzy controller models proposed above to take the decision regarding which train should be given green signal first. Various parameters are considered while taking the decision.

The Decision Parameters:

1) **Waiting Time (WT):** The time for which a particular train is waiting for the green signal.

2) **Late Status (LS):** Whether the train is running on time or not, and if it is running late, then how much it is late.

3) **Distance Travelled(D):** The distance covered by the train till now from its starting point.

4) **Train Category (C):** All the trains are divided into different categories depending on their type. We divide them into three categories:

(a) **Emergency:** medical facilities train, emergency train for incidental help, etc

(b) **Very Important (VI):** Rajdhani, Shatabdi, Durrant, Palace on Wheels, etc.

(c) **Important (I):** Other superfast trains.

(d) **General (G):** The remaining express, passenger, local trains and cargo trains.

4.1. Application of the proposed model:

We first execute the steps 1-4 mentioned in 3.1.1. and take care of the trains having waiting time greater than T_{max} and the trains of EMERGENCY and VERY IMPORTANT category.

We now describe how to compute the priority measure of the trains belonging to IMPORTANT and GENERAL categories.

4.1.1. FUZZIFICATION:

The fuzzy sets corresponding to WT(waiting time), D(distance travelled) and C(customer category) ,LS(late status) are

WT	LS	**D**	**C**
Small (S) Medium (M) Long (L)	On Time (OT) Late (LT) Very Late (VL)	Small (S) Medium (M) Long (L)	Emergency(E) Very Important(VI) Important (I) General (G)

Table 1

The corresponding output parameter is the degree of priority. The fuzzy sets corresponding to the output are **Very High**(1), **High**(0.75), **Medium**(0.5), **Low**(0.25), **Very Low**(0). The number in the bracket represents the weights corresponding to each output.

The crisp input values are fuzzified using membership functions for different fuzzy sets corresponding to the input parameters. The membership functions for different fuzzy sets are given below:

For Waiting time(WT):

$$M_S(x) = \begin{cases} 0 & , x < 0 \\ 1 & , 0 \le x \le 5 \\ (10-x)/(10-5) & , 5 < x \le 10 \\ 0 & , x > 10 \end{cases} \quad , \quad M_M(x) = \begin{cases} 0 & , x < 5 \\ (x-5)/(10-5) & , 5 \le x < 10 \\ 1 & , 10 \le x \le 15 \\ (20-x)/(20-15) & , 15 < x \le 20 \\ 0 & , x > 20 \end{cases} \quad , \quad M_L(x) = \begin{cases} 0 & , x < 15 \\ (x-15)/(20-15) & , 15 \le x \le 20 \\ 1 & , x > 20 \end{cases}$$

For Late status(LS):

$$M_{OT}(x) = \begin{cases} 0 & , x < 0 \\ 1 & , 0 \le x \le 10 \\ (20-x)/(20-10) & , 10 < x \le 20 \\ 0 & , x > 20 \end{cases} \quad , \quad M_{LT}(x) = \begin{cases} 0 & , x < 10 \\ (x-10)/(20-10) & , 10 \le x < 20 \\ 1 & , 20 \le x \le 40 \\ (60-x)/(60-40) & , 40 < x \le 60 \\ 0 & , x > 60 \end{cases} \quad , \quad M_{VL}(x) = \begin{cases} 0 & , x < 40 \\ (x-40)/(60-40) & , 40 \le x \le 60 \\ 1 & , x > 60 \end{cases}$$

For Distance traveled(D):

$$M_S(x) = 0 \quad , x < 0 \qquad , \qquad M_M(x) = 0 \quad , x < 100$$

$$\begin{cases} 1 & , 0 \le x \le 100 \\ (200-x)/(200-100), & 100 < x \le 200 \\ 0 & , x > 200 \end{cases} \qquad \begin{cases} (x-100)/(200-100), & 100 \le x < 200 \\ 1 & , 200 \le x \le 300 \\ (400-x)/(400-300), & 300 < x \le 400 \\ 0 & , x > 400 \end{cases}$$

$$M_L(x) = \begin{cases} 0 & , x < 300 \\ (x-300)/(400-300), & 300 \le x \le 400 \\ 1 & , x > 400 \end{cases}$$

The graph of these membership functions are shown below:

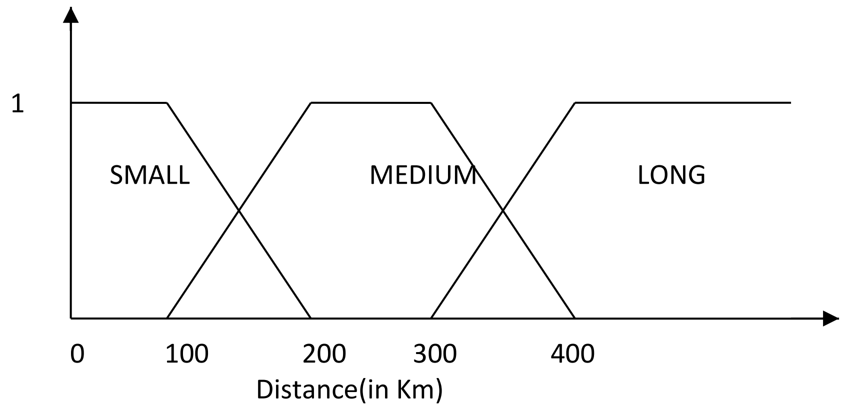

Figure 1: Fuzzy sets for

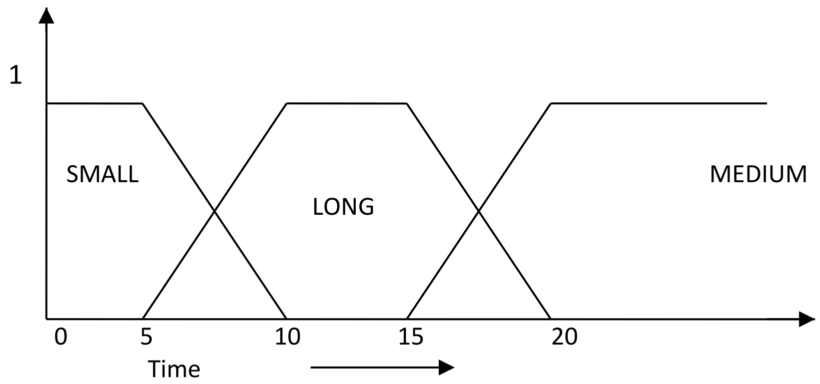

Figure 2: Fuzzy sets for WT

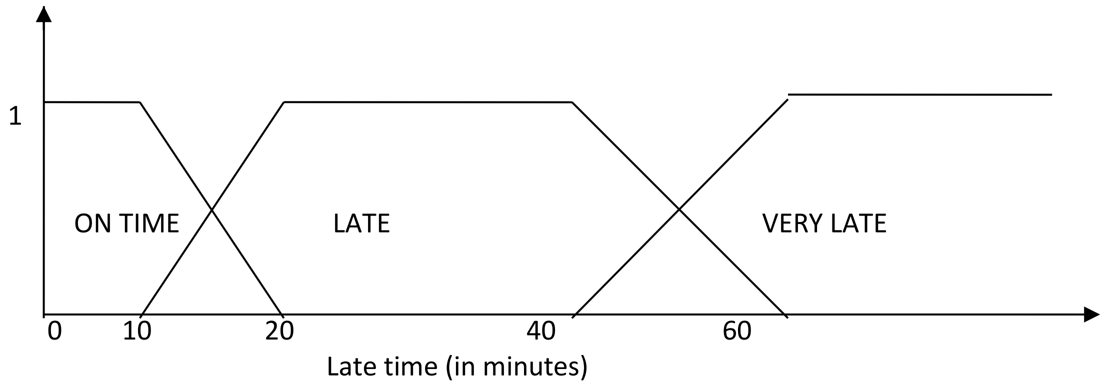

Figure 3: Fuzzy sets for LS

4.1.2. INFERENCE ENGINE:

The rule base for this rail traffic problem under the first model is given below. Here different rule bases are constructed for different categories. As for example, the first rule states that *"IF Train Status (TS) is IMPORTANT AND LS is OT AND D is Small (S) AND Waiting Time(WT) is Small(S), THEN 'Priority' is Very High (VH)."*

IF C is IMPORTANT AND							***IF C is GENERAL AND***					
IF					***THEN***		***IF***					***THEN***
LS	***AND***	***D***	***AND***	***WT***	***Priority***		***LS***	***AND***	***D***	***AND***	***WT***	***Priority***
OT	***AND***	S	***AND***	S	VH		OT	***AND***	S	***AND***	S	VH
OT	***AND***	S	***AND***	M	VH		OT	***AND***	S	***AND***	M	VH
OT	***AND***	S	***AND***	L	VH		OT	***AND***	S	***AND***	L	VH
OT	***AND***	M	***AND***	S	VH		OT	***AND***	M	***AND***	S	VH
OT	***AND***	M	***AND***	M	VH		OT	***AND***	M	***AND***	M	VH
OT	***AND***	M	***AND***	L	VH		OT	***AND***	M	***AND***	L	VH
OT	***AND***	L	***AND***	S	VH		OT	***AND***	L	***AND***	S	VH
OT	***AND***	L	***AND***	M	VH		OT	***AND***	L	***AND***	M	VH
OT	***AND***	L	***AND***	L	VH		OT	***AND***	L	***AND***	L	VH
LT	***AND***	S	***AND***	S	L		LT	***AND***	S	***AND***	S	VL

LT	***AND***	S	***AND***	M	M		LT	***AND***	S	***AND***	M	VL
LT	***AND***	S	***AND***	L	H		LT	***AND***	S	***AND***	L	L
LT	***AND***	M	***AND***	S	L		LT	***AND***	M	***AND***	S	VL
LT	***AND***	M	***AND***	M	M		LT	***AND***	M	***AND***	M	L

LT	***AND***	M	***AND***	L	H		LT	***AND***	M	***AND***	L	M
LT	***AND***	L	***AND***	S	M		LT	***AND***	L	***AND***	S	L
LT	***AND***	L	***AND***	M	H		LT	***AND***	L	***AND***	M	M
LT	***AND***	L	***AND***	L	VH		LT	***AND***	L	***AND***	L	M
VL	***AND***	S	***AND***	S	VL		VL	***AND***	S	***AND***	S	VL
VL	***AND***	S	***AND***	M	L		VL	***AND***	S	***AND***	M	VL
VL	***AND***	S	***AND***	L	M		VL	***AND***	S	***AND***	L	M
VL	***AND***	M	***AND***	S	M		VL	***AND***	M	***AND***	S	L
VL	***AND***	M	***AND***	M	H		VL	***AND***	M	***AND***	M	M
VL	***AND***	M	***AND***	L	VH		VL	***AND***	M	***AND***	L	H
VL	***AND***	L	***AND***	S	H		VL	***AND***	L	***AND***	S	H
VL	***AND***	L	***AND***	M	VH		VL	***AND***	L	***AND***	M	VH
VL	***AND***	L	***AND***	L	VH		VL	***AND***	L	***AND***	L	VH

Table 2

4.1.3. DEFUZZIFICATION:

Now, the priority measure is computed as described in 3.1.4. After calculating the priority for all the trains waiting in the queue, the train with the maximum priority is selected to get the green light.

Example 1:

Let we have 3 trains waiting in a queue to get access to a particular platform. The following information is available:
Train 1: Rajdhani Express, D = 570 km, WT = 8 min, LS = 5 min.
Train 2: Gitanjali (super fast) Express, D = 270 km, WT = 18 min, LS = 15 min.
Train 3: Lalgola Passenger, D = 325 km, WT = 21 min, LS = 45 min.

From the graphs of the membership functions for the fuzzy sets corresponding to D, WT and LS, we can easily compute the following:

For Train 1:

Category is VERY IMPORTANT,

D is L with membership grade 1,

WT is S with membership grade 0.4 and M with membership grade 0.6,

LS is OT with membership grade 1.

For Train 2:

Category is IMPORTANT,

D is M with membership grade 1,

WT is M with membership grade 0.4 and L with membership grade 0.6,

LS is LT with membership grade 0.5 and OT with membership grade 0.5.

For Train 3:

Category is GENERAL,

D is M with membership grade 0.75 and L with membership grade 0.25,

WT is L with membership grade 1,

LS is LT with membership grade 0.75 and VL with membership grade 0.25.

Obviously, Train 1 is a VERY IMPORTANT category train and should get access to the platform before Train 2 and 3. We now compute the priority of the other trains.

Now we execute the applicable rules from the rule base in 4.1.2.(Table 2) to find the priority of Train 2 and Train 3.

For Train 2:

From the rule base we find that the following four rules are applicable.

If C is IMPORTANT and D is M with 1, WT is M with 0.4 and LS is LT with 0.5, then priority is M with membership grade = min{1, 0.4, 0.5} = 0.4.

If C is IMPORTANT and D is M with 1, WT is M with 0.4 and LS is OT with 0.5, then priority is VH with membership grade = min{1, 0.4, 0.5} = 0.4.

If C is IMPORTANT and D is M with 1, WT is L with 0.6 and LS is LT with 0.5, then priority is H with membership grade = min{1, 0.6, 0.5} = 0.5.

If C is IMPORTANT and D is M with 1, WT is L with 0.6 and LS is OT with 0.5, then priority is VH with membership grade = min{1, 0.6, 0.5} = 0.5.

Therefore,

Priority = {(0.4x0.5) + (0.4 x1) + (0.5x0.75) + (0.5x1)}/(0.4 + 0.4 + 0.5 + 0.5) = **0.82**

For Train 3:

If C is GENERAL and D is M with 0.75, WT is L with 1 and LS is LT with 0.75, then priority is M with membership grade = min{0.75, 1, 0.75} = 0.75.

If C is GENERAL and D is M with 0.75, WT is L with 1 and LS is VL with 0.25, then priority is H with membership grade = min{0.75, 1, 0.25} = 0.25.

If C is GENERAL and D is L with 0.25, WT is L with 1 and LS is LT with 0.75, then priority is M with membership grade = min{0.25, 1, 0.75} = 0.25.

If C is GENERAL and D is L with 0.25, WT is L with 1 and LS is VL with 0.25, then priority is VH with membership grade = min{0.25, 1, 0.25} = 0.25.

Therefore,

Priority = {(0.75x0.5) + (0.25 x0.75) + (0.25x0.5) + (0.25x1)}/(0.75 + 0.25 + 0.25 + 0.25)

= **0.63**

Hence, we can see that Train 2 has higher priority than Train 3 and hence Train 2 should get the access to the platform before Train 3. So, Train 1 will get access to the platform first, followed by Train 2 and Train 3 respectively.

CONCLUSION:

In this paper we have proposed a fuzzy logic controller for queuing control based on priority and keeping the objective of minimizing the customer inconvenience in mind, in addition to the aim of minimizing the costs. We have kept the category of each customer in mind while making the rule base and hence get different rule bases for different categories.

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A Mathematical Model to Determine Sustainable Harvesting Strategy of Forestry Biomass for Forestry Dependent Industrialization

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Abstract

Forestry is an important natural resource. Various industries depend on forestry biomass. Though renewable, forestry management for sustainable socio-economic development is a challenging issue in the modern world. The major concern with the consumption of forestry biomass is that it takes quite a long time to grow back to the economically profitable level. Uncontrolled exploitation of forestry biomass will not only create high environmental pollutions it may also create irreversible natural hazards such as climate change, global warming etc. Therefore, for sustainable development we have to find an optimal strategy for forestry management so that the forestry dependent industries can survive without overexploiting and depleting the forestry biomass. In this purpose we have considered a deterministic mathematical model involving pre-mature trees (not suitable for industries and restricted for industrial usage), mature trees (suitable for industrial usage and can be cut for industrial usage) and industrialization as the state variables. The harvesting of the trees is considered to be age – structured, as only mature trees can be cut. We have considered modified Leslie-Gower type industrial growth to incorporate the alternatives that industries may use when there is a shortage or low growth of mature trees. We have used *Pontryajin's Maximum Principle* to determine sustainable harvesting strategy of the forestry biomass so that forestry related industries can grow without depleting the forestry biomass. Numerical verification of analytical results has also been studied and mathematical results have been interpreted bionomically.

Keywords: Deterministic mathematical model, modified Leslie–Gower function, stability analysis, optimal control, sensitivity analysis, data fitting.

Mathematics Subject Classification: 34D, 34H, 90A, 92B.

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1 Introduction

In today's world natural resources are highly essential for the all-round development of any human civilization. Coal and Fossil Fuel are two of the most important natural resources. They are inseparable parts of socio – economic developments in any human society. However, these resources are not renewable and due to unplanned & uncontrolled exploitation in near future coal and fossil fuel reserve will be exhausted all over the world. Therefore, in coming times renewable natural resources such as water, wind, biofuel and forestry etc. will play pivotal roles in the development of human civilization. Apart from water and wind, forestry is an important natural resource. Forestry biomass helps us to mitigate the issue of climate change, ensures adequate supply of fresh water by regulating rainfall, enhances biodiversity, provides sustainable income and livelihood to the people, preserves food security, generates vital air to breath, stops soil erosion etc. United Nations declared the year 2011 as the year of ''Forests'' to generate awareness against depletion of forestry biomass and to strengthen the conservation, adequate management and sustainable development of all types of forests for the benefit of not only of the present generations but also for the future (http:// www.un.org/ en/ events/iyof2011/).

Forestry biomass-based industries are one of the major aspects of the forest associated economy. Various industries such as fruit, juice, honey, rubber, tea and coffee etc. are completely dependent on forestry biomass. But in this type of agriculture associated industries forestry biomass is not affected directly as we do not have to cut the trees completely for these types of industries. However, there are other important industries where felling of the forestry biomass is a must. For example, industries related to formation of sandal wood, timber & roundwood for construction & furniture, production of paper, plywood & other allied industries etc. For the growth of these industries the forestry biomass needed to be cut and that may cause a havoc environmental hazard unless managed properly. Therefore, it is a challenging issue to determine an effective policy so that those important industries that cause loss of forestry biomass, can be sustained and the environmental balance is not hampered too.

1.1 Present situation of forestry biomass in India

India State of Forest Report 2021 **[1]** states that the total forest cover of the country is at present 713789 km^2 that is 21.71% of the total geographical area of India. The current assessment indicates that the forest cover has increased 1540 km^2 (0.22%), the tree cover has increased 721 km^2 (0.76%) and total forestry biomass has increased 2261 km^2 (0.28%) from the statistics reported in India State of Forest Report 2019 **[2]**. Moreover, according to **[1]** the total forest cover in the Northeast region is 169521 km^2, which shows a decrease of 1020 km^2 (0.6%) forest area from 2019. The total growing stock of wood in the country is estimated to be 6167.5 $million\ m^3$ comprising 4388.15 $million\ m^3$ inside forest area and 1779.35 $million\ m^3$ outside the recorded forestry biomass.

There is no doubt that the India's forest and tree cover are experiencing a steady growth in the last two decades. However, there is a standing deficit between timber production and its demand for industrial use in India & export in other foreign countries. Therefore, India imports large quantities of timber, especially roundwood, for industrial use. This trend began in the 1980s when roundwood production was in the range of 10 to 15 $million\ m^3$ per year. The main reason of this gradual decline in production has been due to increased monitoring for conservation of forests after the notification of the National Forest Policy, 1988. According to the Statistical Yearbook 2021 of the Food and Agricultural Organization (FAO) of United Nations **[3]** the Compounded Annual Growth Rate (CAGR) in India declined every year during the decade 1991 to 2000. For industrial coniferous roundwood it was 0.7%. Whereas, for industrial non-coniferous roundwood the decline was 1.15%, for coniferous sawn – wood 8.72%, 8.39% for non-coniferous sawn – wood and 5.09% for veneer. With the stringent restrictions placed by the Supreme Court of India on harvesting from forests, initially in the North-eastern states and later to other states of India, the trend of decline became sharper in the following decades.

1.2 Present situation of forestry biomass-based industries in India

The demand for wood products is always high despite the available alternatives such as iron, steel, plastic and aluminium, etc and the price is quite high. This trend may be observed due to the high social acceptability of wood in every compartment of Indian society and the availability of forestry biomass. The timber industry in India is substantially significant and it serves a wide variety of end – uses that include construction, paper, plywood and panels, furniture, agricultural implements, handicrafts, and toys. The "*Make in India*" initiative of Government of India is also expanding the demand for wood from forestry biomass. The Indian timber market is quite dependent on the harvesting of timber from the following four main sources, namely, (a) the state forest departments, (a) forest development corporations, (c) privately owned forests known as Trees Outside Forests (TOF), and (d) imports. Trees outside Forests are either created by industries on their own land, or on farmers' land under buy-back arrangements, or agro-forestry practices in farmers' fields. The Indian economy depends highly on the wood-based or forestry biomass-based industries. It significantly contributes to rural economy and provides employment for the urban population, both permanent as well as seasonal.

Therefore, it will be interesting to determine an optimal harvesting policy of the forestry biomass so that forestry biomass related industries can be sustained without hampering the ecological balance, not only in India but with respect to the world too.

2 Model formulation

Extensive researches have been performed by the various researchers to mitigate the issue of sustainable socio – economic development of the human society conserving the environmental balance. The effects of deforestation on climate change and the impact of climate change on forestry biomass have been studied by **[4]**. In their study they have shown that the annual rate of deforestation is 0.14 % and as a result the world has lost 2.3 million square kilometres of forest cover has been lost between 2000 and 2012. Shukla et al. **[5]** proposed and analysed a mathematical model for regeneration of forestry resource and provided some realistic conservation mechanisms. The study of Agarwal et al. **[6]** and Chaudhary et al. **[7]** have shown that forestry resources deplete alarmingly due to the pressure of industrialization. The later study also has determined the effort required for optimal harvesting of forestry biomass. The studies of Mirsa et al. **[8]** and Lata et al. **[9]** deals with the effect of population pressure on forestry biomass. They have shown that the population pressure has significant influence on the existing forestry biomass and can result in depletion of forestry biomass. Another mathematical model has been proposed and analysed by Misra et al. **[10]** to observe the effect of technological effort on the conservation of forestry biomass. Dhar **[11]** studied a mathematical model that considers a two-patch habitat for forestry biomass resources.

2.1 Basic assumptions

We construct a deterministic mathematical model with the help of nonlinear differential equations with the following assumptions:

A1. Forestry biomass or basically commercially usable trees in biomass – based industries are divided into two groups, namely, premature trees, whose population density at any time t is denoted by the state variable $P(t)$ and the mature tress with population density at any time t given by $M(t)$. Therefore, we consider an age – structured model. The density of forestry biomass – based industries at any given time t is represented by the state variable $I(t)$.

A2. Premature trees grow logistically with an intrinsic growth rate r up to environment's carrying capacity k towards the premature trees.

A3. The interaction between the forestry biomass and forestry biomass – based industrialization is equivalent to the prey – predator type interactions as the later one feed off the former for survival. But the forestry – biomass can grow independently similar to a prey.

A4. The premature trees become mature at a rate α and move to the mature class. The new plantation rate of trees is denoted by the parameter λ. It should be noted that the new plantation depends on the existing number of premature trees.

A5. For industrial purpose only the mature trees are harvested with harvesting rate q and harvesting effort E. It is restricted to harvest the premature trees. We assume that the harvesting is density dependent, i.e., harvesting depends on the existing density of the mature trees. Moreover, it will tend to a constant value for large t. The parameter a is the half saturation constant.

A6. The rate of natural depletion of mature trees due to natural calamities is represented by the parameter μ_1.

A7. It is assumed that the growth of the biomass – based industries follow the modified Leslie – Gower function that was introduced by Leslie **[12,13]**. In context of the prey – predator interactions the construction of Leslie – Gower function is based on the assumption that reduction in the density of the predator population has a reciprocal effect on the per capita availability of its preferred prey. But according to the modified Leslie Gower functional response, if the preferred prey of the predator facing severe scarcity, then the predator can opt for other available alternative preys. However, it will have an adverse effect on the growth of predator, as it is unable to consume its favoured prey **[12,13,14,15]**. We consider modified Leslie – Gower function to represent the growth of the biomass – based industries depending on forestry biomass with β be the maximum reduction rate of that industries can attain sustainably, the parameter γ represent the maximum value for the reduced rate of industrialization, θ^{-1} is the average rate of governmental protection towards the sustainable conservation of forestry – biomass. The objective is to incorporate the fact that biomass – based industries can survive for a time period, depending on some alternative resources or governmental support, when their primary forestry biomass resource is absent due to some governmental conservation policies or some natural calamities. This assumption very important for sustainability of the forestry biomass – based industrialization.

A8. The forestry biomass – based industries are shut down or decrease in absence of forestry – resource at a rate μ_2.

A9. All the parameters are positive.

The schema diagram based on the above assumptions depicting the interactions between the state variables is given in Fig. 1.

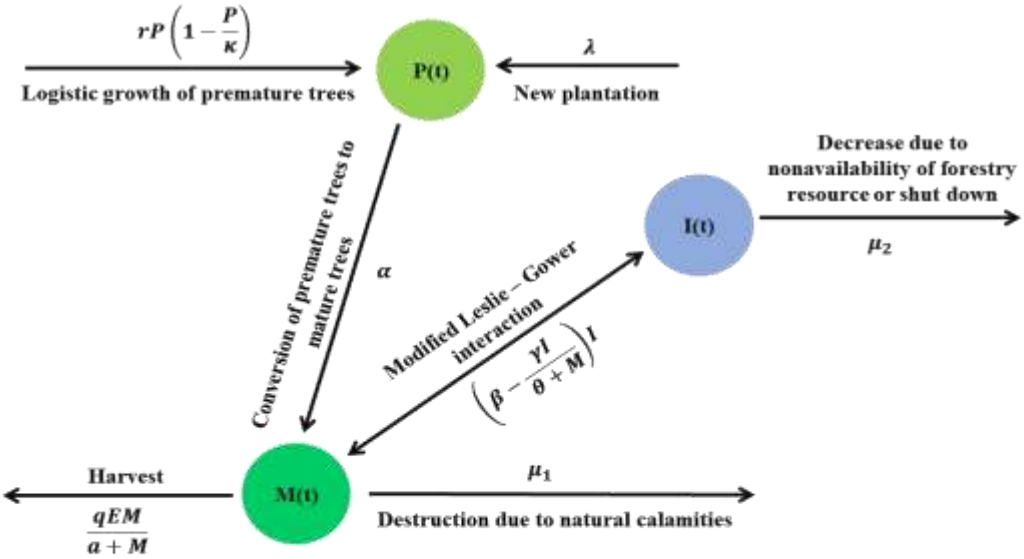

Figure 1. *Schema diagram of the interactions between the state variables according to the assumptions stated in A1 – A9.*

In view of the assumptions A1 – A9 and the schema diagram given in Fig. 1 we formulate the following model system (1):

$$\left.\begin{aligned} \frac{dP}{dt} &= rP\left(1-\frac{P}{\kappa}\right) - \alpha P + \lambda P, \\ \frac{dM}{dt} &= \alpha P - \frac{qEM}{a+M} - \mu_1 M, \\ \frac{dI}{dt} &= \left(\beta - \frac{\gamma I}{\theta + M}\right) I - \mu_2 I. \end{aligned}\right\} \tag{1}$$

Initial Condition: $P(0) > 0, M(0) > 0, I(0) > 0.$ (2)

3 Analytical results

In this section we derive various conditions such as positivity, boundedness of the solutions and permanence of the system (1).

3.1 Positivity and boundedness of the solutions

Proposition 3.1 *All the solutions of the model system (1) are positively invariant and ultimately bounded in the region* Γ *for sufficiently large* t *where* Γ *is given by:*

$$\Gamma = \left\{(P,M,I) \in \mathbb{R}_+^3 :\ 0 < P(t) \le \frac{\kappa(r+\lambda)}{r}, 0 <\ M(t) \le \frac{\alpha\kappa(r+\lambda)}{r\mu_1},) < I(t) \le \frac{\beta[\theta r\mu_1 + \alpha\kappa(r+\lambda)]}{\gamma r\mu_1}\right\}.$$

Proof. First, we show that all the solutions of the system (1) starting with initial conditions (2) are positive, using a lemma proposed by **Nagumo [16].**

Lemma 3.1 *Consider a system* $\dot{X} = F(X)$ *where* $F(X) = [F_1(X), F_2(X), \dots, F_n(X)]$, $X \in \mathbb{R}^n$ *with initial condition* $X(0) = X_0 \in \mathbb{R}^n$. *If for* $X_i = 0, i = 1,2,\dots,n$ *we get* $F_i(X)|_{X_i=0} \geq 0$, *then any solution of* $\dot{X} = F(X)$ *with given initial condition, say,* $X(t) = X(t; X_0)$ *will be positive i.e.,* $X(t) \in \mathbb{R}^n_+$.

From model system (1), one can easily see that $\frac{dP}{dt} = 0, \frac{dM}{dt} = 0, \frac{dI}{dt} = 0$ when $P = M = I = 0$. Hence following Lemma 3.1 we conclude that all solutions of model system (1) is positively invariant in $\mathbb{R}^3_+$.

Again, to establish the boundedness of the solutions of (1) first we state the following lemma proposed by **Chen [17]**:

Lemma 3.2 *If* $a, b > 0$ *and* $\frac{dX}{dt} \leq (or \geq) X(t)(a - bX(t))$ *with* $X(0) > 0$, *then* $\limsup_{t\to\infty} X(t) \leq \frac{a}{b}\left(or\ \liminf_{t\to\infty} X(t) \geq \frac{a}{b}\right)$.

From the first equation of (1) following Lemma 3.2 and (2) we obtain

$$\frac{dP}{dt} \leq rP\left(1 - \frac{P}{\kappa}\right) + \lambda P = P\left([r+\lambda] - \left[\frac{r}{\kappa}\right]P\right)$$

$$\Rightarrow \limsup_{t\to\infty} P(t) \leq \frac{\kappa(r+\lambda)}{r}. \quad (3)$$

Again, using (3) the second equation of (1) yields

$$\frac{dM}{dt} \leq \frac{\alpha\kappa(r+\lambda)}{r} - \mu_1 M$$

$$\Rightarrow \limsup_{t\to\infty} M(t) \leq \frac{\alpha\kappa(r+\lambda)}{r\mu_1}. \quad (4)$$

Finally, from the third equation of (1) and applying (4) we get,

$$\frac{dI}{dt} \leq \left(\beta - \frac{\gamma I}{\theta + M}\right) I \leq \left(\beta - \left[\frac{\gamma r\mu_1}{\theta r\mu_1 + \alpha\kappa(r+\lambda)}\right] I\right) I.$$

Further using Lemma 3.2 and (2) we calculate,

$$\limsup_{t\to\infty} I(t) \leq \frac{\beta[\theta r\mu_1 + \alpha\kappa(r+\lambda)]}{\gamma r\mu_1}. \quad (5)$$

Hence the proposition is proved.

3.2 Permanence of the system

According the definition of permanence prescribed by Pal et. al. **[18]** the model system (1) along with the initial conditions given in (2) will be permanent if there exist positive constants K_1 and K_2 satisfying $0 < K_1 < K_2$ such that each positive solution $(P(t, P_0, M_0, I_0),\ M(t, P_0, M_0, I_0),\ I(t, P_0, M_0, I_0))$ of (1) where (P_0, M_0, I_0) conforms the initial condition stated in (2), satisfies

$$min\left\{\liminf_{t\to\infty} P(t, P_0, M_0, I_0), \liminf_{t\to\infty} M(t, P_0, M_0, I_0), \liminf_{t\to\infty} I(t, P_0, M_0, I_0)\right\} \geq K_1 \quad (6)$$

and

$$max\left\{\limsup_{t\to\infty} P(t, P_0, M_0, I_0), \limsup_{t\to\infty} M(t, P_0, M_0, I_0), \limsup_{t\to\infty} I(t, P_0, M_0, I_0)\right\} \leq K_2. \quad (7)$$

Proposition 3.2 *The model system (1) with the initial conditions (2) is permanent if all the solutions originate within the interior of the region Γ as specified in Proposition 3.1 and $r > \alpha, E < \frac{\alpha\kappa(r-\alpha)}{rq}$ and $\beta > \mu_2$.*

Proof. For any solution of (1) originating within Γ we obtain:

$$\limsup_{t\to\infty} P(t) \leq \frac{\kappa(r+\lambda)}{r}, \limsup_{t\to\infty} M(t) \leq \frac{\alpha\kappa(r+\lambda)}{r\mu_1},\ \limsup_{t\to\infty} I(t) \leq \frac{\beta[\theta r\mu_1 + \alpha\kappa(r+\lambda)]}{\gamma r\mu_1}.$$

Define, $K_2 = max\left\{\frac{\kappa(r+\lambda)}{r}, \frac{\alpha\kappa(r+\lambda)}{r\mu_1}, \frac{\beta[\theta r\mu_1+\alpha\kappa(r+\lambda)]}{\gamma r\mu_1}\right\}$. Then

$$0 < max\left\{\limsup_{t\to\infty} P(t), \limsup_{t\to\infty} M(t), \limsup_{t\to\infty} I(t)\right\} \leq K_2.$$

Hence the condition of permanence as given in (7) is satisfied for the solutions of (1) within Γ with respect to any initial condition satisfying (2). Now, applying Lemma 3.1 on the first equation of (1) one can calculate that

$$\frac{dP}{dt} \geq rP\left(1 - \frac{P}{\kappa}\right) - \alpha P = P\left([r-\alpha] - \frac{r}{\kappa}P\right) \Rightarrow \liminf_{t\to\infty} P(t) \geq \frac{\kappa(r-\alpha)}{r} = P_{inf}. \quad (8)$$

Applying (8) in the second equation of (1) can be expressed as:

$$\frac{dM}{dt} = \alpha P - \frac{qEM}{a+M} - \mu_1 M \geq \frac{\alpha\kappa(r-\alpha)}{r} - qE - \mu_1 M$$
$$\Rightarrow \liminf_{t\to\infty} M(t) \geq \frac{\alpha\kappa(r-\alpha) - rqE}{r\mu_1} = M_{inf}. \tag{9}$$

Next, we use (9) in the third equation of (1) and hence applying Lemma 3.2 we get:

$$\frac{dI}{dt} = \left(\beta - \frac{\gamma I}{\theta + M}\right) I - \mu_2 I \geq \left[(\beta - \mu_2) - \frac{r\mu_1\gamma}{r\mu_1\theta + \alpha\kappa(r-\alpha) - rqE} I\right]$$
$$\Rightarrow \liminf_{t\to\infty} I(t) \geq \frac{(\beta-\mu_2)[r\mu_1\theta + \alpha\kappa(r-\alpha) - rqE]}{r\mu_1\gamma} = I_{inf}. \tag{10}$$

Clearly from (8), (9) and (10) we get, if $r > \alpha, E < \frac{\alpha\kappa(r-\alpha)}{rq}$ and $\beta > \mu_2$ then $P_{inf}, M_{inf}, I_{inf}$ are positive. We define $K_1 = min\{P_{inf}, M_{inf}, I_{inf}\} > 0$ and then $K_1 < K_2$ with

$$min\left\{\liminf_{t\to\infty} P(t), \liminf_{t\to\infty} M(t), \liminf_{t\to\infty} I(t)\right\} \geq K_1.$$

Eventually the condition of permanence as given in (6) is satisfied for the solutions of (1) within Γ with respect to any initial condition satisfying (2). Hence the proposition is proved.

3.3 Equilibrium points and their existence conditions

Model system (1) has four equilibrium points.

(A) The trivial equilibrium point $E_0(0,0,0)$ that always exists.

(B) The industry – free equilibrium $E_1(\bar{P}, \bar{M}, 0)$ where $\bar{P} = \frac{\kappa}{r}[(r+\lambda) - \alpha]$ and $\bar{M}$ is given by the positive root of the quadratic equation $A_1\bar{M}^2 + A_2\bar{M} + A_3 = 0$ where $A_1 = \mu_1 > 0, A_2 = a\mu_1 + qE - \alpha\bar{P}, A_3 = -\alpha a\bar{P} < 0$. Clearly this quadratic equation will have exactly one positive root $\bar{M} = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$ provided $A_2^2 - 4A_1A_3 > 0$.

Therefore, the equilibrium E_1 will exist if $(i)\ \lambda > \lambda^*$ and $(ii)\ \Delta_1(\lambda) > 0$ (11)

where $\lambda^* = \alpha - r,\ \Delta_1(\lambda) = A_2^2 > 4A_1A_3$.

(C) The biomass – free equilibrium $E_2(0,0,\hat{I})$ where $\hat{I} = \frac{\theta(\beta-\mu_2)}{\gamma}$.

It is easy to check that the equilibrium E_2 exists if $\beta > \beta^*$, where $\beta^* = \mu_2$. (12)

(D) The coexistence equilibrium $E^*(P^*, M^*, I^*)$

where $P^* = \frac{\kappa}{r}[(r+\lambda) - \alpha], M^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}$, $I^* = \frac{(\theta + M^*)(\beta - \mu_2)}{\gamma}$ with $B_1 = \mu_1 > 0, B_2 = a\mu_1 + qE - \alpha P^*, B_3 = -\alpha a P^* < 0$.

The equilibrium E^* will exist if $(i)\ \lambda > \lambda^*$, $(ii)\ \beta > \beta^*$, $(iii)\ \Delta_2(\lambda) > 0$, (13)

where $\Delta_2(\lambda) = B_2^2 > 4B_1B_3$ and the expressions of λ^* and β^* are given in (11) and (12).

3.4 Local stability analysis of the equilibrium points

In this section we derive the stability conditions of different equilibrium points of system (1) using the method of linearization. The Jacobian of the model system (1) is evaluated to be

$$V(P, M, I) = \begin{pmatrix} r - \frac{2rP}{\kappa} - \alpha + \lambda & 0 & 0 \\ \alpha & -\frac{qEa}{(a+M)^2} - \mu_1 & 0 \\ 0 & \frac{\gamma I^2}{(\theta + M)^2} & \beta - \frac{2\gamma I}{\theta + M} - \mu_2 \end{pmatrix}. \quad (14)$$

Proposition 3.3 *The trivial equilibrium $E_0(0,0,0)$ always exists and is locally asymptotically stable if $(i)\ \lambda < \lambda^*$ and $(ii)\ \beta < \beta^*$ where the expressions of λ^* and β^* are given in (11) and (12) respectively.*

Proof. The characteristic equation of the Jacobian matrix evaluated at E_0 is given by

$$[\xi - (r + \lambda - \alpha)] \left[\xi + \left\{\frac{qEa}{(a+M)^2} + \mu_1\right\}\right] [\xi - (\beta - \mu_2)] = 0.$$

Therefore, the eigenvalues of the Jacobian matrix evaluated at E_0 are $\xi_1 = (r+\lambda) - \alpha, \xi_2 = -\left\{\frac{qEa}{(a+M)^2} + \mu_1\right\} < 0, \xi_3 = \beta - \mu_2$. Thus E_0 will be locally asymptotically stable when $\xi_1 < 0$ and $\xi_3 < 0$ that is when if $(i)\ r + \lambda < \alpha \Rightarrow \lambda < \lambda^*$ and $(ii)\ \beta < \mu_2 \Rightarrow \beta < \beta^*$ where λ^*, β^* are given in (11) and (12) respectively. Hence the proposition is proved.

Remark 3.1 From (11), (12) and (13) it is clear that if E_0 is locally asymptotically stable, then E_1, E_2 and E^* will not exist.

Proposition 3.4 *The industry–free equilibrium $E_1(\bar{P}, \bar{M}, 0)$ if exists following the conditions given in (11), will be locally asymptotically stable if $\beta < \beta^*$ where β^* is given by (12).*

Proof. The characteristic equation of the Jacobian matrix (14) evaluated at E_1 is given by:

$$\left(\xi + \frac{r\bar{P}}{\kappa}\right)\left(\xi + \left\{\frac{qEa}{(a+\bar{M})^2} + \mu_1\right\}\right)(\xi - \{\beta - \mu_2\}) = 0. \qquad (15)$$

The eigenvalues are evaluated to be $\xi_1 = -\frac{r\bar{P}}{\kappa} < 0, \xi_2 = -\left(\frac{qEa}{(a+\bar{M})^2} + \mu_1\right) < 0$ and $\xi_3 = \beta - \mu_2$. All the eigenvalues will have a negative real part if $\beta - \mu_2 < 0 \Rightarrow \beta < \beta^*$ where β^* is given by (12). Therefore, the industry–free equilibrium E_1, if exists following (11), will be locally asymptotically stable if $\beta < \beta^*$. Hence the proposition is proved.

Remark 3.2 From (12), (13), and Proposition 3.3, it is evident that if E_1 is locally asymptotically stable, then E_2 & E^* will not exist and E_0 will be unstable.

Proposition 3.5 *The biomass–free equilibrium $E_2(0,0,\hat{I})$ if exists following the conditions given in (12), will be locally asymptotically stable if $\lambda < \lambda^*$ where λ^* is given by (11).*

Proof. The characteristic equation of the Jacobian matrix (14) evaluated at E_2 is given by:

$$(\xi - \{r + \lambda - \alpha\})\left(\xi + \left\{\frac{qE}{a} + \mu_1\right\}\right)(\xi + \{\beta - \mu_2\}) = 0. \qquad (16)$$

The eigenvalues are evaluated to be $\xi_1 = r + \lambda - \alpha, \xi_2 = -\left(\frac{qE}{a} + \mu_1\right) < 0$ and $\xi_3 = -(\beta - \mu_2) < 0$ (following the existence condition of E_2 that is given in (12)). All the eigenvalues will have a negative real part if $r + \lambda < \alpha \Rightarrow \lambda < \lambda^*$ where λ^* is given by (11). Therefore, the biomass–free equilibrium E_2, if exists following (12), will be locally asymptotically stable if $\lambda < \lambda^*$. Hence the proposition is proved.

Remark 3.3 From (11), (13) and Proposition 3.3 we assert that if E_2 is locally asymptotically stable, then E_1 & E^* will not exist and E_0 will be unstable.

Proposition 3.6 *The coexistence equilibrium $E^*(P^*, M^*, I^*)$ if exists following the conditions specified in (13) will be locally asymptotically stable.*

Proof. The characteristic equation of the Jacobian matrix (14) evaluated at E^* is given by:

$$\left(\xi+\frac{rP^*}{\kappa}\right)\left(\xi+\left\{\frac{qEa}{(a+M^*)^2}+\mu_1\right\}\right)\left(\xi+\frac{\gamma I^*}{\theta+M^*}\right)=0. \qquad (16a)$$

The eigenvalues are evaluated to be $\xi_1=-\frac{rP^*}{\kappa}<0, \xi_2=-\left(\frac{qEa}{(a+M^*)^2}+\mu_1\right)<0$ and $\xi_3=-\frac{\gamma I^*}{\theta+M^*}<0$. Hence all the eigen values have negative real part. Therefore, the coexistence equilibrium E^*, if exists following (13), will be locally asymptotically stable. Hence the proposition is proved.

3.5 Analysis of the bionomical equilibrium

We now study the bionomical equilibrium associated with the model system (1). Bionomical equilibrium is defined as the level at which the total revenue (TR) generated by selling the harvested trees or biomass in an economic equilibrium stage is equal to the total cost (TC) that has been incurred to harvest the biomass or mature trees in our case. In fact, bionomic equilibrium is achieved when the economic rent is completely dissipated.

Let, p be the selling price per unit of biomass and c is the cost incurred to harvest per unit of biomass. Then the net economic revenue generated at any time t is given by

$$\Pi(P,M,I,E,t)=\left(\frac{qpM}{a+M}-c\right)E. \qquad (17)$$

We derive the bionomic equilibrium $\mathcal{E}_{BN}(P_\infty,M_\infty,I_\infty,E_\infty)$ by solving the following equations:

$$\left.\begin{aligned} &rP\left(1-\frac{P}{\kappa}\right)-\alpha P+\lambda P=0,\\ &\alpha P-\frac{qEM}{a+M}-\mu_1 M=0,\\ &\left(\beta-\frac{\gamma I}{\theta+M}\right)I-\mu_2 I=0.\\ &\left(\frac{qpM}{a+M}-c\right)E=0. \end{aligned}\right\} \qquad (18)$$

Hence,

$$P_\infty=\frac{\kappa}{r}[(r+\lambda)-\alpha]\,, M_\infty=\frac{ac}{pq-c}\,, I_\infty=\frac{[(\theta pq+ac)-\theta c][\beta-\mu_2]}{\gamma(pq-c)},$$
$$E_\infty=\frac{\alpha(a+M_\infty)}{q}\left[\frac{P_\infty}{M_\infty}-\frac{\mu_1}{\alpha}\right]. \qquad (19)$$

Therefore, we obtain the following proposition:

Proposition 3.6 *The model system (1) will have a feasible bionomic equilibrium* $\mathcal{E}_{BN}(P_\infty, M_\infty, I_\infty, E_\infty)$ *if* $(i)\ \lambda > \lambda^*$, $(ii)\ q > q^*$, $(iii)\ \beta > \beta^*$, $(iv)\ \frac{P_\infty}{M_\infty} > \frac{\mu_1}{\alpha}$. *The expressions of* $P_\infty, M_\infty, I_\infty, E_\infty$ *are given in (19),* $q^* = \frac{c}{p}$. *The expressions of* λ^* *and* β^* *are given in (11) and (12) respectively.*

3.6 Optimal harvesting policy

In this section we analyse the optimal harvesting policy of a renewable resource in presence of industrialization related harvesting. The main objective of this study is to determine a harvesting policy so that the biomass-based industries can survive sustainably without depleting or over exploiting the biomass. This analysis is very much essential for sustainable socio-economic development of any population dependent on biomass-based industries. Let us denote the present valuation of a continuous time stream of revenue generated by harvesting biomass as Q. Then Q which is the objective functional, is given by:

$$Q = \int_0^\infty e^{-\delta t}\left(\frac{qpM(t)}{a+M(t)} - c\right)E(t)dt. \tag{20}$$

Here δ is considered to be the instantaneous rate of annual discount. We use *Pontryajin's Maximum Principle* to maximize Q subject to the state equations given in (1) and control constraints $0 \le E \le E_{max}$. The associated Hamiltonian is given by:

$$H = e^{-\delta t}\left[\frac{qpM}{a+M} - c\right]E + \lambda_1\left[rP\left(1 - \frac{P}{\kappa}\right) - \alpha P + \lambda P\right] + \lambda_2\left[\alpha P - \frac{qEM}{a+M} - \mu_1 M\right]$$
$$+\lambda_3\left[\left(\beta - \frac{\gamma I}{\theta + M}\right)I - \mu_2 I\right]. \tag{21}$$

Here $\lambda_1, \lambda_2, \lambda_3$ are the adjoint variables. Next, we define the switching function $\sigma(t)$ as:

$$\sigma(t) = e^{-\delta t}\left[\frac{qpM}{a+M} - c\right] - \frac{\lambda_2 qM}{a+M}. \tag{22}$$

According to the $Pontryajin's\ Maximum\ Principle$, the optimal control $E(t)$ that maximizes H must satisfy the following conditions:

(i) $E = E_{max}$ when $\sigma(t) > 0 \Rightarrow \lambda_2 e^{\delta t} < p - \frac{c(a+M)}{qM}$. (23)

(ii) $E = 0$ when $\sigma(t) < 0 \Rightarrow \lambda_2 e^{\delta t} > p - \frac{c(a+M)}{qM}$. (24)

It should be noted that the usual shadow price is given by $\lambda_2 e^{\delta t}$ and the net economic revenue generated per unit harvest is $p - \frac{c(a+M)}{qM}$. It implies that if the shadow price is less than the total generated economic revenue per unit harvest, then $E = E_{max}$. Whereas, if the shadow price is higher than the net economic revenue generated per unit harvest, then $E = 0$. Moreover, if the shadow price becomes equal to the net economic revenue per unit harvest, then $\sigma(t) = 0$ and in this case H becomes independent of the control variable E, that is, $\frac{\partial H}{\partial E} = 0$. This condition is necessary for a singular control $\mathcal{E}^*$ to exist and to become optimal over the interval $(0, E_{max})$. Hence the optimal harvesting policy is defined as:

$$E(t) \quad \begin{cases} = E_{max}\ , & \sigma(t) > 0 \\ = \mathcal{E}^*, & \sigma(t) = 0. \\ = 0\ , & \sigma(t) < 0. \end{cases} \tag{25}$$

Now when $\sigma(t) = 0$ then one can see that

$$\lambda_2 = e^{-\delta t}\left[p - \frac{c(a+M)}{qM}\right]. \tag{26}$$

According to $Pontryajin's Maximum\ Principle$ in order to determine the singular control the adjoint variables must satisfy:

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial P}\ , \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial M}\ , \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I}\ , \tag{27}$$

under the following conditions for optimal solution:

$$rP\left(1 - \frac{P}{\kappa}\right) - \alpha P + \lambda P = 0,$$

$$\alpha P - \frac{qEM}{a+M} - \mu_1 M = 0, \tag{27a}$$
$$\left(\beta - \frac{\gamma I}{\theta + M}\right) I - \mu_2 I = 0.$$

Hence the equations described in (27) with the help of (27a) can be written as:

$$\frac{d\lambda_1}{dt} = \frac{\lambda_1 rP}{\kappa} - \alpha e^{-\delta t}\left[p - \frac{c(a+M)}{qM}\right], \tag{28}$$

$$\frac{d\lambda_2}{dt} = -\frac{e^{-\delta t} pqaE}{(a+M)^2} + \lambda_2\left[\frac{qEa}{(a+M)^2} + \mu_1\right] - \lambda_3 \frac{\gamma I^2}{(\theta+M)^2}, \tag{29}$$

$$\frac{d\lambda_3}{dt} = \left(\frac{\gamma I}{\theta + M}\right)\lambda_3. \tag{30}$$

Solving (28) we get:

$$\lambda_1 = \frac{\alpha\kappa e^{-\delta t}}{\delta\kappa + rP}\left[p - \frac{c(a+M)}{qM}\right] + K_3 e^{\frac{rPt}{\kappa}}. \tag{31}$$

Here K_3 is an arbitrary constant. It should be noted that when $t \to \infty$, then the shadow price $\lambda_1 e^{\delta t}$ must be bounded. Hence, we consider $K_3 = 0$ and thus from (31) we get at optimal level

$$\lambda_1 = \frac{\alpha\kappa e^{-\delta t}}{\delta\kappa + rP}\left[p - \frac{c(a+M)}{qM}\right]. \tag{32}$$

Again solving (30) one can obtain:

$$\lambda_3 = K_4 e^{\left(\frac{\gamma I}{\theta + M}\right)t}. \tag{33}$$

Here K_4 is an arbitrary constant. Noting that λ_3 is bounded as $t \to \infty$ we consider the constant $K_4 = 0$. Hence, from (33) we obtain $\lambda_3 = 0$ at optimal harvesting level. Now substituting $\lambda_3 = 0$ in (29) and solving for λ_2 one can get:

$$\lambda_2 = -e^{\delta t}\frac{pqaE}{(a+M)^2} + K_5 e^{\left[\frac{qEa}{(a+M)^2} + \mu_1\right]t}. \tag{34}$$

Here K_5 is an arbitrary constant. As λ_2 is bounded as $t \to \infty$ we set $K_5 = 0$. Hence from (34) we obtain:

$$\lambda_2 = -e^{\delta t}\frac{pqaE}{(a+M)^2}. \tag{35}$$

Equating (26) and (35) one can easily calculate the optimal harvesting effort:

$$E_{optimal} = \mathcal{E}^* = \frac{(a+M)^2}{pqa}\left[\frac{c(a+M)}{qM} - p\right]. \tag{36}$$

4 Numerical simulations

In this section we perform numerical simulations of the model system (1) with the help of MATLAB2015a software. First we estimate the parameter values using Root Mean Square Error (RMSE) technique by fitting the year wise real time data of production of round wood timber, in $million\; m^3\; RWE$ (Round Wood Equivalent) unit, from Indian forestry resources during 2010 – 2020, obtained in **[19]** with the state variable $M(t)$, of system (1) i.e., the mature trees that are allowed to cut for use in the biomass based industries. The fitted $M(t)$ is shown in Fig. 2. It can be clearly seen that our model is a good fit with the real time data for the following parameter set:

$$r = 0.1,\quad \kappa = 10,\quad \alpha = 0.5,\quad q = 1.1,\quad E = 5.5,\quad a = 5,\; \mu_1 = 0.1, \tag{37}$$
$$\gamma = 2.2;\quad \theta = 2.3,\quad \mu_2 = 0.1,\quad \lambda = 0.45,\quad \beta = 0.5.$$

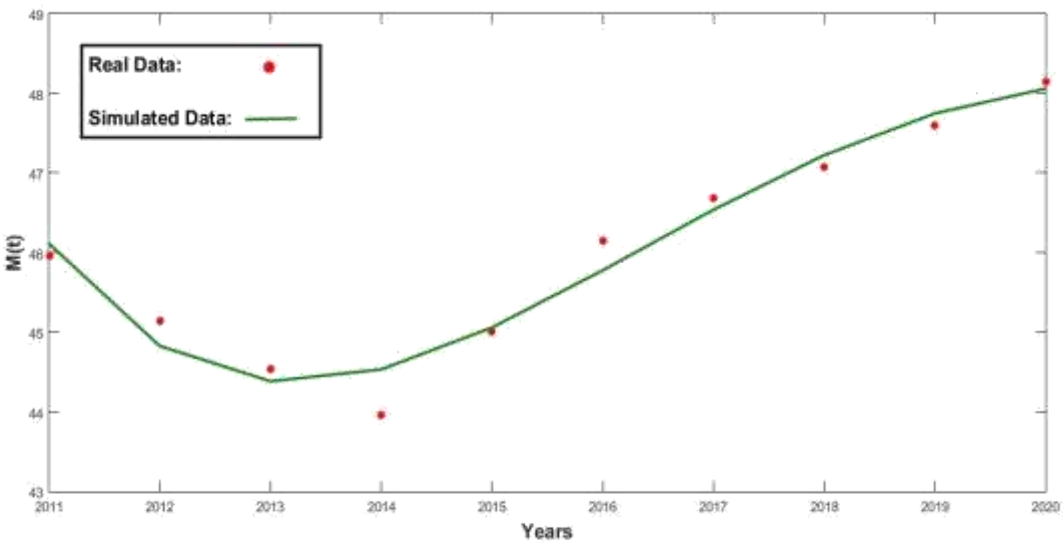

Figure 2. *Data fitting of the state variable $M(t)$, of system (1) i.e., the mature trees that are allowed to cut for use in the biomass-based industries, against real time data of production of round wood timber, in million m^3 RWE (Round Wood Equivalent) unit, from Indian forestry resources during 2010 – 2020, obtained in* ***[19]*** *using Root Mean Square Error (RMSE) technique.*

To identify the sensitive parameters of system (1) we have performed PRCC (Partial Rank Correlation Coefficient) analysis with < 0.0001 . The sensitivity diagram is given in Fig. 3.

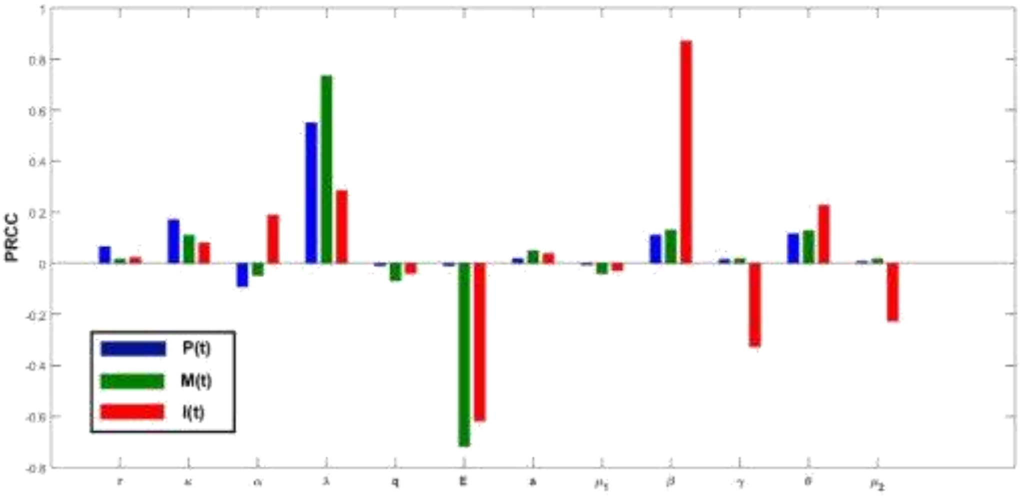

Figure 3. *PRCC (Partial Rank Correlation Coefficient) analysis with $p < 0.0001$ of the parameters of the system (1).*

From Fig. 3 it can be easily seen that λ, i.e., the parameter measuring the rate of new plantation, E i.e., rate of harvesting of mature trees and β i.e., the maximum reduction rate of that industries can attain sustainably are the most sensitive parameters. We therefore, verify the effects of these parameters on system (1). For the parameter values given in (37) using (11) & (12), one can calculate $\lambda^* = 0.4$ and $\beta^* = 0.1$. Moreover, using (13) the value of $\Delta_2(\lambda) = 21.4025 > 0$. Hence the existence conditions of E^* as listed in (13) are satisfied by the parameter values given in (37). Again, following Proposition 3.6 the coexistence equilibrium E^* is stable. The time evolution of system (1) for parameter values (37) is depicted in Fig. 4. It shows all the solution trajectories converge to locally asymptotically stable $E^*(5\,, 2.8814\,, 0.9421)$.
Again for $\lambda = 0.45(> \lambda^* = 0.4)$ one can calculate $\Delta_1(\lambda) = 21.4025 > 0$. Therefore, following (11) the industry – free equilibrium $E_1(5\,,\ 2.8814,\ 0)$ exists. We fix $\beta = 0.05$ to satisfy $\beta < \beta^* = 0.1$ keeping all parameters as in (37). Therefore, according to Proposition 3.4 the equilibrium E_1 is locally asymptotically stable. This case is represented in Fig. 5. It shows for these chosen parameter values all trajectories converge to stable E_1 i.e, $P(t), M(t)$ persists in stable condition in long run. But $I(t)$ tends to zero for sufficiently large t.

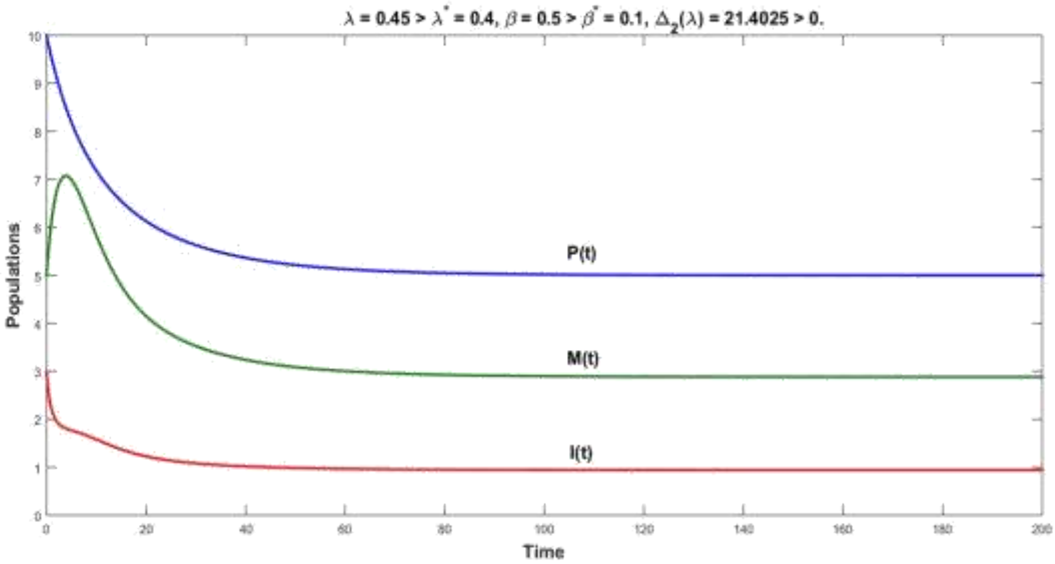

Figure 4. *Time evolution of the populations of model system (1). The parameter* $\lambda = 0.45, \beta = 0.5$ *and other parameters are as in (37).*

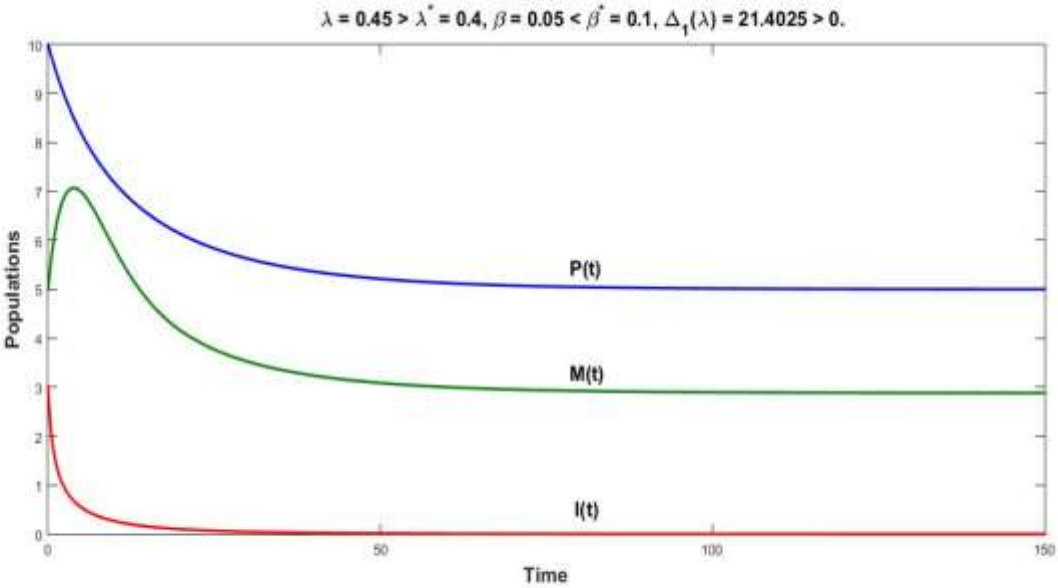

Figure 5. *Time evolution of the populations of model system (1). The parameter* $\lambda = 0.45, \beta = 0.05$ *and other parameters are as in (37).*

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Next, we consider $\lambda = 0.35 < \lambda^* = 0.4$ and $\beta = 0.5 > \beta^* = 0.1$ keeping other parameters same as in (37). Following (12) and Proposition 3.5, in this case, the biomass – free equilibrium E_2 is locally asymptotically stable and all system trajectories converge to E_2 $(0\ ,\ 0,\ 0.4182)$ for sufficiently large t. This case is shown in Fig. 6. It should be noted that, in this case, even though forestry biomass is completely depleted, the forestry biomass – based industries survive, though in a very low level. The reason for that, is the alternative resources that the forestry biomass – based industries can avail when the primal resource of forestry – biomass is absent or scarce.

This phenomenon is due to consideration of modified Leslie – Gower interaction between industry ready mature trees and forestry biomass-based industries.

Next, we consider $\lambda = 0.35 < \lambda^* = 0.4$ and $\beta = 0.5 > \beta^* = 0.1$ keeping other parameters same as in (37). Following (12) and Proposition 3.5, in this case, the biomass – free equilibrium E_2 is locally asymptotically stable and all system trajectories converge to $E_2\ (0\ ,\ 0,\ 0.4182)$ for sufficiently large t. This case is shown in Fig. 6. It should be noted that, in this case, even though forestry biomass is completely depleted, the forestry biomass – based industries survive, though in a very low level. The reason for that, is the alternative resources that the forestry biomass – based industries can avail when the primal resource of forestry – biomass is absent or scarce. This phenomenon is due to consideration of modified Leslie – Gower interaction between industry ready mature trees and forestry biomass-based industries.

Finally, we set $\lambda = 0.35 < \lambda^* = 0.4$ and $\beta = 0.05 < \beta^* = 0.1$ keeping other parameters same as in (37). According to Proposition 3.3 the biomass – free and industry – free equilibrium E_0 is locally asymptotically stable for this choice of parameters. The time series solution of system (1) in this case is depicted in Fig. 7 that shows system trajectories converge to stable E_0 (0,0,0) for sufficiently large t.

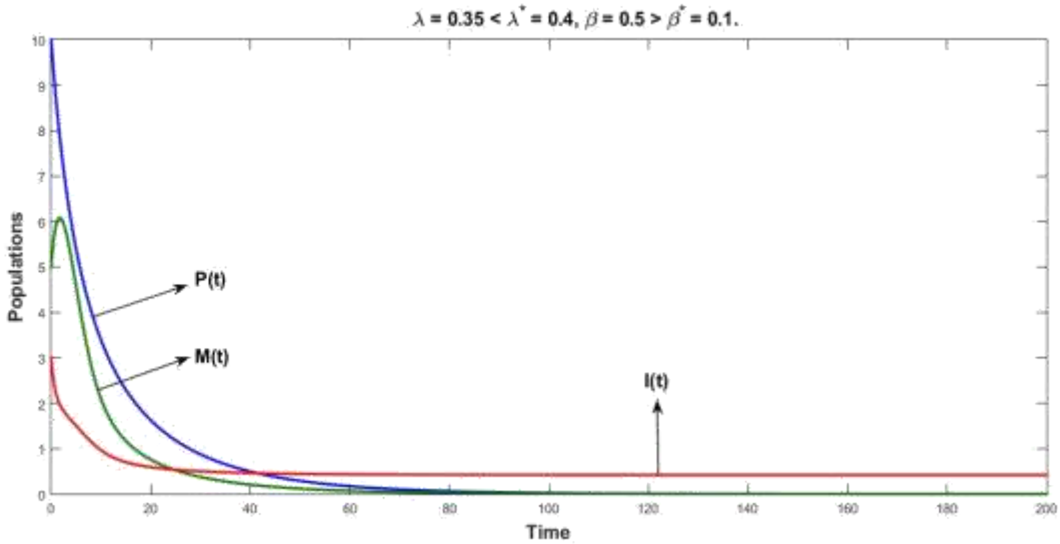

Figure 6. *Time evolution of the populations of model system (1). The parameter* $\lambda = 0.35, \beta = 0.5$ *and other parameters are as in (37).*

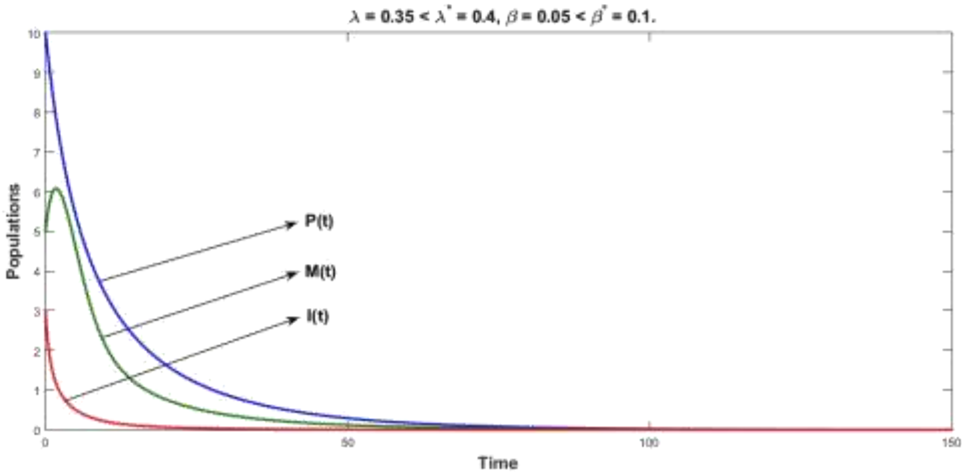

Figure 7. *Time evolution of the populations of model system (1). The parameter* $\lambda = 0.35, \beta = 0.05$ *and other parameters are as in (37).*

To check the effect of harvesting effort we plot the solution trajectories of the state variable $I(t)$ representing industrialization for different values of E in Fig. 8. We have taken $E = 0.5, 2.5$ and 4.5. The value of $\lambda^* = 0.4$ and $\beta^* = 0.1$ for each of these chosen values of E. Moreover, $\Delta_2(\lambda) = 7.1025, 5.5625$ and 13.7025 respectively for $E = 0.5, 2.5$ and 4.5. Therefore, following (13) and Proposition 3.6 the coexistence equilibrium E^* exists and stable for each of these chosen values of E. From Fig. 8 one can see that for increasing effort the density of mature trees (Fig. 8a) and biomass-based industries (Fig. 8b) decreases. Due to higher effort, mature trees are cut at a higher rate and we can see that initially the density of the industries increase for a short period of time. But increasing felling rate of mature trees is not reciprocal to the growth of premature trees. Therefore, after sufficient time, there is an disparity between the number of trees becoming mature from prematurity and the number of mature trees cut for industrialization. Gradually the later one overcomes the former. Therefore, after some time the industries depending on mature tree biomass decrease due to unavailability of mature trees.

However, the population density of industries does not tend to zero, due to availability of alternative resources.

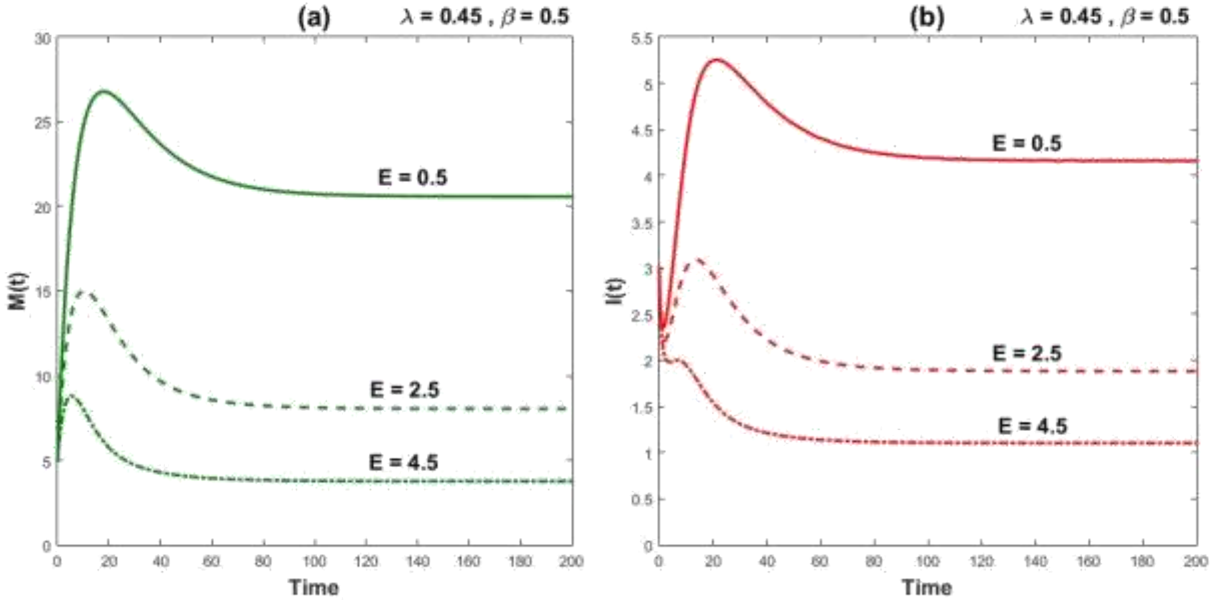

Figure 8. *Time evolution of the $M(t)$ and $I(t)$ populations of model system (1) for different values of $E = 0.5, 2.5, 4.5$. The parameter $\lambda = 0.45, \beta = 0.5,$ and other parameters are as in (37).*

According to the above analysis we assert that just by increasing effort we will not be able to sustain forestry biomass – based industries. Therefore, to optimize industrialization we have to apply optimal harvesting strategy. For that purpose, we consider $p = 0.1$ and $c = 0.01$ where p be the selling price per unit biomass and c is the cost incurred to harvest per unit biomass. Using (36) we have calculated that $E_{optimal} = \mathcal{E}^* = 2.0522$ for the values of the parameter as specified in (37). In this case $\lambda = 0.45 > \lambda^* = 0.4, \beta = 0.5 > \beta^* = 0.1,$ $\Delta_2(\lambda) = 9.07 > 0$. Hence, according to Proposition 3.6 the coexistence equilibrium E^* is locally asymptotically stable. Using these parameter values, in Fig. 9 we have fitted the simulated data with respect to the state variable $M(t)$ with the estimated requirement data of round wood timber, in $million\, m^3\, RWE$ (Round Wood Equivalent) unit, in India during 2021 – 2030 that has been obtained from **[19]**. It shows when $E = \mathcal{E}^* = 3.6522$ then our simulated data provides a good fit to the estimated data in 95% confidence interval. Therefore, our analysis shows that if the harvesting of mature trees is performed with optimal harvesting effort $E = E_{optimal} = \mathcal{E}^* = 2.0522$, then the estimated requirement of round wood timber in India during 2021 – 2030 can be fulfilled.

5 Summary and discussions

In this article we have proposed a deterministic mathematical model using nonlinear differential equations to determine sustainable harvesting strategy of forestry biomass for forestry dependent industrialization. The main objective of this study is to analyse a dynamical mathematical model to determine strategies so that forestry biomass and forestry biomass-based industries both can sustain for socioeconomic development of any society. We know that forestry biomass is constantly being depleted due to manmade issues such as harvesting, increasing forestry-based industries, soil pollution etc. or natural calamities. Therefore, an optimal and adequate harvesting strategy should be prepared for sustainable development. In view of this, we have considered an age structured forestry biomass model that incorporates the interactions among pre-mature trees, mature trees and forestry biomass – based industries. We have assumed that for industrial purpose only mature trees can be harvested. Premature trees are restricted for use in industrial activities. Further, we have considered modified Leslie – Gower response function to incorporate the effect of alternative resources for the biomass – based industries in case the biomass resource for industries is scarce.

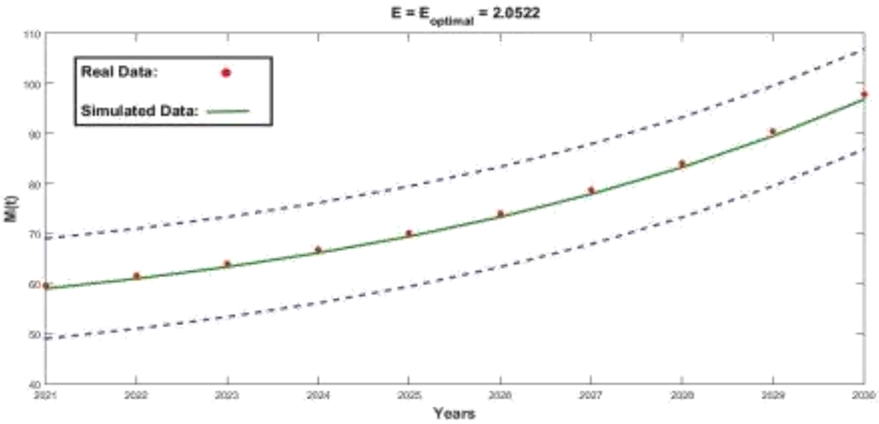

Figure 9. *Data fitting of the state variable $M(t)$, of system (1) i.e., the mature trees that are allowed to cut for use in the biomass-based industries, against the estimated requirement data of round wood timber, in $million\, m^3$ RWE (Round Wood Equivalent) unit, in Indian forestry resources during 2021 – 2030, obtained in **[19]** using Root Mean Square Error (RMSE) technique and 95% confidence interval (shown by the dotted lines).*

We have determined positivity, boundedness and permanence of the solutions of system (1). The local asymptotic stability conditions of the equilibrium points have been analysed using linearization technique. Through PRCC sensitivity analysis we have found that the parameters representing the rate of new plantations (λ), maximum reduction rate of that industries can attain (β) and the harvesting effort E are the most sensitive parameters and can significantly alter the system dynamics. We have shown that the coexistence equilibrium E^* will exist in stable mode if the rate of plantation of new trees λ is greater than a critical value λ^* and maximum reduction rate of that industries can attain i.e., β is higher than a threshold value β^* along with $\Delta_2(\lambda) > 0$. If $\lambda < \lambda^*, \beta > \beta^*$ then due to low number of new plantations, forestry biomass dies and system converges to stable biomass – free equilibrium E_2. It is interesting to observe that, even if in absence of forestry biomass, biomass – based industries may survive due to the availability of alternative resources that has been incorporated through the modified Leslie – Gower function. However, if $\lambda > \lambda^*, \beta < \beta^*, \Delta_1(\lambda)$ are satisfied, then as the maximum reduction rate of that industries can attain is lower than the critical value β^* the biomass – based industries will die out in long run and system converges to stable industry – free equilibrium E_1. But if new plantation rate and maximum reduction rate of that industries can attain are both less than the critical values,

i.e., if $\lambda < \lambda^*, \beta < \beta^*,$ then all the populations die out. In this case system trajectories converge to stable trivial equilibrium E_0. Assuming p be the selling price per unit biomass and c is the cost incurred to harvest per unit biomass, we have analysed the existence of bionomic equilibrium E_∞ and shown that this equilibrium will exist if new plantation rate, the maximum reduction rate of that industries can attain and the rate of harvesting q is higher than the evaluated threshold values provided the ratio of the density of $P(t)$ and $M(t)$ at E_∞ exceeds a critical value. Parameters have been estimated using the real time data of production of round wood timber, in million m^3 RWE (Round Wood Equivalent) unit, from Indian forestry resources during 2010 – 2020, obtained in **[19]** using Root Mean Square Error (RMSE) technique. It has been shown that for estimated parameters our model is a good fit. Extensive numerical simulations have been performed with the estimated parameters to determine the effects of the sensitive parameters on the system dynamics. It has been shown that increasing effort to harvest mature trees will not result in increasing industries. Rather it will decrease the biomass – based industries, as rate of harvesting in general will not reciprocate the rate of maturation of premature trees. Therefore, for sustainability harvesting should be performed optimally. We have determined the expression of the optimal harvesting effort in this purpose using Pontryajin's Maximum Principle. Moreover, with the help of data of future estimate of demand of timber in India during 2021 – 2030 obtained from **[19]** it has been shown with 95% confidence interval, that if the harvesting effort is in accordance with the optimal harvesting level, then the future requirement of timber in India, during the time period 2021 – 2030 can be fulfilled. This finding is an important one form the application point of view of our model.
Therefore, the findings of our study are interesting and can be used to form harvesting strategies so that forestry – biomass and forestry-based industries both can sustain simultaneously, by fulfilling the need of sustainable socio – economic development.

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Interval Valued Intutionistic Fuzzy Soft Sets Similarity Measure For Knowledge Discovery In Decision Making Problems

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Abstract: In this article we further study several distances between two interval valued intutionistic fuzzy soft sets(IVIFSsets). IVIFSset is a combination of soft set and interval valued intutionistic fuzzy set. Based on these distances, similarity measure between two IVIFS sets is calculated. An algorithm is developed for decision making problems. Lastly an example is given to show the possible application of similarity measure for knowledge discovery in COVID-19 patients..

Key words: soft set, interval valued intuitionistic fuzzy soft set, similarity measure, decision making.

AMS classification: 03E72

1. Introduction:

In 1965 Prof. L. A. Zadeh[**22**] pioneered the concept of fuzzy set theory. After then several researchers have extended this concept in many directions. As a result interval valued fuzzy set[23], intuitionistic fuzzy set[1], interval valued intuitionistic fuzzy set[2], soft set[7,11], fuzzy soft set[12], interval valued fuzzy soft set[21], intuitionistic fuzzy soft set[13], interval valued intuitionistic fuzzy soft set[6], etc have been introduced. Researchers like S. M. Chen [4], Hu and Li[5] etc. have studied the problem of similarity measure between fuzzy sets and vague sets. P. Majumdar and S. K. Samanta[8,9,10] have studied the similarity measure of soft sets, fuzzy soft sets and intuitionistic fuzzy soft sets. W. K. Min[14] also introduced similarity in soft set theory. Cagman and Deli[3] studied similarity measure of intuitionistic fuzzy soft sets [19]. A. Mukherjee and S. Sarkar introduced the concept of similarity measure for interval valued fuzzy soft sets[15] and interval valued intutionistic fuzzy soft sets[16]. Similarity measures are very useful to make good decisions. For Further study we refer the papers [17],[18] and [20]. In this manuscript different similarities like Normalized Hamming Distance and Normalized Euclidean Distance are used. These techniques are very useful to deal with the problems whose attributes are numerous. The proposed similarity measures applied to diagnose the Covid-19 will be extremely useful to case history. In medical diagnosis, decision-making .It is very much complicated because some medical tests are too expensive and time taking. Some time condition of the patient is not stable, the doctor did not have enough time to wait for the results of a medical test . So they should take urgent steps to save the patient's life. In such conditions.These kinds of studies help them to take effective decisions to save a patient's life. In the days of COVID-19 patients are large in number and the medical test is very much expensive, time-taking . The laboratories are not enough to fulfll such a big task. Some patients had financial problems and the condition of some patients iscritical they did not have enough time to wait. To handle that pandemic situation, select an ideal case (a patient who wassuffering from COVID-19). By calculation of similarity measure of any suspected patient with an ideal case to diagnosis the COVID-19. Through this technique doctors can take fast a decision to diagnosis the disease and create a helping environment in medical diagnosis..

In this work, we further study several distances between two interval valued intutionistic fuzzy soft sets. Based on these distances, similarity measure between two interval valued intutionistic fuzzy soft sets is calculated. An algorithm is developed for decision making problem. . Lastly an example is given to show the possible application of similarity measure for knowledge discovery in COVID-19 patients.

2. Preliminaries

In this section we briefly review some basic definitions related to interval-valued intuitionistic fuzzy soft sets which will be used in the rest of the paper.

Definition 2.1[22] Let X be a non empty collection of objects denoted by x. Then a **fuzzy set** *(FS for short)* α in X is a set of ordered pairs having the form $\alpha = \{(x, \mu_\alpha(x)) : x \in X\}$,

Where the function $\mu_\alpha : X \to [0,1]$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in α .The interval M = $[0,1]$ is called membership space.

Definition 2.2[23] Let X be a non empty set and D be the set of closed subintervals of the interval [0, 1]. Then an ***interval-valued fuzzy set*** *A* in X is an expression A given by

$A = \{(x, M_A(x)) : x \in X\}$, where M_A:X $\to D$.

Definition 2.3[7,11] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subseteq E$. Then the pair (F, A) is called a ***soft set*** over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.4[12] Let U be an initial universe and E be a set of parameters. Let I^U be the set of all fuzzy subsets of U and A⊆ B . Then the pair (F,A) is called **a *fuzzy soft set*** over U, where F is a mapping given by $F : A \to I^U$.

Definition 2.5[21] Let U be an initial universe and E be a set of parameters, a pair(F,E) is called an ***interval valued- fuzzy soft set*** over F(U), where F is a mapping given by F:E→ F(U),

An interval-valued fuzzy soft set is a parameterized family of interval-valued fuzzy subsets of U, thus, its universe is the set of all interval-valued fuzzy sets of U, i.e. F(U). An interval-valued fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to F(U),∀ e ∈ E , F(U) is referred as the interval fuzzy value set of parameters e , it is actually an interval-valued fuzzy set of U where $x \in U$ and $e \in E$, it can be written as:

$$F(e) = \left\{(x, \mu_{F(e)}(x)) : x \in U\right\}$$

where, F(U) is the interval-valued fuzzy membership degree that object x holds on parameter.

Definition 2.6[1] Let X be a non empty set. **An *intuitionistic fuzzy set*** (*IFS* in short) α in X is a set of ordered triples given by,

$\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) : x \in X\}$

where, the functions $\mu_\alpha : X \to [0,1]$ and $\gamma_\alpha : X \to [0,1]$ called degree of membership and the degree of non-membership of each element $x \in X$ to the set α respectively and $0 \le \mu_\alpha(x) + \gamma_\alpha(x) \le 1$ for each $x \in X$.

Definition 2.7[2] **An interval-valued intuitionistic fuzzy set** (*IVIFSet* in short) α over a universe X is defined as the object of the form $\alpha = \left\{\left\langle x, \mu_\alpha(x), \gamma_\alpha(x)\right\rangle : x \in X\right\}$, where

$\mu_\alpha(x): X \to D([0,1])$ and $\gamma_\alpha(x): X \to D([0,1])$ are functions such that the condition: $\forall x \in X \quad 0 \le sup\mu_\alpha(x) + sup\gamma_\alpha(x) \le 1$ is satisfied (where $D([0,1])$ is the set of all closed intervals of $[0,1]$).

Definition 2.8 [**13**] Let U be an initial universe and E be a set of parameters. Let IF^U be the set of all intuitionistic fuzzy subsets of U and $A \subseteq E$. Then the pair (F, A) is called an ***intuitionistic fuzzy soft set*** over$U$, where F is a mapping given by $F: A \to IF^U$.

Definition 2.9[6] Let U be an initial universe and E be a set of parameters. Let $IVIFS^U$ be the set of all interval valued intuitionistic fuzzy sets on U and $A \subseteq E$. Then the pair (F, A) is called an ***interval-valued intuitionistic fuzzy soft set*** *(IVIFSset) for short)* over U, where F is a mapping given by $F: A \to IVIFS^U$.

3. Similarity measure based on distance

In this section we further study several distances between two interval-valued intuitionistic fuzzy soft sets and based on this distances similarity measure is defined. Also study some basic properties of similarity measure with application in decision making problems.

Definition 3.1 Let $U = \{x_1, x_2, x_3, ..., x_n\}$ be the universe, $E = \{e_1, e_2, e_3, ..., x_m\}$ be the set of parameters, $\mathrm{A, B} \subseteq \mathrm{E}$ and (F, A), (G, B) be two IVIFS Sets on U with their intuitionistic fuzzy approximation functions $F_A(e_i) = \{(x, \mu_A(x), \upsilon_A(x)) : x \in U\}$ and

$G_B(e_i)=\{(x,\mu_B(x),\upsilon_B(x)):x\in U\}$ respectively, where μA(x): U→D[0,1] and ν_A(x): U→D[0,1] are functions such that ∀x∈U, supμA(x)+supν_A(x)≤1 and D[0,1] is the set of all closed subintervals of [0,1]. If A = B then we define the following distances between (F,A) and (G,B):

1. Hamming distance:

$$d_H(F,G)=\frac{1}{2n}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left|\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right|+\left|\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right|\right\}$$

(when n >m)

$$d_H(F,G)=\frac{1}{2m}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left|\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right|+\left|\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right|\right\}$$

(when m >n)

2. Normalized Hamming distance:

$$d_{NH}(F,G)=\frac{1}{2mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left|\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right|+\left|\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right|\right\}$$

3. Euclidean distance:

$$d_E(F,G)=\left[\frac{1}{2n}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left(\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right)^2+\left(\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right)^2\right\}\right]^{\frac{1}{2}}$$

(when n >m)

$$d_E(F,G)=\left[\frac{1}{2m}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left(\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right)^2+\left(\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right)^2\right\}\right]^{\frac{1}{2}}$$

(when m >n)

4. Normalized Euclidean distance:

$$d_{NE}(F,G)=\left[\frac{1}{2mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left(\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right)^2+\left(\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right)^2\right\}\right]^{\frac{1}{2}}$$

Where $\bar{\mu}_F(e_i)(x_j)=\frac{1}{2}\left\{\sup\mu_F(e_i)(x_j)+\inf\ \mu_F(e_i)(x_j)\right\}$

$$\bar{\mu}_G(e_i)(x_j)=\frac{1}{2}\left\{\sup\mu_G(e_i)(x_j)+\inf\ \mu_G(e_i)(x_j)\right\}$$

$$\bar{\nu}_F(e_i)(x_j)=\frac{1}{2}\left\{\sup\nu_F(e_i)(x_j)+\inf\ \nu_F(e_i)(x_j)\right\}$$

$$\bar{\nu}_G(e_i)(x_j)=\frac{1}{2}\left\{\sup\nu_G(e_i)(x_j)+\inf\ \nu_G(e_i)(x_j)\right\}$$

Definition 3.2 Let U be universe and E be the set of parameters and (F,A), (G,B) be two IVIFS Sets on U , where A = B ⊆ E. Then based on these distances defined in definition 3.1 similarity measure of (F,A) and (G,B) is defined as

$$\mathrm{Sm}(F,G) = \frac{1}{1+d(F,G)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(3.1)}$$

Another similarity measure of (F, A) and (G,B) can also be defined as

$$\mathrm{Sm}(F,G) = e^{-\alpha d(F,G)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(3.2)}$$

where $d(F,G)$ is the distance between the IVIFS Sets (F,A) and (G,B) and α is a positive real number.

Definition 3.3 Let (F,A) and (G,A) be two IVIFS Sets defined on the universe U . Then the we define following distances between (F,A) and (G,A) as,

$$d(F,G) = \left[\frac{1}{2n}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left|\bar{\mu}_F(e_i)(x_j) - \bar{\mu}_G(e_i)(x_j)\right|^k + \left|\bar{\upsilon}_F(e_i)(x_j) - \bar{\upsilon}_G(e_i)(x_j)\right|^k\right\}\right]^{\frac{1}{k}} \ldots\ldots \mathbf{(3.3)}$$

when n > m

$$d(F,G) = \left[\frac{1}{2m}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left|\bar{\mu}_F(e_i)(x_j) - \bar{\mu}_G(e_i)(x_j)\right|^k + \left|\bar{\upsilon}_F(e_i)(x_j) - \bar{\upsilon}_G(e_i)(x_j)\right|^k\right\}\right]^{\frac{1}{k}}$$

when m > n

$$d(F,G)=\left[\frac{1}{2mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\left|\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right|^k+\left|\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right|^k\right\}\right]^{\frac{1}{k}} \quad \ldots.\ (\mathbf{3.4})$$

where k > 0. If k = 1 then equation (3.3) and (3.4) are respectively reduced to Hamming distance and Normalized Hamming distance. Again if k = 2 then equation (3.3) and (3.4) are respectively reduced to Euclidean distance and Normalized Euclidean distance.

The weighted distance is defined as

$$d^w(F,G)=\left[\frac{1}{2n}\sum_{i=1}^{m}\sum_{j=1}^{n}w_j\left\{\left|\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right|^k+\left|\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right|^k\right\}\right]^{\frac{1}{k}} \quad \ldots(\mathbf{3.5})$$

when n > m,

$$d^w(F,G)=\left[\frac{1}{2m}\sum_{i=1}^{m}\sum_{j=1}^{n}w_j\left\{\left|\bar{\mu}_F(e_i)(x_j)-\bar{\mu}_G(e_i)(x_j)\right|^k+\left|\bar{\upsilon}_F(e_i)(x_j)-\bar{\upsilon}_G(e_i)(x_j)\right|^k\right\}\right]^{\frac{1}{k}}$$

when m > n, where $w_1, w_2, w_3, \ldots, w_n$ are the weights of $x_1, x_2, x_3, \ldots, x_n$ respectively, $\sum_{i=1}^{n} w_i = 1$ and k > 0. Especially, if k =1 then (3.5) is reduced to the weighted Hamming distance and If k = 2, then (3.5) is reduced to the weighted Euclidean distance.

Definition 3.4 Based on the weighted distance between two IVIFS Sets (F,A) and (G,A) given by equation (3.5), the similarity measure between (F,A) and (G,A) is defined as

$$\text{Sm(F,G)} = \frac{1}{1+d^w(F,G)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \ (\mathbf{3.6})$$

Theorem 3.5 If Sm(F,G) be the similarity measure between two IVIFS Sets (F,E) and (G,E) then

(i) Sm(F,G) = Sm(G,F)

(ii) $0 \leq Sm(F,G) \leq 1$

(iii) Sm(F,G) = 1 if and only if (F,E) = (G,E).

Proof: Obvious from the definition 3.2.

Definition 3.6 Let (F,A) and (G,B) be two IVIFS Sets over U. Then (F,A) and (G,B) are said be **α-similar**, denoted by $(F,A)\overset{\alpha}{\approx}(G,B)$ if and only if Sm((F,A),(G,B)) > α for α ∈ (0,1). We call the two IVIFS **Sets significantly similar** if Sm((F,A),(G,B)) > 0.6 Where A = B is the set of parameters. If Sm((F,A),(G,B)) < 0.6 then the two sets **are not significantly** similar. For Sm((F,A),(G,B)) = 0.6 no **decision can be taken immediately**.

Example 3,7 : Consider an interval valued intutionistic fuzzy soft set (F,A) U is a set of six houses under the consideration of a decision maker to purchase. It is denoted by $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$. A is asset of parameters, $A = \{e_1, e_2, e_3, e_4, e_5\}$ = { beautiful, expensive, wooden, in good repair, in green surrounding}. The IVFS set (F,A) describes the attractiveness of the houses to the decision maker .

Suppose

$$F(e_1) = \{\langle h_1, [0.5, 0.8], [0.1, 0.2]\rangle, \langle h_2, [0.7, 0.8], [0.1, 0.2]\rangle, \langle h_3, [0.5, 0.7], [0.2, 0.3]\rangle,$$
$$\langle h_4, [0.65, 0.78], , [0.1, 0.2]\rangle, \langle h_5, [0.5, 0.6], [0.2, 0.4]\rangle, \langle h_6, [0.6, 0.8], [0.1, 0, 2]\rangle\}$$

$$F(e_2)=\{\langle h_1,[0.6,0.8],[0.1,0.2]\rangle,\langle h_2,[0.6,0.7],[0.2,0.3]\rangle,\langle h_3,[0.4,0.6],[0.2,0.4]\rangle,$$
$$\langle h_4,[0.6,0.7],[0.2,0.3]\rangle,\langle h_5,[0.6,0.7],[[0.1,0.2]\rangle,\langle h_6,[0.7,0.8],[0.1,0.2]\rangle\}$$
$$F(e_3)=\{\langle h_1,[0.7,0.8],[0.1,0.2]\rangle,\langle h_2,[0.5,0.6],[0.2,0.4]\rangle,\langle h_3,[0.5,0.7],[0.2,0.3]\rangle,$$
$$\langle h_4,[0.65,0.7],[0.1,0.3]\rangle,\langle h_5,[0.7,0.8],[0.1,0.2]\rangle,\langle h_6,[0.6,0.7],[0.1,0.2]\rangle\}$$
$$F(e_4)=\{\langle h_1,[0.8,0.9],[0.05,0.1]\rangle,\langle h_2,[0.6,0.7],[0.1,0.2]\rangle,\langle h_3,[0.5,0.6],[0.2,0.3]\rangle,$$
$$\langle h_4,[0.65,0.75],[0.15,0.25]\rangle,\langle h_5,[0.7,0.8],[0.1,0.2]\rangle,\langle h_6,[0.6,0.7,[0.05,0.15]]\rangle\}$$
$$F(e_5)=\{\langle h_1,[0.7,0.85],[0.05.0.1]\rangle,\langle h_2,[0.5,0.6],[0.2,0.3]\rangle,\langle h_3,[0.6,0.7],[0.1,0.2]\rangle,$$
$$\langle h_4,[0.5,0.7],[0.1,0.25]\rangle,\langle h_5,[0.7,0.8],[0.025,0.1]\rangle,\langle h_6,[0.7,0.8],[0.1,0.2]\rangle\}$$

The IVFS set (F,A) is a parameterized family $\{F(e_i), i=1,2,3,4,5\}$ of interval valued intuitionistic fuzzy sets of U and (F,A)=

Type equation here.

$$F(e_1)=\{\langle h_1,[0.5,0.8],[0.1,0.2]\rangle,\langle h_2,[0.7,0.8],[0.1,0.2]\rangle,\langle h_3,[0.5,0.7],[0.2,0.3]\rangle,$$
$$\langle h_4,[0.65,0.78],,[0.1,0.2]\rangle,\langle h_5,[0.5,0.6],[0.2,0.4]\rangle,\langle h_6,[0.6,0.8],[0.1,0,2]\rangle\}$$

$$F(e_2)=\{\langle h_1,[0.6,0.8],[0.1,0.2]\rangle,\langle h_2,[0.6,0.7],[0.2,0.3]\rangle,\langle h_3,[0.4,0.6],[0.2,0.4]\rangle,$$
$$\langle h_4,[0.6,0.7],[0.2,0.3]\rangle,\langle h_5,[0.6,0.7],[[0.1,0.2]\rangle,\langle h_6,[0.7,0.8],[0.1,0.2]\rangle\}$$
$$F(e_3)=\{\langle h_1,[0.7,0.8],[0.1,0.2]\rangle,\langle h_2,[0.5,0.6],[0.2,0.4]\rangle,\langle h_3,[0.5,0.7],[0.2,0.3]\rangle,$$
$$\langle h_4,[0.65,0.7],[0.1,0.3]\rangle,\langle h_5,[0.7,0.8],[0.1,0.2]\rangle,\langle h_6,[0.6,0.7],[0.1,0.2]\rangle\},$$

$$F(e_4)=\{\langle h_1,[0.8,0.9],[0.05,0.1]\rangle,\langle h_2,[0.6,0.7],[0.1,0.2]\rangle,\langle h_3,[0.5,0.6],[0.2,0.3]\rangle,$$
$$\langle h_4,[0.65,0.75],[0.15,0.25]\rangle,\langle h_5,[0.7,0.8],[0.1,0.2]\rangle,\langle h_6,[0.6,0.7,[0.05,0.15]]\rangle\},$$

$$F(e_5)=\{\langle h_1,[0.7,0.85],[0.05.0.1]\rangle,\langle h_2,[0.5,0.6],[0.2,0.3]\rangle,\langle h_3,[0.6,0.7],[0.1,0.2]\rangle,$$
$$\langle h_4,[0.5,0.7],[0.1,0.25]\rangle,\langle h_5,[0.7,0.8],[0.025,0.1]\rangle,\langle h_6,[0.7,0.8],[0.1,0.2]\rangle\}.$$

The tabular representation of (F,A), Table-1

U	e_1	e_2	e_3	e_4	e_5
h_1	[0.5,0.8], [0.1,0.2]	[0.6,0.8], [0.1,0.2]	[0.7,0.8], [0.1,0.2]	[0.8,0.9], [0.05,0.1]	[0.7,0.85], [0.05,0.1]
h_2	[0.7,0.8], [0.1,0.2]	[0.6,0.7], [0.2,0.3]	[0.5,0.6], [0.2,04]	[0.6,0.7], [0.1,0.2]	[0.5,0.6], 0.2,0.3]
h_3	[0.5.0.7]. [0.2,0.3]	[0.4,0.6], [0.2,0.4]	[0.5,0.7], [0.2,0.3]	[0.5,0.6], [0.2,0.3]	[0.6,0.7], [0.1,0.2]
h_4	[0.65,0.78], [0.1,0.2]	[0.6,0.7], [0.2,0.3]	[0.65,0.7], [0.1,0.3]	[0.65,0.75], [0.15,0.25]	[0.5,0.7], [0.1,0.25]
h_5	[0.5,0.6], [0.2,0.4]	[0.6,0.7], [0.1,0.2]	[0.7,0.8], [0.1,0.2]	[0.7,0.8], [0.1,0.2]	[0.7,0.8], [0.025,0.1]
h_6	[0.6,0.8], [0.1,0.2]	[0.7,0.8], [0.1,0.2]	[0.6,0.7], [0.1,0.2]	[0.6,0.7], [0.05,0.15]	[0.7,0.8], [0.1,0.2]

3.8: Consider the example **3.7**. We construct the following algorithm

Step 1. Find the average of the membership interval and the average of the non-membership interval.

Step 2. Calculate their differences D_i , i=1,2,3,4,5.

Step 3. Find the row sum.

Step 4. Select the maximum of the row sum.

Table-2

U	**e_1**	**e_2**	**e_3**	**e_4**	**e_5**
h_1	**0.65, 0.15**	**0.7, 0.15**	**0.75, 0.15**	**0.85,0.075**	**0.775,0.075**
h_2	**0.75,0.15**	**0.65,0.25**	**0.55,0.3**	**0.65,0.25**	**0.55,0.25**

h_2	**0.75,0.15**	**0.65,0.25**	**0.55,0.3**	**0.65,0.25**	**0.55,0.25**
h_3	**0.6, 0.25**	**0.5,0.3**	**0.6,0.25**	**0.55, 0.25**	**0.65,0.15**
h_4	**0.715,0.15**	**0.65,0.25**	**0.675,0.2**	**0.7,0.2**	**0.6,0.175**
h_5	**0.55,0.3**	**0.65,0.15**	**0.75,0.15**	**0.75,0.15**	**0.75,.0625**
h_6	**0.7,0.15**	**0.75,0.15**	**0.65,0.15**	**0.65,0.1**	**0.75.0.15**

Table-3

U	D_1	D_2	D_3	D_4	D_5
h_1	0.5	0.55	0.6	0.775	0.7
h_2	0.6	0.4	0.25	0.4	0.3
h_3	0.35	0.2	0.35	0.3	0.5
h_4	0.565	0.4	0.475	0.5	0.425
h_5	0.25	0.5	0.6	0.6	0.6775
h_6	0.55	0.6	0.5	0.55	0.6

Row sums of h_i 's , i=1,2,3,4,5,6.

3.125 , 1.95 , 1.70 , 2.365 , 2.625 . 2.80

The customer will purchase the house h $_1$.

4. Application in decision-making problem:

In this section we developed an algorithm in interval-valued intuitionistic fuzzy soft setting using similarity measure for a decision making problem(COVID-19 patients).. For this we have to construct an IVIFS set for the ideal alterative and IVIFS sets for the available alternatives. Then we calculate the similarity measure between ideal alternative and available

alternatives.

The steps of the algorithm of this method are as follows:

Step 1: Construct an IVIFS set for the ideal alternative

Step 2: Construct IVIFS sets for available alternatives

Step 3: Calculate Normalized Hamming distances between ideal alternative and available alternatives.

Step 4: Calculate similarity measure.

Step 5: Estimate result by using the similarity.

Example 4.1 Here we are giving an example of a decision-making method in interval-valued intuitionistic fuzzy soft set setting using similarity measure.

To handle that pandemic situation, select an ideal case (a patient who was suffering from covid-19). By calculation of similarity measure of any suspected patient with an ideal case to diagnosis the covid-19. Through this technique, doctors can take fast a decision to diagnosis the disease and create a helping environment in medical diagnosis.

In this case, let us consider two a set, the first one is an ideal case (a patient who was suffering from covid-19). and the second is a set of and the second is a set of

suspected patients P~=fP1, P2, P3, P4 }.

Let U be the universal set, which contains only two elements x_1 and x_2 i.e. U={x_1,x_2} and

E is the set of parameters c_1,c_2,c_3,c_4,e_5 i.e. E={c_1,c_2,c_3,c_4,e_5}.

The major problem of COVID-19 disease which is affected many people in India . The following are the common symptoms found in India..

e_1= Fever, e_2= Dry cough, . e_3 =Tiredness, e_4= Headache, e_5 =Loss of taste or smell, e_6= difficulty breathing or shortness of breath.
we choose a range belongs to [0,1]. For measuring the symptoms e_1 to e_5 as Low, Moderate, Highly moderate, High, Very high respectively (according to the medical experts and according to the range). For example the range [0.7, 0.9] is highly effected....

Step 1: Construct IVIFS Set (G,E) for ideal alternative:

Although the ideal alternative does not exist in real world, we construct the IVIFS

Set for ideal alterative by taking membership value = 1, non-membership value =

Table-4

(G ,E)	e_1	e_2	e_3	e_4	e_5
x_1	**[1,1], [0,0]**	**[1,1], [0,0]**	**[1,1], [0,0]**	**[1,1], [0,0]**	**[1,1], [0,0]**
x_2	**[1,1], [0,0]**	**[1,1], [0,0]**	**[1,,}], [0,0]**	**[1,1], [0,0]**	**[1,1], [0,0]**

Tabular representation of IVFSset (G,E) for ideal alternative

Step 2: Construct IVIFS Sets (P_1, E), (P_2, E), (P_3, E), (P_4, E) for the four alternatives, which can be constructed by taking data from the medical experts.

Table-5

(P_1, E)	e_1	e_2	e_3	e_4	e_5
x_1	**[0.4,0.5], [0.3,0.4]**	**[0.3,0.4], [0.4,0.6]**	**[0.2,0.3], [0.4,0.7]**	**[0.4,0.6], [0.2,0.4]**	**[0.4,0.5], [0.4,0.5]**
x_2	**[0.2,0.3], [0.6,0.7]**	**[0.5,0.6], [0.2,0.4]**	**[0.5,0.6], [0.3,0.4]**	**[0.7,0.8], [0.0,0.2]**	**[0.4,0.6], [0.2,0.4]**

Tabular representation of IVFSset (P_1, E)

Table-6

Table-6

(P_2, E)	e_1	e_2	e_3	e_4	e_5
x_1	**[0.35,0.5], [0.25,0.45]**	**[0.2,0.4], [0.5,0.6]**	**[0.15,0.35], [0.4,0.65]**	**[0.4,0.6], [0.2,0.4]**	**[[0.3,0.5], [0.4,0.5]**
x_2	**[0.1,0.25], [0.5,0.75]**	**[0.4,0.6], [0.3,0.4]**	**[0.4,0.55], [0.3,0.45]**	**[0.6,0.75], [0.1,0.2]**	**[0.3,0.5], [0.1,0.4]**

Tabular representation of IVFSset (P_2, E)

Table-7

(P_3, E)	e_1	e_2	e_3	e_4	e_5
x_1	**[0.1,0.2], [0.7,0.8]**	**[0.2,0.4], [0.5,0.7]**	**[0.3,0.4], [0.5,0.6]**	**[0.4,0.6], [0.3,0.5]**	**[0.2,0.4], [0.4,0.6]**
x_2	**[0.2,0.3], [0.6,0.7]**	**[0.2,0.4], [0.5,0.7]**	**[0.1,0.3], [0.5.,0.7}**	**[0.1,0.3],[0.5,0.7]**	**[0.1,0.2], [0.6,0.8]**

Tabular representation of IVFSset (P_3, E)

Table-8

(P_4, E)	e_1	e_2	e_3	e_4	e_5
x_1	**[0.1,0.3], [0.5,0.7]**	**[0.2,0.3], [0.5,0.7]**	**[0.2,0.4], [0.5,0.7]**	**[0.1,0.3, [0.5,0.7]**	**[0.1,0.2], [0.6,0.8]**
x_2	**[0.2,0.4], [0.4,0.6}**	**[0.1,0.2], [0.7,0.8]**	**[0.2,0.3], [0.6,0.7]**	**[0.1,0.3], [0.5,0.7]**	**[0.1,0.2], [0.5,0.7]**

Tabular representation of IVFSset (P_4, E)

Step 3: Calculate Hamming distances:

Now by definition **3.1** the Normalised Hamming distance between G and P_1, P_2, P_3, P_4 are given by

$d_H(G,P_1)= 0.4975$, $d_H(G,P_2)=0.4675$, $d_H(G,P_3)=0.69$, $d_H(G,P_4)= 0.75$

Step 4: Calculate similarity measure:

Now by equation (**3.1**) similarity measure between G and P_1, P_2, P_3, P_4 are given by,

$Sm(G,P_1) = 0.609$, $Sm(G,P_2) = 0.67$, $Sm(G,P_3) = 0.58$, $Sm(G,P_4) = 0.506$.

Step 5: Estimate the result:

From the values of similarity measure the raking order of the four alternatives is

$$\mathbf{P_2 > P_1 > P_3 > P_4}$$

Step 6: After that calculation the results are

Patients Results-

P_1	**keep him/ her under observation**
P_2	**+ve**
P_3	**+ve**
P_4	**–VE**

5. Conclusion:

In this paper we further study several distances between two interval-valued intuitionistic fuzzy soft sets. The similarity measure is defined between two interval-valued intuitionistic fuzzy soft sets. An algorithm is developed using for decision-making problem. An example is given to demonstrate the possible application of the proposed algorithm in COVID-19 patients. Similarity measures for interval valued intuitionistic fuzzy soft sets can also be applied to pattern recognition problem, medical diagnosis problem, image processing, image recognition, coding theory and several other problems that contain uncertainties.

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A Note on Neutrosophic Compact Space via Grills

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Abstract

The aim of this paper is to introduce compactness via grill $\mathcal{G}$ in a neutrosophic topological space. We call it neutrosophic $\mathcal{G}$- compactness. We iintend to reveal its some basic properties and characterization theorems. We also obtain its relationship with neutrosophic compactness and other known things. Lastly, we exhibit a new method for one - point neutrosophic compactification of a locally compactness.

Key Words: $\mathcal{G}$- compact, $\mathcal{G}$- cocover, neutrosophic topological space, grill, neutrosophic set.
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1. Introduction:

Everyday we are facing real life problems due to uncertainty. In order to handle such problems, Zadeh [15] introduced fuzzy set (FS, in short). Thereafter, Atanassov [1] invented intuitionistic fuzzy set (IFS, in short) by adding non - membership value along with membership value. But it was not sufficient to solve all real life problems due to uncertainty. In order to handle some types of problems on decision making under uncertainty, Smarandache [12] introduced the notion of neutrosophic set (NS, in brief) consisting with membership, non - membership and indeterminacy function defined on the universal set. These three functions are completely independent. Smarandache [13] further investigated on the applications of it. One can resolve real life problem in complex situation with the help of NS. The idea of neutrosophic topological space (NTS, in short) was referred in [9, 10].

The brilliant reveal of a grill was initiated by Choquet [2]. Subsequently, it became important convenient weapon for various topological studies. It is invented that grills are more suitable than nets and filters. Roy and Mukherjee [8] introduced the notion of compactness in topological space via grills. Pal et al [6] introduced the notion of grill in NTS and neutrosophic minimal space. Pal and Dhar [5] introduced the notion of compactness in neutrosophic minimal space. Further, different researchers [3, 4, 7, 11, 14] investigated in NTS. Following their works, we have motivated to introduce and investigate basic properties and results of compactness via grills in NTS. We shall also focus to construct new types of neutrosophic spaces through grills. The study reveals as follows. We shortly mention some known definitions and results related to NS and NTS in next section. In section 3, we unfold neutrosophic $\mathcal{G}$- compact space. We also investigate some basic properties and theorems of this space. Section 4 reveals the method of construction of neutrosophic space via grills and focuses their basic properties. Section 5 and 6 indicate the conclusion and future motivation of the work.

2. Preliminaries:

We recollect some basic concepts and results for study of this article.

Definition 2.1. [2] Set X as non - empty collection and be its collection of subsets. Take A, $B \subseteq X$. Then $\mathcal{G}$ is termed grill of X provided it obeys the axioms:

(i) A is a member of $\mathcal{G}$ as well as subset of B implies that B is a member of $\mathcal{G}$,

(ii) the union of A and B is a member of it indicates either A is a member of $\mathcal{G}$ or B is a member of $\mathcal{G}$.

Definition 2.2. [8] Take $\mathcal{G}$ as a grill on topological space (X, τ). A cover $\{O_i : i \in I\}$ of it is a $\mathcal{G}$.- cover when we can obtain I_0 as finite subset of I for which $X \setminus \cup_{i\in I_0} O_i$ is not a member of $\mathcal{G}$.

Definition 2.3. [9] An NS K in a whole set W is a set where each element consists with truthness, falseness and indeterminacy membership values appear from three independent functions, denoted by f_K, g_K, h_K in [0,1]. It is as below:

$$K = \{(x, f_K(x), g_K(x), h_K(x)) : x \in W\}$$

and each of $f_K(x)$, $g_K(x)$, $h_K(x))$ is a member of unit closed interval with the condition that sum of them lies between 0 to 3.

Definition 2.4. [9] We mean 0_N and 1_N as empty and whole NSs respectively on W where

(i) $0_N = \{(w, 0, 1, 1) : w \in W\}$.

(ii) $1_N = \{(w, 1, 0, 0) : w \in W\}$.

Definition 2.5. [9] Consider N a collection of NSs of W. Then N is termed neutrosophic topology (NT, in brief) of W provided the following axioms are satisfied:

(i) The null and whole NSs are members of N.

(ii) N is closed with respect to finite intersections.

(iii) N is closed with respect to arbitrary unions.

Here (W, N) is said to be NTS of W. Elements of N are known as neutrosophic open sets (NOSs, in short) whereas complements of them are known as neutrosophic closed sets (NCSs, in short).

Definition 2.6. [9] Consider V as NS in (W, N). The neutrosophic interior and neutrosophic closure, denoted by $N_{int}(V)$ and $N_{cl}(V)$ respectively of V are given by

$N_{int}(V) = \cup\{K : K$ is a NOS in W where $K \subseteq V\}$,

$N_{cl}(V) = \cap\{H : H$ is a NCS in W whee $V \subseteq H\}$.

Remark 2.7. [9] Clearly $N_{cl}(V)$ ($N_{int}(V)$) is the smallest (largest) neutrosophic closed (open) set on W.

Proposition 2.8. [9] For any NS V in (W, N), we have

(i) $N_{int}(V^c) = (N_{cl}(V))^c$.

(ii) $N_{cl}(V^c) = (N_{int}(V))^c$.

Definition 2.9. [9] Consider U a non - null set. A sub - collection $\mathcal{G}$ of NSs on U (not containing 0_N) and $F, H \subseteq U$ is called a grill of U when $\mathcal{G}$ obeys following axioms:

(i) $F \in \mathcal{G}$, $H \subseteq B$ gives $H \in \mathcal{G}$,

(ii) $F \cup H \in \mathcal{G}$ gives $F \in \mathcal{G}$ or $H \in \mathcal{G}$.

3. Neutrosophic $\mathcal{G}$ - compactness:

We propose and invent idea of neutrosophic - compactness in NTS..

Definition 3.1. Consider the grill $\mathcal{G}$ on NTS (X, T). A cover $\{V_\alpha : \alpha \in \mathrm{J}\}$ of X is termed as neutrosophic $\mathcal{G}$ - cover if $\exists$ a finite subset J_0 of J such that $X \setminus \bigcup_{\alpha \in \mathrm{J}_0} V_\alpha \notin \mathcal{G}$.

Definition 3.2. Consider the grill $\mathcal{G}$ on NTS (X, T). It is known as neutrosophic $\mathcal{G}$ - compact if every open cover of X is neutrosophic - cover. Henceforth, we denote neutrosophic compact by N - compact and neutrosophic - compact by $N_{\mathcal{G}}$ - compact on a NTS (X, T)..

Remark 3.3. (i) Clearly each N - compact is $N_{\mathcal{G}}$ - compact.

(ii) Take $\mathcal{G} = P(X) \setminus 0_N$, then $N_{\mathcal{G}}$ - compactness converts to N - compact.

(iii) If $(X, T_{\mathcal{G}})$ is $N_{\mathcal{G}}$ - compact, so (X, T) is N - compact and $N_{\mathcal{G}}$ - compact.

Theorem 3.4. A grill $\mathcal{G}$ is $N_{\mathcal{G}}$ - compact on (X, T) iff $(X, T_{\mathcal{G}})$ is $N_{\mathcal{G}}$ - compact.

Proof. As $T \subseteq T_{\mathcal{G}}$, (X, T) is $N_{\mathcal{G}}$ - compact if $(X, T_{\mathcal{G}})$ is N - compact

Consider $(X, T_{\mathcal{G}})$ as $N_{\mathcal{G}}$ - compact and $\{V_\alpha : \alpha \in \mathrm{J}\}$ be a basic T - open cover. Every $\alpha \in \mathrm{J}$, $U_\alpha = V_\alpha \setminus A_\alpha$ where $V_\alpha \in T$ and $A_\alpha \notin G$. Then $\{V_\alpha : \alpha \in \mathrm{J}\}$ is neutrosophic T - open cover. So $N_{\mathcal{G}}$ - compactness of (X, T), $\exists$ finite subset J_0 of J where $X \setminus \cup_{\alpha\in \mathrm{J}_0} V_\alpha \notin \mathcal{G}$.

Now, $X \setminus \cup_{\alpha\in \mathrm{J}_0} U_\alpha = X \setminus \cup_{\alpha\in \mathrm{J}_0} (V_\alpha \setminus A_\alpha) \subseteq \left(X \setminus \cup_{\alpha\in \mathrm{J}_0} V_\alpha\right) \cup \left(\cup_{\alpha\in \mathrm{J}_0} A_\alpha\right) \notin \mathcal{G}$ (as $A_\alpha \notin \mathcal{G}$, $\forall\ \alpha \in \mathrm{J}_0$). Hence $(X, T_{\mathcal{G}})$ is $N_{\mathcal{G}}$ - compact.

Remark 3.5. For a NTS (W, ρ), the following implication diagram holds:

$$\begin{array}{ccc} (W, \rho) \text{ is } N \text{ - compact} & \Longleftarrow & (W, \rho_{\mathcal{G}}) \text{ is } N \text{ - compact} \\ \Downarrow & & \Downarrow \\ (W, \rho) \text{ is } N_{\mathcal{G}} \text{ - compact} & \Leftrightarrow & (W, \rho_{\mathcal{G}}) \text{ is } N_{\mathcal{G}} \text{ - compact} \end{array}$$

Definition 3.6. A NTS (X, T) is referred neutrosophic quasi H - closed (shortly, NQHC) when each open cover U, $X = \cup(cl(U): U \in U_0)$ where U_0 denotes finite sub - collection.

Theorem 3.7. Consider a grill $\mathcal{G}$ on a NTS (X, T) where $T \setminus 0_N \subseteq G$. If (X, T) is $N_{\mathcal{G}}$ - compact. Then (X, T) is NQHC.

Proof. Consider $\{U_\alpha : \alpha \in \mathrm{J}\}$ as open cover of NTS. So for $N_{\mathcal{G}}$ - compactness, $\exists$ finite set J_0 of J where $\left(X \setminus \cup_{\alpha\in \mathrm{J}_0} U_\alpha\right) \notin \mathcal{G}$.Thus $\mathrm{int}\left(X \setminus \cup_{\alpha\in \mathrm{J}_0} U_\alpha\right) = 0_N$. For otherwise, $\mathrm{int}\left(X \setminus \cup_{\alpha\in \mathrm{J}_0} U_\alpha\right) \in T \setminus 0_N$ and hence $\left(X \setminus \cup_{\alpha\in \mathrm{J}_0} U_\alpha\right) \in \mathcal{G}$, a contradiction. Hence $X = \cup_{\alpha\in \mathrm{J}_0} \mathrm{cl}(U_\alpha)$ and X is NQHC.

Theorem 3.8. A NQHC (*X*, *T*) is neutrosophic $\mathcal{G}_\delta$ - compact, where $\mathcal{G}_\delta = \{A \subseteq X : \text{int}(\text{cl}(A)) \neq 0_N\}$, is a grill on *X*.
Proof. Take $\{V_\alpha : \alpha \in J\}$ as cover of (*X*, *T*). Then by NQHC, there is a finite set J_0 of J where $X \setminus \cup_{\alpha \in J_0} \text{cl}(V_\alpha) = 0_N$. Take $(X \setminus \cup_{\alpha \in J_0}(V_\alpha) \notin \mathcal{G}_\delta$.In fact, $(X \setminus \cup_{\alpha \in J_0}(V_\alpha)) \in \mathcal{G}_\delta \Rightarrow \text{int}(\text{cl}(X \setminus \cup_{\alpha \in J_0}(V_\alpha) \neq 0_N \Rightarrow X \setminus \cup_{\alpha \in J_0} Cl(V_\alpha) \neq 0_N$ a contradiction. Hence (*X*, *T*) is neutrosophic $\mathcal{G}_\delta$ - compact.

Definition 3.9. Consider grill $\mathcal{G}$ on NTS (*X*, *T*). Then *X* is called neutrosophic $\mathcal{G}$ - regular ($N_\mathcal{G}$ $-$ regular, in brief) if for any NCS *F* in *X* with neutrosophic point (NP, in short) $x \notin F$, ∃ disjoint NOS *U* and *V* such that $x \in U$ and $F \setminus V \notin \mathcal{G}$.
Theorem 3.10. Consider grill $\mathcal{G}$ on NQHC (*X*, *T*). If (*X*, *T*) is $N_\mathcal{G}$ – regular, then it is $N_\mathcal{G}$ - compact.

Proof. Take *V* as a neutrosophic open cover of (*X*, *T*). Every NP $x \in X$, there is some $V_x \in V$ where $x \in V_x$. So $x \notin (X \setminus V_x)$ where $(X \setminus V_x)$ is a NCS. Hence for $N_\mathcal{G}$ - regularity, ∃ disjoint NOSs G_x and H_x where $(X \setminus U_x) \setminus H_x \notin \mathcal{G}$ and $x \in G_x$. Let $A_x = (X \setminus V_x) \setminus H_x$. Now, $\text{cl}(G_x) \cap H_x = 0_N \Rightarrow \text{cl}(G_x) \subseteq X \setminus H_x \subseteq (X \setminus H_x) \cup V_x = [X \setminus (H_x \cup V_x)] \cup V_x = A_x \cup V_x$.
Again, $\{G_x : x \in X\}$ being a neutrosophic open cover of neutrosophic *H* - closed space, ∃NPs $x_1, x_2, \ldots\ldots, x_n$ in *X* such that $X = \cup_{i=1}^{n} \text{cl}(G_{x_i})$. Then $X = \cup_{i=1}^{n} \text{cl}(G_{x_i}) \subseteq \cup_{i=1}^{n}(A_{x_i} \cup U_{x_i}) \Rightarrow X \setminus \cup_{i=x}^{n} U_{x_i} \subseteq \cup_{i=1}^{n} A_{x_i} \notin \mathcal{G}$ .Hence (*X*, *T*) is $N_\mathcal{G}$ - compact.

Theorem 3.11. A $N_\mathcal{G}$ - compact neutrosophic Housdroff space (*W*, $\mathcal{H}$) is $N_\mathcal{G}$ - regular.

Proof. Take *F* as a NCS of *W* and NP $x \notin W$. By Housdorffness, every NP $y \in F$, ∃ disjoint NOSs U_y and V_y such that $x \in U_y$ and $y \in V_y$. Now, $\{V_y : y \in F\} \cup \{X \setminus F\}$ is a neutrosophic open cover of *X*. So for $N_\mathcal{G}$ - compactness of *X*, ∃ NP $y_1, y_2 \ldots\ldots, y_n$ in *F* such that $X \setminus [(\cup_{i=1}^{n} V_{y_i}) \cup (X \setminus F)] \notin \mathcal{G}$. Let $\mathcal{G} = X \setminus \cup_{i=1}^{n} \text{cl}(V_{y_i})$and $H = \cup_{i=1}^{n} V_{y_i}$. Then *F*and *H* are disjoint non - empty NOSs in *X* where $x \in G, F \setminus H = F \cap (X \setminus \cup_{i=1}^{n} V_{y_i}) = X \setminus [(\cup_{i=1}^{n} V_{y_i}) \cup (X \setminus F)] \notin \mathcal{G}$. Hence (*X*, *T*) is $N_\mathcal{G}$ - regular.

Corollary 3.12. A $N_\mathcal{G}$ - compact neutrosophic Housdroff space (*W*, $\mathcal{H}$) is is neutrosophic *H* - closed and $N_\mathcal{G}$ - regular. Here $\mathcal{H} \setminus 0_N \subseteq \mathcal{G}$.

Theorem 3.13. A $N_\mathcal{G}$ - regular neutrosophic *H* - closed space (*W*, $\mathcal{H}$) is $N_\mathcal{G}$ - compact.

Proof. Take V as a neutrosophic open cover of W. So every NP $x \in W$, there is some $V_x \in V$ where $x \in V_x$. Thus $x \notin (W \setminus V_x)$ where $(W \setminus V_x)$ is a NCS. By $N_{\mathcal{G}}$ - regularity of W, ∃ disjoint NOSs G_x and H_x where $(W \setminus U_x) \setminus H_x \notin \mathcal{G}$, $x \in G_x$. Take $A_x = (W \setminus U_x) \setminus H_x$. So $clG_x \cap H_x = 0_N \Rightarrow clG_x \subseteq X \setminus H_x \subseteq (X \setminus H_x) \cup U_x = [X \setminus (H_x \cup U_x)] \cup U_x = A_x \cup U_x$. Further $\{G_x : x \in X\}$ being neutrosophic open cover, there are NPs $x_1, x_2, ..., x_n$ in W where $W = \bigcup_{i=1}^{n} clG_{x_i}$. So $W = \bigcup_{i=1}^{n} clG_{x_i} \subseteq \bigcup_{i=1}^{n} (A_{x_i} \cup U_{x_i}) \Rightarrow W \setminus \bigcup_{i=1}^{n} U_{x_i} \subseteq \bigcup_{i=1}^{n} A_{x_i} \notin \mathcal{G}$ (as $A_{x_i} \notin \mathcal{G}$ for $i = 1, 2, ..., n$). Thus $(W, \mathcal{H})$ is $N_{\mathcal{G}}$ - compact. From Corollary 3.12.and Theorem 3.13., we concludefollowing:

Corollary 3.14. Consider $\mathcal{G}$ on neutrosophic Housdroff space $(W, \mathcal{H})$ where $\mathcal{H} \setminus 0_N \subseteq \mathcal{G}$. Then W is $N_{\mathcal{G}}$ - compact iff W is neutrosophic H - closed and $N_{\mathcal{G}}$ -regular.
Theorem 3.15. Consider U a neutrosophic open sub-base of NTS (X, T). Then neutrosophic open $\mathcal{G}$ - cover of X corresponds a $\mathcal{G}$ - cover with elements of U.

Proof. Take $\mathcal{C}$ as all neutrosophic open $\mathcal{G}$ - covers of X. Evidently, $\mathcal{C}$ is non – empty and take $\{P_\alpha\}$ as a linearly ordered neutrosophic subset of it. So $\bigcup_\alpha P_\alpha$ is neutrosophic covering. Consider it as neutrosophic $\mathcal{G}$ - covering. Otherwise ∃ $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n \in \bigcup P_\alpha$ where $X \setminus \bigcup_{i=1}^{n} (\mathcal{G}_i) \notin \mathcal{G}$. Now, ∃ a $P_\beta \in C$ such that $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n \in P_\beta$. So $P_\beta \notin C$, a contradiction. Thus, a maximal element P is there in C. Take $\mathcal{K}$ as open where $\notin \mathcal{G}$. So ∃ finitely many $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n \in P$ where $X \setminus (\mathcal{K} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n) \notin \mathcal{G}$. Take $\mathcal{K}_1, \mathcal{K}_2 \in T$ and $\mathcal{K}_1, \mathcal{K}_2 \notin P$. Thus, $X \setminus (\mathcal{K}_1 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n) = A_1 \notin \mathcal{G}$ and $X \setminus (\mathcal{K}_2 \cup V_1 \cup V_2 \cup \cdots \cup V_m) = A_2 \notin \mathcal{G}$, for sub - collections $\{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n\}$, $\{V_1, V_2, ..., V_m\}$ of P. Take $B = X \setminus [(\mathcal{K}_1 \cap \mathcal{K}_2) \cup (\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n) \cup (V_1 \cup V_2 \cup \cdots \cup V_m)]$. So $B \subseteq A_1 \cup A_2$. As $A_1 \cup A_2 \notin \mathcal{G}$, so $B \notin \mathcal{G}$. Hence $(\mathcal{K}_1 \cap \mathcal{K}_2) \in T \setminus P$. Take $\mathcal{K} \notin P$ and $\mathcal{K} \subseteq \mathcal{G}$, where $\mathcal{G}$ and $\mathcal{K}$are NOSs. Then $X \setminus (\mathcal{K} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n) \notin \mathcal{G}$ for finitely many $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n \in P$. Thus $X \setminus (\mathcal{G} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n) \notin \mathcal{G}$ and hence $\mathcal{G} \in T \setminus P$. Now it is quite enough to prove $U \cap P$ is a neutrosophic $\mathcal{G}$ - cover of X. Let x be a NP where $x \in X$. Since P is a neutrosophic open cover of X, ∃ a $\mathcal{G} \in P$ where $x \in \mathcal{G}$. Since U is a neutrosophic sub-base for X, ∃ $\mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_n \in U$ where $x \in \mathcal{K}_1 \cap \mathcal{K}_2 \cap \cdots \cap \mathcal{K}_n \subseteq \mathcal{G}$. So ∃ an $\mathcal{K}_i$ where $\mathcal{K}_i \in P$. If $\mathcal{K}_i \notin P$, then $\bigcup_{i=1}^{n} (\mathcal{K}_i) \notin P$. Thus $\mathcal{G} \notin P$, a contradiction. Hence $x \in \mathcal{K}_i \in U \cap P$ and thus, $U \cap P$ is a neutrosophic $\mathcal{G}$ - cover.

4. Construction of new neutrosophic topological space via grills:
Here we explore a new compact neutrosophic Hausdorff space.

Theorem 4.1. Consider a grill $\mathcal{G}$ on a NTS (X, T). Take $X^* = X \cup x$ where $x \notin X$. Then f $\acute{P}(X^*) \rightarrow \acute{P}(X^*)$, where

$$f(\bar{A}) = cl\bar{A}, \text{ if } cl\bar{A} \notin \mathcal{G}, \text{ for } A \subseteq X$$
$$f(\bar{A}) = cl\bar{A} \cup x, \text{ if } cl\bar{A} \in \mathcal{G}, \text{ for } \bar{A} \subseteq X$$
$$f(\bar{A}) = (cl\,\bar{A} \setminus x) \cup x, \text{ if } x \in \bar{A}$$

is Kuratowski neutrosophic operator,

where
(i) each T - open is $T*$ - open in X,
(ii) if K is T^* - open, $K \cap X$ is T - open.

Proof. Firstly, explore f satisfies Kuratowski neutrsophic closure axioms. Clearly $f(0_N) = 0_N$ (as $0_N \notin \mathcal{G}$) and for $\bar{A} \subseteq X^*$, $\bar{A} \subseteq f(\bar{A})$,. We verify $f(\bar{A} \cup \mathscr{B}) = f(\bar{A}) \cup f(\mathscr{B})$ where $\mathscr{B}$ in X^*.

Case - I. Take $\mathrm{cl}(\bar{A} \cup \mathscr{B}) \notin \mathcal{G}$. So $f(\bar{A} \cup \mathscr{B}) = \mathrm{cl}(\bar{A} \cup \mathscr{B}) = \mathrm{cl}\,\bar{A} \cup \mathrm{cl}\,\mathscr{B} = f(\bar{A}) \cup f(\mathscr{B})$.
Further $\mathrm{cl}(\bar{A} \cup \mathscr{B}) \in \mathcal{G}$, so $\mathrm{cl}\bar{A}$ or $\mathrm{cl}\mathscr{B} \in \mathcal{G}$ and $f(\bar{A} \cup \mathscr{B}) = \mathrm{cl}(\bar{A} \cup \mathscr{B}) \cup x = \mathrm{cl}\,\bar{A} \cup \mathrm{cl}\mathscr{B} \cup x = f(\bar{A}) \cup f(\mathscr{B})$.

Case-II. Take $x \in \mathscr{B}$ and $\mathrm{cl}\bar{A} \notin \mathcal{G}$. Hence $f(\bar{A} \cup \mathscr{B}) = \mathrm{cl}((\bar{A} \cup \mathscr{B}) \setminus x) \cup x = \mathrm{cl}(\bar{A} \cup (\mathscr{B} \setminus x)) \cup x = \mathrm{cl}\,\bar{A} \cup \mathrm{cl}(\mathscr{B} \setminus x) \cup x = f(\bar{A}) \cup f(\mathscr{B})$. If $\mathrm{cl}\,\bar{A} \in \mathcal{G}$, then $f(\bar{A} \cup \mathscr{B}) = \mathrm{cl}((\bar{A} \cup \mathscr{B}) \setminus x) \cup x = \mathrm{cl}\,\bar{A} \cup x \cup \mathrm{cl}(\mathscr{B} \setminus x) \cup x = f(\bar{A}) \cup f(\mathscr{B})$.

Case-III. $x \in \bar{A}$ and $x \in \mathscr{B}$ Here $f(\bar{A} \cup \mathscr{B}) = \mathrm{cl}((\bar{A} \cup \mathscr{B}) \setminus x) \cup x = \mathrm{cl}(\bar{A} \setminus x) \cup \mathrm{cl}(\mathscr{B} \setminus x) \cup x = f(\bar{A}) \cup f(\mathscr{B})$. Now we prove $f(f(\bar{A})) = f(\bar{A})$, where $\bar{A} \subseteq X^*$.
Case - (i): $\bar{A} \subseteq X$. If $\mathrm{cl}\,\bar{A} \notin \mathcal{G}$, $f(f(\bar{A})) = f(\mathrm{cl}\,\bar{A}) = \mathrm{cl}\,\bar{A} = f(\bar{A})$ and $\mathrm{cl}\bar{A} \in \mathcal{G}$, then $f(f(\bar{A})) = f(\mathrm{cl}\bar{A} \cup x) = f(\mathrm{cl}\bar{A}) \cup f(x) = \mathrm{cl}\,\bar{A} \cup x = f(\bar{A})$.
Case - (ii): $x \in \bar{A}$. If $\mathrm{cl}(\bar{A} \setminus x) \notin \mathcal{G}$, then $f(f(\bar{A})) = f[\mathrm{cl}(\bar{A} \setminus x) \cup x] = f[\mathrm{cl}(\bar{A} \setminus x)] \cup f(x) = \mathrm{cl}(\bar{A} \setminus x) \cup x = f(\bar{A})$. If $\mathrm{cl}(\bar{A} \setminus x) \in \mathcal{G}$, then $f(f(\bar{A})) = f[\mathrm{cl}(\bar{A} \setminus x) \cup x] = f[\mathrm{cl}(\bar{A} \setminus x)] \cup f(x) = \mathrm{cl}(\bar{A} \setminus x) \cup x = f(\bar{A})$. It implies f is a Kuratowski neutrosophic closure operator on X^* and gives a NT T^* on X^* where $f(\bar{A}) = T^*$ - $\mathrm{cl}\,\bar{A}$, for any $\bar{A} \subseteq X^*$.
(a) Take $\dot{U} \subseteq X$ neutrosophic T - open. Then $f(X^* \setminus \dot{U}) = \mathrm{cl}[(X^* \setminus \dot{U}) \cup x] \cup x = \mathrm{cl}(X \setminus \dot{U}) \cup x = (X \setminus \dot{U}) \cup x = X^* \setminus \dot{U}$, so $\dot{U}$ is neutrosophic T^* - open.
(b) As $\dot{U}$ is neutrosophic T^* - open, $f(X^* \setminus \dot{U}) = X^* \setminus \dot{U}$ ... (i). Now, $x \notin \dot{U} \Rightarrow \mathrm{cl}[(X^* \setminus \dot{U}) \cup x] \cup x = X^* \setminus \dot{U} \Rightarrow \mathrm{cl}(X \setminus \dot{U}) \cup x = X^* \setminus \dot{U} \Rightarrow \mathrm{cl}(X \setminus \dot{U}) = (X \setminus \dot{U}) \Rightarrow (X \setminus \dot{U})$ is neutrosophic T - closed $\Rightarrow \dot{U}$ $(=\dot{U} \cap X)$ is neutrosophic T - open. Further, $x \in \dot{U} \Rightarrow \mathrm{cl}(X^* \setminus \dot{U}) = X^* \setminus \dot{U}$ and since $x \in (X^* \setminus \dot{U})$ $\Rightarrow \mathrm{cl}[(X \cup x) \cap (X^* \setminus \dot{U})] = (X \setminus x) \cap (X^* \setminus \dot{U}) \Rightarrow \mathrm{cl}[X \cap (X \setminus \dot{U})] = X \cap (X \setminus \dot{U}) \Rightarrow \mathrm{cl}(X \setminus (\dot{U} \cap X)) = X \setminus \dot{U} \cap X \Rightarrow \dot{U} \cap X$ is neutrosophic T - open.

Theorem 4.2. Consider grill $\mathcal{G}$ on a NTS (X, T) where each NP $x \in X$, $x \notin \mathcal{G}$. Adjoin to X a new NP y where $y \notin X$. Then $\exists$ a NT on $X^* = X \cup y$ obeying below:
(a) X^* is neutrosophicT_1.
(b) X is neutrosophic dense in X^*.

Proof. Take (X^*, T^*) like as Theorem 4.1. For NP x where $x \in X$, $f(x) = x$ as $\mathrm{cl}(x) = x \notin \mathcal{G}$ and $f(y) = \mathrm{cl}(y) \cup y = y$. It shows (a). Again $\mathrm{cl}X = X \in \mathcal{G}$, $f(X) = \mathrm{cl}X \cup y = X \cup y = X^*$, showing (b).

Theorem 4.3. Consider $\mathcal{G}$ on neutrosophic Housdroff space (W, $\mathcal{H}$) where for every NP α, $\alpha \in W$, $\alpha \notin \mathcal{G}$.. If U is open neighbourhood of α where $\mathrm{cl}U \notin \mathcal{G}$, then $\mathcal{H}^* = \mathcal{H} \cup \beta$ ($\beta \notin \mathcal{H}$) obeying as below:

(a) $\mathcal{H}^*$ is neutrosophic Hausdorff.

(b) $\mathcal{H}$ is neutrosophic dense in $\mathcal{H}^*$.

Proof. Take (W^*, $\mathcal{H}^*$) of Theorem 4.1. We note $\mathrm{cl}W \in G$, T^* - $\mathrm{cl}W = W^*$ and so (b). For (a), α, β as distinct NPs of W. For neutrosophic Hausdorffness of (W, $\mathcal{H}$)), α and β are strongly separated through $\ddot{U}$, $\tilde{V}$ which are NOSs in W and W^*. By hypothesis, for any NP α, there is a neutrosophic $\mathcal{H}$ - open neighbourhood $\ddot{U}$ of α such that $\mathrm{cl}\ddot{U} \notin \mathcal{G}$. Let $N = W^* \setminus \ddot{U}$. Since $\mathrm{cl}\ddot{U} \notin \mathcal{G}$, we have $f(\ddot{U}) = \ddot{U}$. Thus N is a neutrosophic open neighbourhood of β in W^*. Consequently, $\ddot{U}$ and N are the required disjoint $\mathcal{H}^*$ - open neutrosophic neighbourhoods of α and β respectively in W^*. Hence W^* is Hausdorff, proving (a).

Now we state the following theorem without proof. The theorem can be proved with the help of general techniques.

Theorem 4.4. Consider a neutrosophic Hausdorff space (W, $\mathcal{H}$) which is neutrosophic locally compact,. Introducing NP α (where $\alpha \notin W$), construct extension space $W^* = W \cup \alpha$ having the following:

(a) W^* is neutrosophic Hausdorff.

(b) W is neutrosophic dense in W^*.

(c) W^* is neutrosophic compact.

5. Conclusion:

In this article, we have defined compactness in neutrosophic topological space with respect to a grill. We have called it neutrosophic $\mathcal{G}$ - compactness. We have investigated some basic properties of this newly defined compactness. Some characterization theorems of neutrosophic $\mathcal{G}$ - compactness have also been established in neutrosophic topological space. We have also established the relationship of neutrosophic $\mathcal{G}$ - compactness with neutrosophic H - closed space and other known things. Lastly we have proposed a new method of construction of neutrosophic Hausdorff space and investigated some of its basic properties.

6. Future motivation of the work:

It is expected that the work done will motivate in further investigation of the compactness in neutrosophic topological space with respect to grills. It is hoped that the notion of compactness which have been discussed here can also be extended to neutrosophic supra bi tri, soft, multiset topological spaces, etc.

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SOME STATISTICAL PROPERTIES OF FUZZY REAL-VALUED DOUBLE SEQUENCES

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ABSTRACT: In this article we have studied on the notion of statistically pre-Cauchy fuzzy real-valued double sequences and defined it with Orlicz function. We have proved some relations between statistically Cauchy sequence and statistically pre-Cauchy sequences and established some results in terms of Orlicz function in a different approach.

KEY WORDS: pre-Cauchy, statistically Cauchy, Orlicz function, fuzzy double sequence.

AMS CLASSIFICATION NO. **40A05; 40A25; 40A30; 40C05.**

1. INTRODUCTION AND PRELIMINARIES

The idea Statistical convergence was introduced by Steinhaus [29] and Fast [11] and then reintroduced by Schoenberg [28]. Thereafter different classes of sequences were introduced based on this concept and has become an active area of research such as summability theory, number theory, Fourier analysis and Banach spaces etc. Later on it was studied with summability theory by Fridy [12], Kwon [14], Nuray [20], Salat [27] and some others. In the beginning the study was restricted to real or complex sequences. In 2000 Kwon [14], Nuray [20] and Savas [22] extended the idea to apply to sequences of fuzzy numbers. Bilgin [2] has introduced Δ-statistical and strong Δ-Cesaro convergence of sequences of fuzzy real numbers.

The concept, statistical convergence is based on the natural density of the set of positive integers which is a subset of the set of natural numbers $\mathbb{N}$.

The natural density of a subset *F* of $\mathbb{N}$ is denoted by δ(*K*) and defined by

δ(*F*) = $\lim_{n\to\infty}\frac{1}{n}\left|k<n : k\in F\right|$, where the vertical bars represent the cardinality of the enclosed set.

If $F\subset\mathbb{N}$ is a finite set, then δ(*F*) = 0 and for any set $F\subset\mathbb{N}$, δ(F^c) = 1- δ(*F*).

A sequence *x* = (x_k) of real number is said to be statistically convergent to the number ξ if for every ε > 0, the set $(\{k\in N : |x_k-\xi|\geq\varepsilon\})$ has natural density zero i.e. δ $(\{k\in N : |x_k-\xi|\geq\varepsilon\})$ = 0. Mursaleen and Osama [18] extended this idea to double sequences and established a relation between statistical convergence and strongly Cesàro summable double sequences.The idea statistically pre-Cauchy for real valued sequence was introduced by Connor, Fridy and Kline [5] and established some important results. Colak et al. [4] studied fuzzy Δ-statistically pre-Cauchy and established some properties defied by modulus function.

Definition 1.1 A sequence of real numbers, *x* = (x_k) is said to be statistically pre-Cauchy if for very ε>0, $\lim_{m\to\infty}\frac{1}{m^2}\left|(p,q)<m : |x_p-x_q|\geq\varepsilon\right|$ = 0.

Tripathy [31] introduced the concept of density for subsets of $\mathbb{N}\times\mathbb{N}$ as follows:

Definition 1.2 The density ρ(*F*), of a subset *F* of $\mathbb{N}\times\mathbb{N}$ is defined by $\rho(F)=\lim_{p,q\to\infty}\frac{1}{pq}\sum_{i\leq p}\sum_{j\leq q}\chi_F(i,j)$ and also ρ(F^c) = ρ($\mathbb{N}\times\mathbb{N}$ - *F*) = 1- ρ(*F*).

Definition 1.3 A double sequence of real numbers <a_{nk}> is said to be statistically convergent to a real number *K* if for a given $\varepsilon>0$, the natural density is zero i.e. $\rho(\{(n,k) : |a_{nk}-K|\geq\varepsilon\})=0$.

An Orlicz function M is defined by the mapping M: $[0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If the convexity is replaced by sub-additive property i.e. $M(x + y) \leq M(x) + M(y)$ then it is a modulus function. It is to be noted that $M(0) = 0$ and $M(kx) \leq kM(x)$ for all λ with $0 < k < 1$.

Different sequence spaces with Orlicz function are studied from by some authors. Among them we may mention [8, 9, 26]. Khan and Lohani [13] studied statistically pre-Cauchy with Orlicz function.

The concepts of fuzzy sets and its different operations were introduced by L. A. Zadeh. Subsequently several authors have studied different aspects of theory and application of fuzzy sets such as fuzzy topology, relations and fuzzy orderings, fuzzy measures theory, fuzzy mathematical programming, fuzzy sequences etc. Matloka [16] studied on bounded and convergent sequences of fuzzy numbers and established some important properties. Some remarkable contributions on sequences of fuzzy numbers were found in Nanda [19], Nuray [20], Kwon [14], Savas [23], Wu and Wang [34], Bilgin [2], Basarır and Mursaleen [3], Aytar [1], Fang and Huang [10], Esi [7], Subramanian and Esi [30] and many others.

A fuzzy real number Z is a fuzzy set on $\mathbb{R}$, more precisely can be defined by the mapping Z: $\mathbb{R} \rightarrow I$ (= [0, 1]), which associates each real number t with its grade of membership $Z(t)$ and satisfy the following three conditions:

(i) Z is normal if there exists $t \in \mathbb{R}$ such that $Z(t) = 1$.

(ii) Z is upper semi continuous if for each $\varepsilon > 0$, $Z^{-1}([0, a+\varepsilon))$, is open in the usual topology of $\mathbb{R}$, for all $a \in I$.

(iii) X is convex, if $Z(t) \geq Z(s) \wedge Z(r) = min\ (Z(s), Z(r))$, where $s < t < r$.

We denote the class of all upper-semi-continuous, normal, convex fuzzy real numbers by $R(I)$ and $R(I)^*$ denotes the set of all positive fuzzy real numbers.

Each real number $r\in\mathbb{R}$ can be represent as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(t) = \begin{cases} 1, \text{ for } t = r, \\ 0, \text{ otherwise.} \end{cases}$$

Thus we can say that R can be embedded into $R(I)$. The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Since each fuzzy number X is express as grade of membership, the α-level set of X is defined as follows

$$[X]_\alpha = \begin{cases} \{t\in R : X(t)\geq\alpha\},\ for\ 0<\alpha\leq 1 \\ \overline{\{t\in R : X(t)>\alpha\}},\ for\ \alpha=0. \end{cases}$$

Let L be the set of all closed bounded intervals $Z = [Z^L, Z^R]$ then we write $Z \leq Y$ if and only if $Z^L \leq Y^L$ and $Z^R \leq Y^R$. We can write

$$d(Z, Y) = \max\{|Z^L-Y^L|, |Z^R-Y^R|\}, \text{ where } Z = [Z^L, Z^R] \text{ and } Y = [Y^L, Y^R].$$

It is to be noted that (L, d) is a complete metric space.

We consider a mapping $\bar{d} : R(I) \times R(I) \to \mathbb{R}$ defined by

$$\bar{d}(Z, Y) = \sup_{0\leq\alpha\leq 1} d(Z^\alpha, Y^\alpha), \text{ for } Z, Y\in R(I),$$

and clearly $\bar{d}$ is a metric on $R(I)$.

A fuzzy real-valued double sequence is a double infinite array of fuzzy real numbers denoted by $<Z_{nk}>$, where Z_{nk} are fuzzy real numbers for each $n, k\in N$.

Definition 1.4 A fuzzy real-valued double sequence $<Z_{nk}>$ is said to converge in Pringsheim's sense to a fuzzy real number Z, if for a given $\varepsilon > 0$, there exist real numbers $n_0 = n_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$, so that $\bar{d}(Z_{nk}, Z) < \varepsilon$ for all $n \geq n_0$ and $k \geq k_0$.

Definition 1.5 A fuzzy real-valued double sequence $<Z_{nk}>$ is said to be *bounded* if there exist $\mu \in R(I)^*$ such that $|Z_{nk}| \leq \mu$ for all $n, k \in N$ i.e. $\sup_{n,k} d(Z_{nk}, \bar{0}) < \infty$.

Definition 1.6 A fuzzy real-valued double sequence $<Z_{nk}>$ is said to be statistically convergent to a fuzzy real number Z, if for a given $\varepsilon > 0$, $\rho\left(\left\{(m,n) : d(Z_{nk}, Z) \geq \varepsilon\right\}\right) = 0$.

Savas [25], Tripathy and Dutta ([32, 33]), Dutta [6] studied some important classes of fuzzy real-valued double sequences. Savas and Mursaleen [24] introduced statistically convergent and statistically Cauchy for double sequences of fuzzy numbers.

We define the following notion for fuzzy real valued double sequences.

Definition 1.7 A fuzzy real-valued double sequence $Z = (Z_{ij})$ is said to be statistically pre-Cauchy if

$$\lim_{n,k} \frac{1}{n^2 k^2} \left|(i, j) : i \leq n,\ j \leq k,\ \bar{d}(Z_{ij}, Z_{pq}) \geq \varepsilon\right| = 0.$$

If there exist a $E \subseteq N \times N$ such that it contains '*almost all n and k*' with

$\lim_{p,q \to \infty} \frac{1}{p^2 q^2} \sum_{n \leq p} \sum_{k \leq q} \chi_E(n,k) = 0$, then the sequence is said to be statistically pre-Cauchy whenever $\bar{d}(Z_{ij}, Z_{pq}) < \varepsilon$ for *almost all i and j*.

2. MAIN RESULTS

Theorem 2.1 A statistically convergent fuzzy real valued double sequence is statistically pre-Cauchy. But the converse may not be true.

Proof. Let (X_{nk}) be statistically convergent fuzzy real-valued double sequence. Let $F \subset \mathbb{N}\times\mathbb{N}$ and choose $V \subset F$ such that $F \setminus V$ is finite and $V \subset \{(i, j) : i \le n,\ j \le k, \bar{d}(Z_{ij}, Z_{pq}) < \varepsilon\}$, for some $\varepsilon > 0$. Clearly $\rho(V) = 1$ and we have

$$(\rho(V))^2 = \lim_{p,q\to\infty} \frac{1}{p^2q^2} \sum_{n\le p}\sum_{k\le q} \chi_V(n,k)$$

$$\le \lim_{n,k} \frac{1}{n^2k^2} \left|(i, j) : i \le n,\ j \le k, \bar{d}(Z_{ij,} Z_{pq}) < \varepsilon\right|.$$

Since $\rho(V) = 1$, it follows that

$$\lim_{n\to\infty} \frac{1}{n^2k^2} \left|(i, j) : i \le n,\ j \le k, \bar{d}(Z_{ij,} Z_{pq}) < \varepsilon\right| = 1.$$

Thus we conclude that (Z_{nk}) is statistically pre-Cauchy.

Theorem 2.2 A bounded fuzzy real-valued double sequence (Z_{nk}) is statistically pre-Cauchy if and only if

$$\lim_{n\to\infty} \frac{1}{p^2q^2} \sum_{n\le p}\sum_{k\le q} \bar{d}(Z_{nk}, Z_{ij}) = 0.$$

Proof. First suppose that $\lim_{n\to\infty} \frac{1}{p^2q^2} \sum_{n\le p}\sum_{k\le q} \bar{d}(Z_{nk}, Z_{ij}) = 0$. For each $\varepsilon > 0$, we have

$$\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{n\le p}\sum_{k\le q}\bar{d}(Z_{nk},Z_{ij})\ge \quad \varepsilon \quad \left(\frac{1}{p^2q^2}\left|(n,k): i\le p,\ j\le q,\bar{d}(Z_{nk,}Z_{ij})<\varepsilon\right|\right)\ge$$

Thus $\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{n\le p}\sum_{k\le q}\bar{d}(Z_{nk},Z_{ij}) = 0$.

This implies that (Z_{nk}) is statistically pre-Cauchy.

Conversely, let us suppose that (Z_{nk}) is statistically pre-Cauchy bounded fuzzy real-valued double sequence.

Then $\sup_{n,k}\bar{d}(Z_{nk},\bar{0}) = K < \infty$ and for $n, k \in N$, we have

$$\frac{1}{p^2q^2}\sum_{n\le p}\sum_{k\le q}\bar{d}(Z_{nk},Z_{ij})\le \frac{\varepsilon}{2} + 2K\left(\frac{1}{p^2q^2}\left|(n,k): i\le p,\ j\le q,\bar{d}(Z_{nk,}Z_{ij})\ge \frac{\varepsilon}{2}\right|\right)$$

$< \varepsilon$, for all $n, k > P\in N$.

Thus $\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{n\le p}\sum_{k\le q}\bar{d}(Z_{nk},Z_{ij}) = 0$.

Hence the proof.

Theorem 2.3 Let $Z = (Z_{nk})$ is statistically pre-Cauchy fuzzy real-valued double sequence. If the sequence Z has a subsequence $(Z_{n_ik_j})$ converging to L and $\liminf_{p,q}\frac{1}{pq}\,|\{(n_i,k_j): n_i\le p, k_j\le q; i, j\in N\}| > 0$, then Z is statistically convergent to L.

Proof. Let $u = n_i$ and $v = k_j$ be such that $u, v > K\in N$ for some i and j. Thus for a given $\varepsilon>0$, $\bar{d}(Z_{uv},L)<\frac{\varepsilon}{2}$. Consider $A = \{(n_i, k_j): n_i, k_j > K; i, j\in N\}$ and $B = \{(i, j): \bar{d}(Z_{ij},L)\ge\varepsilon\}$, then

$$\frac{1}{p^2q^2}\left|(i,\, j) : i \le p,\, j \le q; \bar{d}(Z_{ij}, Z_{uv}) \ge \frac{\varepsilon}{2}\right| \ge \frac{1}{p^2q^2}\sum_{i,u\le p}\sum_{j,v\le q}\chi_{A\times B}(i,\, j)$$

$$=\left(\frac{1}{pq}\,|\,\{(n_i, k_j) : n_i \le p, k_j \le q; i, j \in N\}\,|\right)$$

$$\left(\frac{1}{pq}\,|\,(i,\,\mathrm{j}) : i \le p,\, j \le q; |\, Z_{\mathrm{i\,j}} - L\,| \ge \varepsilon\,|\right)$$

Since Z is statistically pre-Cauchy, the L.H.S of the above inequality is zero and since

$$\liminf_{p,q} \frac{1}{pq}\,|\,\{(n_i, k_j) : n_i \le p, k_j \le q; i, j \in N\}\,| > 0,$$

we have

$$\lim_{p,q}\left(\frac{1}{pq}\,|\,(i,\,\mathrm{j}) : i \le p,\, j \le q; |\, Z_{\mathrm{i\,j}} - L\,| \ge \varepsilon\,|\right) = 0.$$

This implies that Z is statistically convergent to L.

Theorem 2.4 Let $Z = (Z_{nk})$ be a fuzzy real-valued double sequence and M be a bounded Orlicz function. Then Z is statistically pre-Cauchy if and only if

$$\lim_{n\to\infty} \frac{1}{p^2q^2}\sum_{n,i\le p}\sum_{k,j\le q} M\left(\frac{\bar{d}(Z_{nk}, Z_{ij})}{\rho}\right) = 0, \text{ for some } \rho > 0.$$

Proof. Let $\lim_{n\to\infty} \frac{1}{p^2q^2}\sum_{n,i\le p}\sum_{k,j\le q} M\left(\frac{\bar{d}(Z_{nk}, Z_{ij})}{\rho}\right) = 0$, for some $\rho > 0$. We have

$$\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{n,i\leq p}\sum_{k,j\leq q}M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right) = \lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{\substack{n,i\leq p\\ \bar{d}(X_{nk},X_{ij})\leq\varepsilon}}\sum_{k,j\leq q}M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right) +$$

$$\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{\substack{n,i\leq p\\ \bar{d}(X_{nk},X_{ij})\geq\varepsilon}}\sum_{k,j\leq q}M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right) \geq$$

$$\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{\substack{n,i\leq p\\ \bar{d}(X_{nk},X_{ij})\geq\varepsilon}}\sum_{k,j\leq q}M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right)$$

$$\geq M(\varepsilon)\left(\frac{1}{p^2q^2}\,|\,(n,k):\bar{d}(Z_{nk},Z_{ij})\geq\varepsilon, n\leq p, k\leq q\,|\right)$$

Since the left hand side is zero, so it gives

$$\frac{1}{p^2q^2}\,|\,(n,k):\bar{d}(Z_{nk},Z_{ij})\geq\varepsilon, n\leq p, k\leq q\,| = 0.$$

Conversely, we assume that Z is statistically Cauchy. Then for a given $\varepsilon > 0$, we choose δ such that $M(\delta) < \frac{\varepsilon}{2}$ and since M is bounded, there exist a positive integer K such that $M(Z) < \frac{K}{2}$, for all $Z \geq 0$.

Thus we have

$$\frac{1}{p^2q^2}\sum_{n,i\leq p}\sum_{k,j\leq q}M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right) = \frac{1}{p^2q^2}\sum_{\substack{n,i\leq p\\ \bar{d}(X_{nk},X_{ij})\leq\delta}}\sum_{k,j\leq q}M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right) +$$

$$\frac{1}{p^2q^2}\sum_{\substack{n,i\le p\\ \bar{d}(X_{nk},X_{ij})\ge\delta}}\sum_{k,j\le q} M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right)$$

$$\le M(\delta) + \frac{1}{p^2q^2}\sum_{\substack{n,i\le p\\ \bar{d}(X_{nk},X_{ij})\ge\delta}}\sum_{k,j\le q} M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right)$$

$$\le \frac{\varepsilon}{2} + \frac{K}{2}\left(\frac{1}{p^2q^2}\,|(n,k):\bar{d}(Z_{nk},Z_{ij})\ge\delta, n\le p, k\le q|\right)$$

$< \varepsilon$, since Z is statistically pre-Cauchy.

Thus we have

$$\lim_{n\to\infty}\frac{1}{p^2q^2}\sum_{n,i\le p}\sum_{k,j\le q} M\left(\frac{\bar{d}(Z_{nk},Z_{ij})}{\rho}\right) = 0.$$

Theorem 2.5 Let $Z = (Z_{nk})$ be a fuzzy real-valued double sequence and M be a bounded Orlicz function. Then Z is statistically convergent to L if and only if

$$\lim_{n\to\infty}\frac{1}{pq}\sum_{n=1}^{p}\sum_{k=1}^{q} M\left(\frac{\bar{d}(Z_{nk},L)}{\rho}\right) = 0.$$

Proof. The proof of this theorem is similar to the previous theorem, so we omit the proof.

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Regular Cesάro Summability of Double Sequences of Bicomplex Numbers

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Abstract

The concept of regular Cesάro summability of double sequences of bicomplex number is presented in this article. The classes of bicomplex sequence space ${}_2b(Ces, c^R)$ and ${}_2b(Ces, c_0^R)$ are introduced and discuss about their characteristics such as solidity, symmetry, completeness, and monotonicity.

Keywords: Double Sequence, bicomplex numbers, Regular convergence, Cesάro summability

MSC 2010 Classification: 46A45, 46B45, 40A05

1. Introduction

Real or complex numbers can be arranged in infinite sequences within a sequence space. Under pointwise addition and scalar multiplication, it is a linear space. Numerous perspectives are being used to study sequence spaces. Summability theory is the study of linear transformations on sequence spaces. In a 1713 letter to C. Wolf, G. Leibniz may have introduced the idea of summability theory. According to G. Leibniz, the oscillatory sequence 1-1+1-1+.. has a sum of 1/2. But G. Leibniz established the requirements for alternating series convergence two years later. The summability theory expanded the field of sequence space research and brought about a significant development. The arithmetic mean approach of summability was first introduced by F. G. Frobenius in 1880. Moreover, in 1890, the technique was generalized by E. Cesάro as the (C, k) method of summability.

For a considerable amount of time, there has been much research conducted on bicomplex numbers. Segre [12] first proposed the idea of bi-complex numbers in 1892. These numbers form an algebra that is isomorphic to the tessarines. Numerous scholars have also studied bi-complex number sequences, including Srivastava and Srivastava [15], Wagh [17], Sager and Sagir [11], and Rochon and Shapiro [10].

Throughout $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ denote the set of real, complex, and bi-complex numbers respectively.

$\xi = z_1 + jz_2 = x_1 + ix_2 + jx_3 + ijx_4$ is the bi-complex number given by Segre [12]. Here, z_1, z_2 are independent units whereas x_1, x_2, x_3, and x_4 are $\mathbb{C}_0$ units. The independent units i, j are such that $i^2 = j^2 = -1$ and $ij = ji$. $\mathbb{C}_2$ represents the set of bi-complex numbers: $\mathbb{C}_2 = \{\xi : \xi = z_1 + jz_2;\, z_1, z_2 \in \mathbb{C}_1(i)\}$, where $\mathbb{C}_1(i) = \{x_1 + ix_2 : x_1, x_2 \in \mathbb{C}_0\}$. Over $\mathbb{C}_1$(i), $\mathbb{C}_2$ is a vector space. $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ are the idempotent elements in $\mathbb{C}_2$, satisfying the relations $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. The unique expression for any bi-complex number $\xi = z_1 + jz_2$ is the product of e_1 and e_2, so $\xi = z_1 + jz_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2 = \mu_1 e_1 + \mu_2 e_2$, where $\mu_1 = (z_1 - iz_2)$ and $\mu_2 = (z_1 + iz_2)$.

A double infinite array of numbers is used to express a double sequence, denoted as (x_{nk}). The concept of convergence of double sequences was first developed in 1900 by Pringsheim [9]. A few early studies on double sequence spaces can be found in Bromwich [2] monograph. Hardy [5] established the idea of regular convergence of double sequences. Over the past century, a great deal of development work on double sequences has been done since then. Numerous scholars have presented diverse types of double sequences and examined their distinct characteristics. Numerous researchers have looked at the double sequence from different perspectives, including Devi and Tripathy [3, 4], Tripathy and Sarma [16], Basarir and Sonalcan [1], and many more.

A double sequence (x_{nk}) is convergent in Pringsheim's sense if x_{nk} converges to the limit L, as n and k tend to ∞ not dependent on one another. i.e.,

$$\lim_{n,k\to\infty} x_{nk} = L.$$

It is found in Moricz [6] that (x_{nk}) converges in Pringsheim's sense if and only if for every $\varepsilon > 0$, there exists an integer $n_o = n_o(\varepsilon)$ such that

$$\left|x_{ij} - x_{nk}\right| \leq \varepsilon, \text{ for all } \min\,(i, j, n, k) \geq n_o.$$

Convergence of ordinary sequence always implies boundedness of the sequence whereas Pringsheim's sense convergence of double sequences doesn't ensure boundedness of the double sequences. The notion of regular convergence of double sequences was introduced by Hardy [5].

A double sequence (x_{nk}) is regularly convergent if (x_{nk}) converges in Pringsheim's sense as well as the limits

$$\lim_{n\to\infty} x_{nk} = L_k, \text{for each } k \in N \text{ and}$$

$$\lim_{k\to\infty} x_{nk} = M_n\,, \text{for each } n \in N \text{ exist.}$$

If the limits $L = L_k = M_n = 0,$ for all $n, k \in N$ in the above definition, we get the definition of regular null double sequences.

Therefore, the definition is identical to

$$\lim_{\max\,(n,k)\to\infty} x_{nk} = 0.$$

Shiue [14] developed the Cesáro sequence space Ces_∞, Ces_p $(1 < p < \infty)$, and it has been demonstrated that $\ell_\infty \subset Ces_p$ is tight for $1 < p < \infty$. Subsequently, the non-absolute Cesáro sequence spaces X_p and X_∞ were Ng and Lee defined [7, 8]. The classes of Cesáro convergence were defined by Sever and Altay [13] in the sense of Pringsheim's sense.

2. Definitions and preliminaries

Definition 2.1. Let E be a subset of ${}_2b(\Omega)$. Then E is called

i) Normal/Solid: If $(x_{mn})\in E \Rightarrow (y_{mn})\in E$, for all (y_{mn}) such that $|y_{mn}| \le |x_{mn}|$, for all $m, n \in \mathbb{N}$.

ii) Monotone: If E contains the canonical pre-images of all its step spaces.

iii) Symmetric: if $(x_{mn})\in E \Rightarrow (x_{\pi(m,n)}) \in E$, for all $m, n \in \mathbb{N}\times\mathbb{N}$.

The bi-complex double-sequence ζ_{mn} is expressed as $\zeta_{mn} = a_{mn} + ib_{mn} + jc_{mn} + ijd_{mn}$.

Definition 2.2. A bi-complex double sequence $\zeta = (\zeta_{mn})$ is said to be regular Cesáro summable if for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{s,t=1}^{m,n} \zeta_{st} = \xi\,, for\ all\ m, n \ge n_0;$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \zeta_{mt} = \xi_m\,, for\ all\ n \ge n_0;$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{s=1}^{m} \zeta_{sn} = \nu_n\,, for\ all\ m \ge n_0.$$

Definition 2.3. A bicomplex double sequence $\zeta = (\zeta_{mn})$ is said to be regular Cesáro null if for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st} = 0\,, for\ all\ m,n \geq n_0;$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \zeta_{mt} = 0\,, for\ all\ n \geq n_0;$$

$$\lim_{m\to\infty} \frac{1}{m}\sum_{s=1}^{m} \zeta_{sn} = 0\,, for\ all\ m \geq n_0.$$

Throughout the article, the classes of all regular Cesáro summable and regular Cesáro null is denoted by ${}_2b(Ces, c^R)$ and ${}_2b(Ces, c_0^R)$ respectively.

1. Main Results

Theorem 3.1. The sequence spaces ${}_2b(Ces, c^R)$ and ${}_2b(Ces, c_0^R)$ are linear spaces.

Proof. (i) Let α, β be the scalars and $\{(\zeta_{mn}), (\eta_{mn})\} \in {}_2b(Ces, c^R)$.

Then,

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st} = \xi\,, for\ all\ m,n \geq n_0;$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\zeta_{mt} = \xi_m\,, for\ all\ n \geq n_0;$$

$$\lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}\zeta_{sn} = \nu_n\,, for\ all\ m \geq n_0.$$

Similarly,

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{s,t=1}^{m,n}\eta_{st} = \rho\,, for\ all\ m,n \geq n_0;$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\eta_{mt} = \rho_m\,, for\ all\ n \geq n_0;$$

$$\lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}\eta_{sn} = \mu_n\,, for\ all\ m \geq n_0.$$

Then, $\lim_{m,n\to\infty}\frac{1}{mn}\sum_{s,t}^{m,n}(\alpha\zeta_{mn} + \beta\eta_{mn}) = \lim_{m,n\to\infty}\frac{1}{mn}\sum_{s,t}^{m,n}\alpha\zeta_{mn} + \lim_{m,n\to\infty}\frac{1}{mn}\sum_{s,t}^{m,n}\beta\eta_{mn}$

$= \alpha\lim_{m,n\to\infty}\frac{1}{mn}\sum_{s,t}^{m,n}\zeta_{mn} + \beta\lim_{m,n\to\infty}\frac{1}{mn}\sum_{s,t}^{m,n}\eta_{mn}.$

Similarly,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}(\alpha\zeta_{mt} + \beta\eta_{mt}) = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}(\alpha\zeta_{mt}) + \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}(\beta\eta_{mt})$$

$$= \alpha\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}(\zeta_{mt}) + \beta\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}(\eta_{mt}).$$

And, $\lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}(\alpha\zeta_{sn}+\beta\eta_{sn}) = \lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}(\alpha\zeta_{sn}) + \lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}(\beta\eta_{sn})$

$$= \alpha\lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}(\zeta_{sn}) + \beta\lim_{m\to\infty}\frac{1}{m}\sum_{s=1}^{m}(\eta_{mt}).$$

Hence, ${}_2b(Ces, c^R)$ is a linear space.

(ii) ${}_2b(Ces, c_0^R)$ follows the same.

Theorem 3.2. The sequence spaces ${}_2b(Ces, c^R)$ and ${}_2b(Ces, c_0^R)$ are normed spaces under the norm

$$\|\zeta\| = sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}\zeta_{st}\right|, \text{for all } m,n \in \mathbb{N} \ldots\ldots..(1)$$

Proof. Let $(\zeta_{mn}) \in {}_2\, b(Ces, c^R)$.

(i) It is obvious that $\zeta_{mn} = 0 \Leftrightarrow \|\zeta_{mn}\| = 0$.

(ii) $\|\zeta_{mn}+\eta_{mn}\| = sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\zeta_{st}+\eta_{st})\right|$

$$\leq sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\zeta_{st})\right| + sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\eta_{st})\right|$$

$$\leq \|\zeta_{mn}\| + \|\eta_{mn}\|.$$

(iii) $\|\lambda\zeta_{mn}\| = sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\lambda\zeta_{st})\right| = \lambda sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\zeta_{st})\right| = \lambda\|\zeta_{mn}\|$.

Hence, ${}_2b(Ces, c^R)$ is a normed space.

Similarly, ${}_2b(Ces, c_0^R)$ is also a normed space.

Theorem 3.3. The bicomplex double sequence spaces ${}_2b(Ces, c_0^R)$ and ${}_2b(Ces, c^R)$ are complete wr.t. normed defined in Equation (1).

Proof. Let $(\zeta_{mn}^l) \in {}_2b(Ces, c^R)$ be a Cauchy sequence. Then, for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$\|\zeta_{mn}^l - \zeta_{mn}^r\| < \varepsilon, \text{ for all } l, r \geq n_0.$$

This implies, $sup\left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\zeta_{st}^l - \zeta_{st}^r)\right| < \varepsilon$.

$\Rightarrow \left|\frac{1}{mn}\sum_{s,t=1}^{m,n}(\zeta_{st}^l - \zeta_{st}^r)\right| < \varepsilon$ ……………………….. (2)

$\Rightarrow \frac{1}{mn}\sum_{s,t=1}^{m,n}\zeta_{st}^l = s_{mn}^l$ is a Cauchy sequence in $\mathbb{C}_2$.

$\Rightarrow \lim_{l\to\infty} s_{mn}^l = s_{mn}$

When $r \to \infty$ in Equation (2), we get $|s_{mn}^l - s_{mn}| < \varepsilon$, for all $l \geq n_0$.

Now, $s_{mn} = |s_{mn}^l - s_{mn}^l + s_{mn}| \leq |s_{mn}^l| + |s_{mn}^l - s_{mn}| \leq (|s_{mn}^l| + \varepsilon) \in {}_2b(Ces, c^R)$.

Hence, ${}_2b(Ces, c^R)$ is a normed space.

Similarly, ${}_2b(Ces, c_0^R)$ is also a normed space.

Result 3.1. The sequence spaces ${}_2b(Ces, c_0^R)$ and ${}_2b(Ces, c^R)$ are not monotone.

The result follows from the example below.

Example 3.1. Consider the bicomplex double sequence (ζ_{mn}) defined by

$$\zeta_{mn} = 3^{mn} + j3^{mn} + ij3^{mn}, for\ all\ m,n \in \mathbb{N}.$$

Then, $s_{mn} = \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st} = 3^{mn} + j3^{mn} + ij3^{mn}, for\ all\ m,n \in \mathbb{N}.$

This implies

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st} = 3 + j3 + ij3;$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \zeta_{mt} = 3 + j3 + ij3\,;$$

$$\lim_{m\to\infty} \frac{1}{m}\sum_{s=1}^{m} \zeta_{sn} = 3 + j3 + ij3.$$

Hence, $(\zeta_{mn}) \in {}_2b(Ces, c^R)$.

Consider (η_{mn}), the preimage of (ζ_{mn}) defined by

$$\eta_{mn} = \begin{cases} 3^{mn} + j3^{mn} + ij3^{mn}, & for\ all\ n = odd \\ 0, & for\ all\ n = even. \end{cases}$$

The Cesrao transformation of $\eta_{mn} = \frac{1}{mn}\sum_{s,t=1}^{m,n}\eta_{st}$ does not converge in Pringsheim's sense since,

$$\frac{1}{mn}\sum_{s,t=1}^{m,n}\eta_{st} = \begin{bmatrix} 3 & \frac{3}{2} & 2 & \frac{3}{2} & \frac{9}{5} & .. \\ 3 & \frac{3}{2} & 2 & \frac{3}{2} & \frac{9}{5} & .. \\ 3 & \frac{3}{2} & 2 & \frac{3}{2} & \frac{9}{5} & .. \\ 3 & \frac{3}{2} & 2 & \frac{3}{2} & \frac{9}{5} & \ldots \\ \ldots. & \vdots & .. & \vdots & \vdots & .. \end{bmatrix}$$

does not tend to a unique limit when $m, n \to \infty$.

Also, $\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\eta_{mt}$ does not exist.

This implies, $(\eta_{mn}) \notin {}_2b(Ces, c^R)$.

Hence, the sequence space ${}_2b(Ces, c^R)$ is not monotone.

Similarly, ${}_2b(Ces, c_0^R)$ is also not monotone.

Result 3.2. The sequence spaces ${}_2b(Ces, c_0^R)$ and ${}_2b(Ces, c^R)$ are not symmetric.

The result follows from the example below.

Example 3.2. Consider the bicomplex double sequence (ζ_{mn}) defined by

$$\zeta_{mn} = \begin{cases} 1, & \text{for all } m+n = even, m, n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

when n is odd.

And,

$$\zeta_{mn} = \begin{cases} 1, & \text{for all } m + n = odd, m, n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

when n is even.

Then, the Cesaro transformation, $s_{mn} = \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st}$ is

$$\frac{1}{mn}\sum_{s,t=1}^{m,n} \eta_{st} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{2}{3} & \frac{1}{2} & \frac{3}{5} & \frac{1}{2} & \frac{4}{7} & \cdots & \square \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & & \square \\ & & & \vdots & & & & \ddots & \vdots \\ & & & \square & & & & \cdots & \square \end{bmatrix}.$$

This implies

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st} = \frac{1}{2} + i\frac{1}{2} + j\frac{1}{2} + ij\frac{1}{2};$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \zeta_{mt} = \frac{1}{2} + i\frac{1}{2} + j\frac{1}{2} + ij\frac{1}{2};$$

$$\lim_{m\to\infty} \frac{1}{m}\sum_{s=1}^{m} \zeta_{sn} = \frac{1}{2} + i\frac{1}{2} + j\frac{1}{2} + ij\frac{1}{2}.$$

Hence, $(\zeta_{mn}) \in {}_2b(Ces, c^R)$.

Consider (η_{mn}), the rearrangement double sequence of (ζ_{mn}) defined by

$$\eta_{mn} = \begin{cases} 1, & \text{for all } n = 4p, p \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

The Cesrao transformation of $\eta_{mn} = \frac{1}{mn}\sum_{s,t=1}^{m,n} \eta_{st}$ does not converge in Pringsheim's sense since,

$$\frac{1}{mn}\sum_{s,t=1}^{m,n} \eta_{st} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots & \square \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & & \square \\ & & & \vdots & & & & \ddots & \vdots \\ & & & \square & & & & \cdots & \square \end{bmatrix}$$

does not tend to a unique limit when $m, n \to \infty$.

Also, $\lim_{m\to\infty} \frac{1}{m}\sum_{s=1}^{m} \eta_{sn} = \begin{cases} 1, & \text{for all } n = 4p, p \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$.

But, $\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \eta_{mt}$ does not converge to a unique limit.

This implies, $(\eta_{mn}) \notin {}_2b(Ces, c^R)$.

Hence, the sequence space ${}_2b(Ces, c^R)$ is not symmetric.

Similarly, ${}_2b(Ces, c_0^R)$ is also not symmetric.

Result 3.3. The sequence spaces ${}_2b(Ces, c_0^R)$ and ${}_2b(Ces, c^R)$ are not solid.

The result follows from the example below.

Example 3.3. Consider the bicomplex double sequence (ζ_{mn}) defined by

$$\zeta_{mn} = \begin{cases} 1, & \text{for all } n = 8p-4, p \in \mathbb{N}; \\ -1, & \text{for all } n = 8p, p \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

when n is even.

Then, $\lim\limits_{m,n\to\infty} \frac{1}{mn}\sum_{s,t=1}^{m,n} \zeta_{st} = 0 = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \zeta_{mt}$.

Also,

$$\lim_{m\to\infty} \frac{1}{m}\sum_{s=1}^{m} \zeta_{sn} = \begin{cases} 1, & \text{for all } n = 8p-4, p \in \mathbb{N}; \\ -1, & \text{for all } n = 8p, p \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the bicomplex sequence of scalars, (α_{mn}) defined by $\alpha_{mn} = \begin{cases} 1, & \text{for } \alpha_{14}; \\ 0, & \text{otherwise.} \end{cases}$

Then, $\frac{1}{mn}\sum_{s,t=1}^{m,n} \alpha_{st}\zeta_{st}$ is given by

$$\frac{1}{mn}\sum_{s,t=1}^{m,n} \alpha_{st}\zeta_{st} = \begin{bmatrix} 0\ 0\ 0\ \frac{1}{4}\ \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7} & \cdots & \square \\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{10}\ \frac{1}{12}\ \frac{1}{14} & \cdots & \square \\ \vdots & \ddots & \vdots \\ \square & \cdots & \square \end{bmatrix}.$$

This implies $\frac{1}{mn}\sum_{s,t=1}^{m,n}\alpha_{st}\zeta_{st}$ doesn't converge to a unique limit in Pringsheim's sense.

Hence, $(\alpha_{mn}\zeta_{mn}) \notin {}_2b(Ces, c^R)$.

Hence, the sequence space ${}_2b(Ces, c^R)$ is not solid.

Similarly, ${}_2b(Ces, c_0^R)$ is also not solid.

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D-valued *p*-harmonic convex function

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Abstract: Hyperbolic number is a most important part of bi-complex number, since the both idempotent parts are real numbers. In this article we define D-valued *p*-harmonic convex set and *D*-valued *p*-harmonic convex function and obtain some results using *D* partial order.

Keywords: Bi-complex number; Hyperbolic number; Partial order; Convex function.

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1. Introduction

Bi-complex numbers are being studied for quite a long time now. Probably Italian school of Segre[6] introduced the bi-complex numbers. For more details on bi-complex numbers and bi-complex functional analysis see [2-8]. Hyperbolic number system has been studied for various reasons. Many researchers developed the hyperbolic numbers.

1.1. Bi-complex Numbers

Segre [6] defined the bi-complex numbers as

$$\xi = z_1 + i_2 z_2$$

$$= x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4,$$

where $z_1, z_2 \in C_1$; $x_1, x_2, x_3, x_4 \in C_0$ and the independent unit i_1, i_2 are such that $i^2 = i^2 = -1$; $i_1i_2 = i_2i_1$, where C_0 and C_1 are set of real and complex numbers respectively.
The set of bi-complex numbers C_2 is defined as

$$C_2 = \{z_1 + i_2z_2 : z_1, z_2 \in C_1(i_1)\},$$

where $C_1(i_1) = \{x_1 + i_1x_2, : x_1, x_2 \in C_0\}$.
The idempotent elements in C_2 are e_1 and e_2, where $e_1 = \frac{1+i_1i_2}{2}$ and $e_2 = \frac{1-i_1i_2}{2}$.
Note that $e_1 + e_2 = 1$ and $e_1e_2 = 0$.
Every bi-complex number can be expressed uniquely as

$$\xi = z_1 + i_2z_2 = \mu_1e_1 + \mu_2e_2,$$

where $\mu_1 = z_1 - i_1z_2, \mu_2 = z_1 + i_1z_2$.
A bi-complex number $\xi = z_1 + i_2z_2$ is said to be singular if $z_1^2 + z_2^2 = 0$ and otherwise it is called non singular. The set of all singular number in C_2 is denoted by $\mathcal{O}_2$.
The Euclidean norm $\|\cdot\|$ on C_2 is defined as

$$\begin{aligned}\|\xi\|_{C_2} &= \sqrt{|z_1|^2 + |z_2|^2};\\ &= \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2};\\ &= \sqrt{\frac{|\mu_1|^2+|\mu_2|^2}{2}},\end{aligned}$$

with this norm, C_2 is a Banach space also C_2 is a commutative algebra. C_2 becomes a modified Banach algebra with respect to this norm in the sense that

$$\|\xi \cdot \eta\|_{C_2} \leq \sqrt{2}\|\xi\|_{C_2} \cdot \|\eta\|_{C_2}.$$

1.2. Hyperbolic Numbers

The number ξ of the form $\xi = x_1 + i_1i_2x_2, x_1, x_2 \in C_0$ is called hyperbolic number. The set of all hyperbolic numbers is denoted by D.
Every hyperbolic numbers can be expressed uniquely as

$$\xi = \mu_1^*e_1 + \mu_2^*e_2,$$

where $\mu_1^* = x_1 + x_2$ and $\mu_2^* = x_2 - x_1$.
Denote, $e_1D = \{\mu_1^*e_1 : \mu_1^*e_1 + \mu_2^*e_2 \in D\}$ and $e_2D = \{\mu_2^*e_1 : \mu_1^*e_1 + \mu_2^*e_2 \in D\}$.

The set of all positive hyperbolic numbers $D^{+} = \{\mu_1^* e_1 + \mu_2^* e_2 : \mu_1^*, \mu_2^* \geq 0\}$.
The norm $\|\cdot\|$ on D is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2}.$$

1.3. Partial order relation on D
Let $\xi,\ \eta \in D$, then ξ is D-larger than η, we write $\eta \leq' \xi$ if $\xi - \eta \in D^{+}$.

Noor et al[1] introduced some new concepts of p-harmonic convex sets and p-harmonic convex function. In this article we have introduced p-harmonic convex sets and p-harmonic convex function using hyperbolic numbers.

Definition 1.1. A set $E \subset D$ is said to be D convex set if for every $\xi, \eta \in D$ with $0 \leq' \alpha \leq' 1$

$$\alpha\xi + (1-\alpha)\eta \in D.$$

Definition 1.2. A function $\Upsilon_D : D \rightarrow \mathrm{D}^{+}$ is called D-valued convex function if for every $\xi, \eta \in D$ with $0 \leq \alpha \leq' 1$ such that

$$\Upsilon_D(\alpha\xi + (1-\alpha)\eta) \leq' \alpha\Upsilon_D(\xi) + (1-\alpha)\Upsilon_D(\eta).$$

Definition 1.3. A set $E \subset D$ is said to be D harmonic convex set if every $\xi, \eta \in D$ with $0 \leq' \alpha \leq' 1$ such that

$$\left(\frac{\xi\eta}{(\alpha\xi + (1-\alpha)\eta)}\right) \in E.$$

Definition 1.4. A function $\Upsilon_D : \mathcal{H}_p(\subset D \setminus \{0\}) \rightarrow \mathrm{D}^{+}$ is said to be D-valued harmonic convex function if

$$\Upsilon_D\left(\left(\frac{\xi\eta}{(\alpha\xi+(1-\alpha)\eta}\right)\right) \leq' \alpha\Upsilon_D(\xi)+(1-\alpha)\Upsilon_D(\eta).$$

Example 1.1. Let $\Upsilon_D : D \setminus \{0\} \to D^+$define by

$$\Upsilon_D(\xi) = \|\xi\|, \forall \xi \in D \setminus \{0\}.$$

Then Υ_D is *D*-valued harmonic convex function.

Definition 1.5. A set $E \subset D$ is said to be *D*-valued p-harmonic convex set if every $\xi, \eta \in D$ with $0 \leq \alpha \leq' 1$ such that

$$\left(\frac{\xi^p\eta^p}{(\alpha\xi^p+(1-\alpha)\eta^p}\right)^{\frac{1}{p}} \in E.$$

Definition 1.6. A function $\Upsilon_D : D \setminus \{0\} \to \mathrm{D}^+$ is said to be D-valued p-harmonic convex function if

$$\Upsilon_D\left(\left(\frac{\xi^p\eta^p}{(\alpha\xi^p+(1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right) \leq' \alpha\Upsilon_D(\xi)+(1-\alpha)\Upsilon_D(\eta).$$

2. Main Result

Theorem 2.1. Let *E* be a *D*-valued p-harmonic convex set. If $\Upsilon_{D^i} : E \to D^+$, (i = 1, 2, 3, ..., m) are *D*-valued p-harmonic convex functions. Then the function

$$\upsilon = \sum_{i=1}^{m} \zeta_i \Upsilon_{D^i}, \qquad \zeta_i \geq' 0$$

is *D*-valued p-harmonic convex functions.

Proof.

$$\upsilon\left(\left(\frac{\xi^p\eta^p}{(\alpha\xi^p+(1-\alpha)\eta^p)}\right)^{\frac{1}{p}}\right)=\sum_{i=1}^{m}\xi_i\Upsilon_{D^i}\left(\left(\frac{\xi^p\eta^p}{(\alpha\xi^p+(1-\alpha)\eta^p)}\right)^{\frac{1}{p}}\right)$$
$$\leq'\sum_{i=1}^{m}(\zeta_i(\alpha\Upsilon_{D^i}(\xi)+(1-\alpha)\Upsilon_{D^i}(\eta)))$$
$$=\sum_{i=1}^{m}\zeta_i\alpha\Upsilon_{D^i}(\xi)+\sum_{i=1}^{m}\zeta_i(1-\alpha)\Upsilon_{D^i}(\eta)$$
$$=\alpha\sum_{i=1}^{m}\zeta_i\Upsilon_{D^i}(\xi)+(1-\alpha)\sum_{i=1}^{m}\zeta_i\Upsilon_{D^i}(\eta)$$
$$=\alpha\upsilon(\xi)+(1-\alpha)\upsilon(\eta).$$

Hence, υ is D-valued p-harmonic convex functions.

Theorem 2.2. Let E be a D-valued p-harmonic convex set. If the function $\Upsilon_D^1: E \to D^+$ is D-valued p-harmonic convex functions and $\Upsilon_D^2: E \to D^+$ is a D-valued linear function, then $\Upsilon_D^1 \circ \Upsilon_D^2$ is a D-valued p-harmonic convex functions.

Theorem 2.3. Let $\Upsilon_D: D \setminus \{0\} \to D^+$be a D-valued p-harmonic convex function and let

$$\Upsilon_D(\xi)=\Upsilon'(\xi e_1)e_1+\Upsilon''(\xi e_2)e_2.$$

Then $\Upsilon'(\xi e_1)$ and $\Upsilon''(\xi e_2)$ are real valued p-harmonic convex function, where $\Upsilon': e_1\mathrm{D} \setminus \{0\} \to C_0$ and $\Upsilon'': e_2\mathrm{D} \setminus \{0\} \to C_0$.

Proof. Let $v_1' = \xi e_1 \in e_1\mathrm{D} \setminus \{0\}$ and $v_2' = \xi e_2 \in e_2\mathrm{D} \setminus \{0\}$. Consider $0 \leq \alpha_1 \leq 1$ and $0 \leq \alpha_2 \leq 1$ such that $0 \leq \alpha \leq 1$, where $\lambda = \lambda_1 e_1 + \lambda_2 e_2$. Since, $\Upsilon_D: D \setminus \{0\} \to D^+$be a D-valued p-harmonic convex function so

$$\Upsilon_D\left(\left(\frac{\xi^p\eta^p}{\alpha\xi^p+(1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)\leq'\alpha\Upsilon_D(\xi)+(1-\alpha)\Upsilon_D(\eta)$$

$$\Rightarrow \Upsilon'\left(\left(\frac{\xi^p\eta^p e_1}{\alpha\xi^p + (1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)e_1 + \Upsilon''\left(\left(\frac{\xi^p\eta^p e_2}{\alpha_1\xi^p + (1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)e_2$$

$$\leq \alpha\{\Upsilon'(\xi e_1)e_1 + \Upsilon''(\xi e_2)e_2\} + (1-\alpha)\{\Upsilon'(\eta e_1)e_1 + \Upsilon''(\eta e_2)e_2\}$$

Multiply by e_1 in the above equation, we get

$$\Upsilon'\left(\left(\frac{\xi^p\eta^p e_1}{\alpha\xi^p + (1-\alpha)\eta^p}\ \frac{1}{p}\right)\right)e_1 \leq' \alpha\Upsilon'(\xi e_1)e_1 + (1-\alpha)\Upsilon'(\eta e_1)e_1 \tag{2.1}$$

From the L.H.S of the above equation, we get

$$\Upsilon'\left(\left(\frac{\xi^p\eta^p e_1}{\alpha\xi^p + (1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)e_1$$

$$= \Upsilon'\left(\left(\frac{\xi^p\eta^p e_1}{(\alpha_1 e_1 + \alpha_2 e_2)\xi^p + (1-\alpha_1 e_1 - \alpha_2 e_2)\eta^p}\right)^{\frac{1}{p}}\right)e_1$$

$$= \Upsilon'\left(\left(\frac{\xi^p\eta^p e_1 e_1}{\{e_1(\alpha_1 e_1 + \alpha_2 e_2)\xi^p + (1-\alpha_1 e_1 - \alpha_2 e_2)\eta^p\}}\right)^{\frac{1}{p}}\right)e_1$$

$$= \Upsilon'\left(\left(\frac{\xi^p\eta^p e_1 e_1}{\alpha_1 e_1\xi^p + (1-\alpha_1 e_1)\eta^p}\right)^{\frac{1}{p}}\right)e_1$$

$$= \Upsilon'\left(\left(\frac{\xi^p\eta^p e_1^p e_1^p}{\alpha_1 e_1^p\xi^p + (1-\alpha_1)e_1^p\eta^p}\right)^{\frac{1}{p}}\right)e_1$$

$$= \Upsilon'\left(\left(\frac{v_1'^p v_2'^p}{\alpha_1 v_1'^p + (1-\alpha_1)v_2'^p}\right)^{\frac{1}{p}}\right)e_1. \qquad (2.2)$$

Similarly from the R.H.S of (2.1), we get

$$\alpha\Upsilon'(\xi e_1)e_1 + (1-\alpha)\Upsilon'(\eta e_1)e_1$$

$$= \alpha_1\Upsilon'(\xi e_1)e_1 + (1-\alpha_1)\Upsilon'(\eta e_1)e_1$$

$$= \alpha_1\Upsilon'(v_1') + (1-\alpha_1)\Upsilon'(v_2')e_1. \qquad (2.3)$$

From (2.2) and (2.3), we get

$$\Upsilon'\left(\left(\frac{v_1'^p v_2'^p}{\alpha_1 v_1'^p + (1-\alpha_1)v_2'^p}\right)^{\frac{1}{p}}\right)e_1 \leq' \alpha_1\Upsilon'(v_1') + (1-\alpha_1)\Upsilon'(v_2')e_1. \qquad (2.4)$$

Which implies that

$$\Upsilon'\left(\left(\frac{v_1'^p v_2'^p}{\alpha_1 v_1'^p + (1-\alpha_1)v_2'^p}\right)^{\frac{1}{p}}\right) \leq \alpha_1\Upsilon'(v_1') + (1-\alpha_1)\Upsilon'(v_2').$$

Hence, $\Upsilon'(\xi e_1)$ is real valued p-harmonic convex function, similarly we can proves the other parts.

Theorem 2.4. Let $\Upsilon_D': e_1 D \to C_0^+$and $\Upsilon_D'': e_2 D \to C_0^+$be any two real valued p-harmonic convex functions, then the sum $\Upsilon_D'(e_1\xi)e_1 + \Upsilon_D''(e_2\xi)e_2$ is a D-valued p - harmonic convex function.

Proof. Let $\xi,\eta \in D$. Let $v_1' = e_1\xi, v_2' = e_1\eta, v_1'' = e_2\xi$ and $v_2'' = e_2\eta$. Then $v_1', v_2' \in e_1D$ and $e_2\xi, e_2\eta \in e_2D$. Further, let us assume that $0 \leq \alpha_1 \leq 1$ and $0 \leq \alpha_2 \leq 1$ be such that $0 \leq' \alpha \leq' 1$, where $\alpha = \alpha_1 e_1 + \alpha_2 e_2$. Since Υ_D' and Υ_D'' are real valued p-harmonic convex function, we have

$$\Upsilon'\left(\left(\frac{{v_1'}^p {v_2'}^p}{\alpha_1 {v_1'}^p + (1-\alpha_1){v_2'}^p}\right)^{\frac{1}{p}}\right) \leq \alpha_1 \Upsilon'(v_1') + (1-\alpha_1)\Upsilon'(v_2') \tag{2.5}$$

$$\Upsilon''\left(\left(\frac{{v_1''}^p {v_2''}^p}{\alpha_1 {v_1''}^p + (1-\alpha_1){v_2''}^p}\right)^{\frac{1}{p}}\right) \leq \alpha_1 \Upsilon''(v_1'') + (1-\alpha_1)\Upsilon''(v_2''). \tag{2.6}$$

Now,

$$\Upsilon'\left(\left(\frac{{v_1'}^p {v_2'}^p}{\alpha_1 {v_1'}^p + (1-\alpha_1){v_2'}^p}\right)^{\frac{1}{p}}\right)$$

$$= \Upsilon'\left(\left(\frac{{v_1'}^p {v_2'}^p e_1}{(\alpha_1 e_1 + \alpha_2 e_2)e_1 {v_1'}^p + (e_1 - \alpha_1 e_1 - \alpha_2 e_2)e_1 {v_2'}^p}\right)^{\frac{1}{p}}\right)$$

$$= \Upsilon'\left(\left(\frac{{v_1'}^p {v_2'}^p e_1}{\alpha e_1 {v_1'}^p + (1-\alpha)e_1 {v_2'}^p}\right)^{\frac{1}{p}}\right) \tag{2.7}$$

Similarly,

$$\Upsilon''\left(\left(\frac{v_1''^p v_2''^p}{\alpha_1 v_1''^p+(1-\alpha_1)v_2''^p}\right)^{\frac{1}{p}}\right)=\Upsilon''\left(\left(\frac{v_1''^p v_2''^p}{\alpha v_1''^p+(1-\alpha)v_2''^p}\right)^{\frac{1}{p}}\right) \tag{2.8}$$

Again,

$$\{\alpha_1\Upsilon'(v_1')+(1-\alpha_1)\Upsilon'(v_2')\}e_1=\{\alpha\Upsilon'(v_1')+(1-\alpha)\Upsilon'(v_2')\}e_1 \tag{2.9}$$

$$\{\alpha_1\Upsilon''(v_1'')+(1-\alpha_1)\Upsilon''(v_2'')\}e_2=\{\alpha\Upsilon''(v_1'')+(1-\alpha)\Upsilon''(v_2'')\}e_2 \tag{2.10}$$

Multiplying the equations (2.5), (2.6) by e_1, e_2 and then adding we get

$$\Upsilon'\left(\left(\frac{v_1'^p v_2'}{\alpha_1 v_1'^p+(1-\alpha_1)v_2'}\right)^{\frac{1}{p}}\right)e_1+\Upsilon''\left(\left(\frac{v_1''^p v_2''^p}{\alpha_1 v_1''^p+(1-\alpha_1)v_2''^p}\right)^{\frac{1}{p}}\right)e_2$$

$$\leq' \ [\alpha_1\Upsilon'(v_1')+(1-\alpha_1)\Upsilon'(v_2')]\mathrm{e}_1+[\alpha_1\Upsilon''(v_1'')+(1-\alpha_1)\Upsilon''(v_2'')]e_2.$$

Using the equations (2.7), (2.8), (2.9) and (2.10), the above inequality becomes

$$\Upsilon'\left(\left(\frac{v_1'^p v_2'^p}{\alpha v_1'^p+(1-\alpha)v_2'}\right)^{\frac{1}{p}}\right)e_1+\Upsilon''\left(\left(\frac{v_1'' v_2''^p}{\alpha v_1''^p+(1-\alpha)v_2''^p}\right)^{\frac{1}{p}}\right)e_2 \leq'$$

$$[\alpha\Upsilon'(v_1')+(1-\alpha)\Upsilon'(v_2')]e_1+[\alpha\Upsilon''(v_1'')+(1-\alpha)\Upsilon''(v_2'')]e_2$$

$$\Rightarrow\ \Upsilon'\left(\left(\frac{\xi^p\eta^p e_1}{\alpha\xi^p+(1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)e_1+\Upsilon''\left(\left(\frac{\xi^p\eta^p e_2}{\alpha\xi^p+(1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)e_2 \leq'$$

$$[\alpha\Upsilon'(\xi e_1)+(1-\alpha)\Upsilon'(\eta e_1)]e_1+[\alpha\Upsilon''(\xi e_2)+(1-\alpha)\Upsilon''(\eta)]e_2$$

$$\Rightarrow (\Upsilon'(e_1.)e_1+\Upsilon''(e_2.)e_2)\left(\left(\frac{\xi^p\eta^p}{\alpha\xi^p+(1-\alpha)\eta^p}\right)^{\frac{1}{p}}\right)\leq'$$

$$\alpha(\Upsilon'(e_1.)e_1 + \Upsilon''(e_2.)e_2)(\xi) + (1-\alpha)(\Upsilon'(e_1.)e_1 + \Upsilon''(e_2.)e_2)(\eta).$$

Hence, $\Upsilon_D'(e_1\xi)e_1 + \Upsilon_D''(e_2\xi)e_2$ is a D-valued p - harmonic convex function.

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