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Peristaltic Transport of Non-Newtonian Fluid inside a Vertical Channel with Soret and Dufour Effects

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Abstract

An analysis has been arranged to examine the peristaltic transport of non-Newtonian fluid in a vertical asymmetric channel. The fluid in the channel is assumed electrically conducted through a porous medium and a uniform magnetic field is applied transversely to the direction of the flow. Simultaneous impacts of heat and mass transfer with Soret and Dufour effects are considered. A Casson rheological model is used to characterize the non-Newtonian behavior of the flow. The equations governing the fluid flow are investigated in a wave frame of reference with a velocity of the wave. Analytic solution is carried out under the assumptions of long wave length and small Reynolds number. The expressions for stream function, temperature, concentration and heat transfer coefficient are obtained. The transformed equations have also been solved numerically by bvp4c function from MATLAB. Effects of various parameters on flow characteristics have been discussed with the help of 2D and 3D graphs.

Keywords: Magnetic field, Heat transfer, Soret number, Porous channel.

1. Introduction:

In recent years the peristaltic transport in channel has practical interest due to wide applications. Peristaltic transport is a form of material transport induced by a progressive wave of area contraction and expansion along the length of a distensible tube, mixing and transporting the fluid in the direction of the wave propagation. Physiological fluids in animal and human bodies are, in general, pumped using this principle. Peristaltic transport occurs widely in the functioning of ureter, food mixing and crime movement in intestine, movement of eggs in fallopian tube, circulation of blood in blood vessel. Latham [1] and Shapiro et al. [2] made early effort regarding peristaltic mechanism of viscous fluid. After that, this topic has been examined for both Newtonian and non-Newtonian fluids [3-10]. Again the porous medium plays an important role in the study of transport process in fluid dynamics, industrial and engineering fields. Fluid movement through a porous medium is widely applicable in lungs, kidneys, bile duct, gall bladder with stones, small blood vessels. In our body, distribution of fatty cholesterol, blood clots in coronary artery, functions of organs are considers as porous medium [11]. Elshehawey et al. [12] discussed the Peristaltic transport in an asymmetric channel through a porous medium. Elangovan and Selvaraj [13] studied MHD peristaltic flow of blood through porous medium with slip effect in the presence of body acceleration.

This mechanism with heat transfer is also used in many biomedical and industry appliances such as finger pump, roller pump, heart-lung machine, blood pump machine and dialysis machine. Heat transfer is also significant in the treatment of cancerous tissues, evaluating skin burns, food processing, radiation between surface and its surrounding environments [14]. Srinivas and Gayathri [15] studied peristaltic transport of Newtonian fluid in a vertical asymmetric channel with heat transfer and porous channel. Mass transfer in peristaltic flow is another vital phenomenon in physiology and industry. It occurs during chemical breakdown of food, distillation process and combustion process [16,17]. In joint heat and mass transfer situations, thermal energy flux resulting from concentration gradients is named diffusion-thermal effect (Dufour), while thermo-diffusion effect (Soret) occurs due to mass fluxes by temperature gradient.

Most of the industrial and physiological fluids (such as blood, food bolus, chyme) are non-Newtonian in nature. So, peristaltic transport of non-Newtonian fluids has been a vital topic for the researchers. These non-Newtonian fluids possess both viscous and elastic properties. But there is not a single constitute equation that can explain the flow properties of these fluids. Due to complex structure of fluids, several models (power-law fluid, Casson model, Couple stress fluid, micropolar fluid) have been proposed [11]. Casson model was introduced by Casson [18]. Human blood, honey, jelly can be presented by Casson's model [19]. To the best of author knowledge the idea of Casson fluid in peristaltic literature have not been discussed yet. For this reason, we have analyzed the heat and mass transfer on peristaltic transport of Casson fluid in a vertical asymmetric channel. Soret and Dufour effects are also accounted. The problem is first formulated and then solved both analytically and numerically. Influences of different important parameters on flow characteristics are presented and discussed.

2. Mathematical Modeling:

Consider the heat and mass transfer effects in the flow of an incompressible and electrically conducting non-Newtonian Casson fluid in a two dimensional asymmetric vertical channel. Soret and Dufour effects are also present. Here X-axis is taken along the length of the channel and Y-axis is normal to it. The channel walls H_1 and H_2 are maintained at constant temperature T_0 and T_1 and constant concentration C_0 and C_1 respectively. A uniform magnetic field B_0 is applied in Y-direction. Here the induced magnetic field is not considered due to small Reynolds numbers. Again, the asymmetry in the channel is produced by selecting the peristaltic wave propagating with constant speed c along the walls defined by

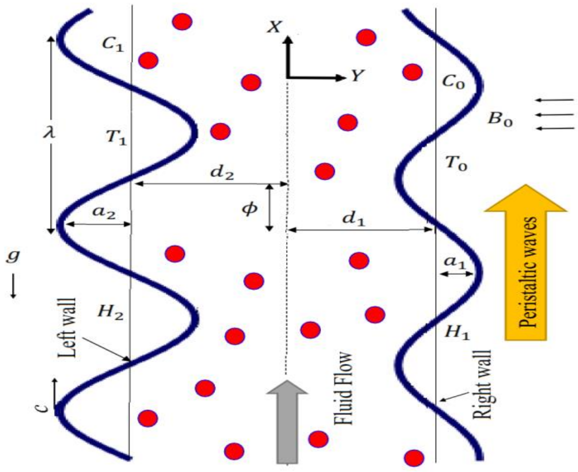

Fig.1. Physical model of the problem

$$\left.\begin{aligned} Y = H_1 = d_1 + a_1 cos\left\{\frac{2\pi}{\lambda}(X - ct)\right\} \\ Y = H_2 = -d_2 - a_2 cos\left\{\frac{2\pi}{\lambda}(X - ct) + \phi\right\} \end{aligned}\right\} \quad (1)$$

where a_1, a_2 denote the waves amplitudes, $d_1 + d_2$ is the channel width, λ is the wave length, t is the time, c is the velocity of propagation and ϕ is the phase difference ($0 \leq \phi \leq \pi$), in which $\phi = 0$ corresponds to symmetric channel with waves out of phase and $\phi = \pi$ corresponds to waves in phase.

The basic laws which governs the present model are

$$\nabla . \bar{q} = 0 \quad (2)$$

$$\rho \frac{D\bar{q}}{Dt} = -\nabla P + \mu \nabla^2 \bar{q} + \bar{J} \times \bar{B} - \frac{\mu}{K'}\bar{q} + \rho g \beta_t (T - T_0) + \rho g \beta_c (C - C_0) \quad (3)$$

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + Q - \nabla q_r + \frac{DK_T}{c_s} \nabla^2 C \quad (4)$$

$$\frac{DC}{Dt} = D \nabla^2 C + \frac{DK_T}{T_m} \nabla^2 T \quad (5)$$

where $\bar{q}$ is the fluid velocity vector, P is the pressure, μ is the viscosity, ρ is the density, $\bar{J}$ is the current density, $\bar{B}$ is the magnetic field, $\bar{J} \times \bar{B}$ is the Lorentz force, g is the acceleration due to gravity, β_t is the thermal expansion coefficient, β_c is the concentration expansion coefficient, C_p is specific heat at constant pressure, k is the thermal conductivity, Q is the heat generation, q_r radiative heat flux, D is the mass diffusivity, K_T is the thermal diffusion ratio, c_s is the concentration susceptibility, T_m is the mean temperature.

The constitute equation for Casson [9,10] fluid is

$$\tau_{ij} = 2\left(\mu_b + \frac{P_y}{\sqrt{2\pi}}\right) e_{ij} \tag{6}$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial y_i}\right)$ is the (i,j)th component of deformation rate, τ_{ij} is the (i,j)th component of the stress tensor, π is the product of the component of deformation rate with itself, and μ_b is the plastic dynamic viscosity. The yield stress P_y is expressed as $P_y = \frac{\mu_b\sqrt{2\pi}}{\beta}$, where β Casson fluid parameter. For non-Newtonian Casson fluid flow $\mu = \mu_b + \frac{P_y}{\sqrt{2\pi}}$ which gives $\vartheta' = \vartheta\left(1 + \frac{1}{\beta}\right)$, where $\vartheta = \frac{\mu_b}{\rho}$ is the kinematic viscosity for Casson fluid. Again the yield stress $P_y = 0$ for Newtonian case.

In the fixed frame, the governing equations for peristaltic motion of Casson fluid through porous medium in a vertical channel are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{7}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) - \frac{\sigma B_0^2}{\rho}U - \nu\left(1 + \frac{1}{\beta}\right)\frac{U}{K'} + g\beta_T(T - T_0) + g\beta_c(C - C_0) \tag{8}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) - \nu\left(1 + \frac{1}{\beta}\right)\frac{V}{K'} \tag{9}$$

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2}\right) + \frac{Q}{\rho C_p} - \frac{1}{\rho C_p}\left(\frac{\partial q_r}{\partial X} + \frac{\partial q_r}{\partial Y}\right) + \frac{DK_T}{c_s}\left(\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2}\right) \tag{10}$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial X} + V\frac{\partial C}{\partial Y} = D\left(\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2}\right) + \frac{DK_T}{T_m}\left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2}\right) \tag{11}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} U = 0,\ T = T_0,\ C = C_0 \quad & \text{at } Y = H_1 \\ U = 0,\ T = T_1,\ C = C_1 \quad & \text{at } Y = H_2 \end{aligned}\right\} \tag{12}$$

The radiative heat flux q_r in X-direction is negligible compared to Y-direction. Using Rosseland approximation for thermal radiation, q_r is defined by

$$q_r = -\frac{16\sigma^* T_0^3}{3k^*}\frac{\partial T}{\partial Y} \tag{13}$$

Where, σ^* is the Stefan-Boltzmann constant and k^* is the absorption coefficient.

The coordinates, velocity, pressure, temperature and concentration in the fixed frame (X, Y) and wave frame (x, y) are related by the following expression

$$\left.\begin{array}{c} x = X - ct, y = Y, u = U - c, v = V, p(x,y) = P(X,Y,t) \\ \bar{T}(x,y) = T(X,Y,t), \bar{C}(x,y) = C(X,Y,t), \end{array}\right\} \quad (14)$$

Where, $u, v, p, \bar{T}$ and $\bar{C}$ are velocity components, pressure, temperature and concentration respectively in wave frame.

Now we introduce the following dimensionless quantities

$$\left.\begin{array}{c} x' = \frac{x}{\lambda}, y' = \frac{y}{d_1}, u' = \frac{u}{c}, v' = \frac{v}{c\delta}, t' = \frac{ct}{\lambda}, p' = \frac{pd_1^2}{\lambda c \mu_b}, \delta = \frac{d_1}{\lambda} \\ h_1 = \frac{H_1}{d_1}, h_2 = \frac{H_2}{d_1}, d = \frac{d_2}{d_1}, a = \frac{a_1}{d_1}, b = \frac{a_2}{d_1}, \theta = \frac{\bar{T}-\bar{T}_0}{\bar{T}_1-\bar{T}_0}, \varphi = \frac{\bar{C}-\bar{C}_0}{\bar{C}_1-\bar{C}_0} \end{array}\right\} \quad (15)$$

The governing equations $(7) - (11)$ under the assumptions of long wave length and low Reynolds number in terms of stream function ψ (dropping the das symbols) become

$$\frac{\partial p}{\partial x} = \left(1 + \frac{1}{\beta}\right)\frac{\partial^3 \psi}{\partial y^3} - M^2\left(\frac{\partial \psi}{\partial y} + 1\right) - \left(1 + \frac{1}{\beta}\right)\frac{1}{K}\left(\frac{\partial \psi}{\partial y} + 1\right) + Gr\theta + Gc\varphi \quad (16)$$

$$\frac{\partial p}{\partial y} = 0 \quad (17)$$

$$(1 + Rd)\theta'' + Q_0 + PrDu\varphi'' = 0 \quad (18)$$

$$\varphi'' + ScSr\theta'' = 0 \quad (19)$$

The dimensionless boundary conditions become

$$\left.\begin{array}{ll} \psi = \frac{F}{2}, \quad \frac{\partial \psi}{\partial y} = -1, \ \theta = 0, \ \varphi = 0 & \text{at } y = h_1 \\ \psi = -\frac{F}{2}, \quad \frac{\partial \psi}{\partial y} = -1, \ \theta = 1, \ \varphi = 1 & \text{at } y = h_2 \end{array}\right\} \quad (20)$$

where $M = \sqrt{\frac{\sigma}{\mu_b}} B_0 d_1$ is the magnetic field parameter, $K = \frac{K\prime}{d_1^2}$ is the permeability parameter, F is the volume flow rate in the wave frame, $\beta = \frac{\mu_b\sqrt{2\pi}}{P_y}$ is the Casson fluid parameter, $Gr = \frac{g\beta_T(\bar{T}_1-\bar{T}_0)d_1^2}{cv}$ is the temperature Grashof number, $Gc = \frac{g\beta_c(\bar{C}_1-\bar{C}_0)d_1^2}{cv}$ is the concentration Grashof number, $Pr = \frac{\rho v C_p}{k}$ is the Prandtl number, $Rd = \frac{16T_0^3\sigma^*}{3kk^*}$ is the radiation parameter, $Q_0 = \frac{Qd_1^2}{k(\bar{T}_1-\bar{T}_0)}$ is the heat generation parameter, $Du = \frac{DK_T(\bar{C}_1-\bar{C}_0)}{\mu_b C_p c_s(\bar{T}_1-\bar{T}_0)}$ is the Dufour number, $Sc = \frac{v}{D}$ is the Schmidt number and $Sr = \frac{DK_T(\bar{T}_1-\bar{T}_0)}{T_m v(\bar{C}_1-\bar{C}_0)}$ is the Soret number.

3. Analytic Solution:

Equation (17) gives that $p \neq p(y)$. Eliminating the pressure terms from (16) we get

$$\frac{\partial^4\psi}{\partial y^4} - \alpha^2 \frac{\partial^2\psi}{\partial y^2} + \frac{\beta}{1+\beta}(Gr\theta' + Gc\varphi') = 0 \quad (21)$$

Solving equations (18) and (19) with the boundary conditions (20), the temperature and concentration are obtained as

$$\theta = -\frac{Ay^2}{2} + A_1 y + A_2 \quad (22)$$

$$\varphi = ScSrA\frac{y^2}{2} + B_1 y + B_2 \quad (23)$$

Using the above solutions and boundary conditions (20) in (21), we get the stream function as

$$\psi = C_1 + C_2 y + C_3 e^{\alpha y} + C_4 e^{-\alpha y} + C_5 y^2 + C_6 y^3 \quad (24)$$

where the constants involved in the solutions are given in the appendix.

The dimensionless mean flow rate Q' in the laboratory frame is related to the dimensionless mean flow rate F in the wave frame by

$$Q' = F + 1 + d \quad (25)$$

in which

$$F = \int_{h_2}^{h_1} u dy \quad (26)$$

Also note the h_1 and h_2 represent the dimensionless forms of the peristaltic walls

$$\left.\begin{aligned} h_1 &= 1 + acos2\pi x \\ h_2 &= -d - bcos(2\pi x + \phi) \end{aligned}\right\} \quad (27)$$

Again the heat transfer coefficient at the right wall $(y = h_1)$ is

$$Z_1 = h_{1x}\theta' = 2\pi a(Ay - A_1)\sin(2\pi x) \quad (28)$$

4. Numerical Solution:

We have also solved the transformed equations by numerical technique bvp4c built in function of MATLAB. The function is strong and beneficial to calculate the solution of model BVPs. In particular, bvp4c is a finite difference code and it provides a C^1-continuous solution which is fourth order accurate uniformly in a given interval. To find numerical result the following parameter values have been used: $a = 0.4, b = 0.5,\ d = 1.3, F = 0.2,\ x = 0.1, M = 1, K = 0.5, \beta = 1,\ Gr = 0.5,\ Rr = 1, Q_0 = 0.5,\ \phi = \pi/4, Gc = 0.3, Du = 0.3, Sc = 0.5, Sr = 0.3$, unless otherwise specified. The value of Prandtl number for human blood is $Pr = 21$ [7]. So Pr is kept 21 throughout the study. The effect of various parameters, such as magnetic field parameter (M), Casson fluid parameter (β), permeability parameter (K), flow rate (F), heat generation parameter (Q_0), radiation parameter (Rd), Dufour number (Du), Soret number (Sr) and Schmidt number (Sc) are displayed using 2D and 3D plots. These plots are sketched to understand and explain the varying activities of model parameters in a better way.

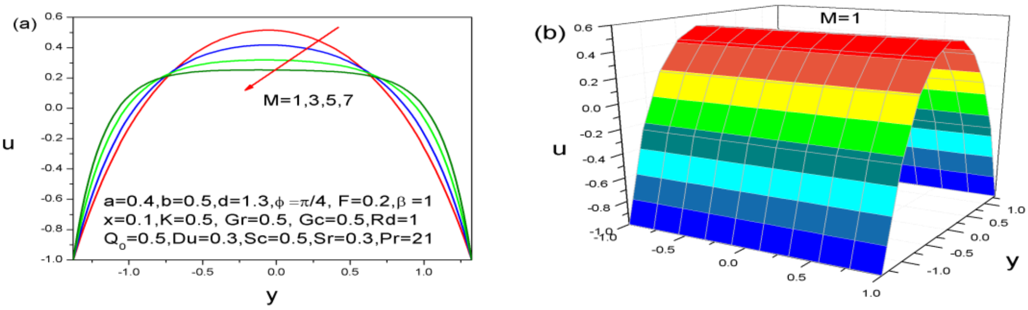

Fig.2. Velocity profiles for different M (a) 2D and (b) 3D

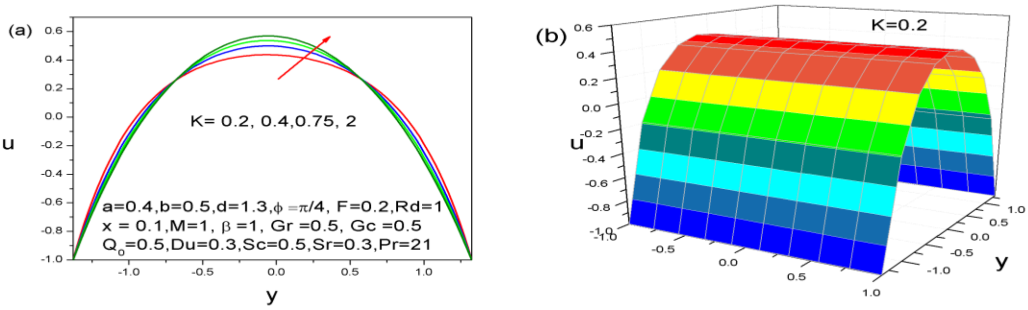

Fig.3. Velocity profiles for different K (a) 2D and (b) 3D

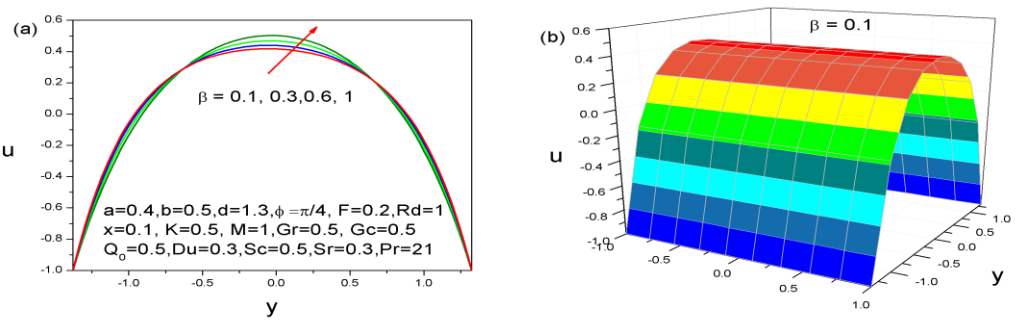

Fig.4. Velocity profiles for different β (a) 2D and (b) 3D

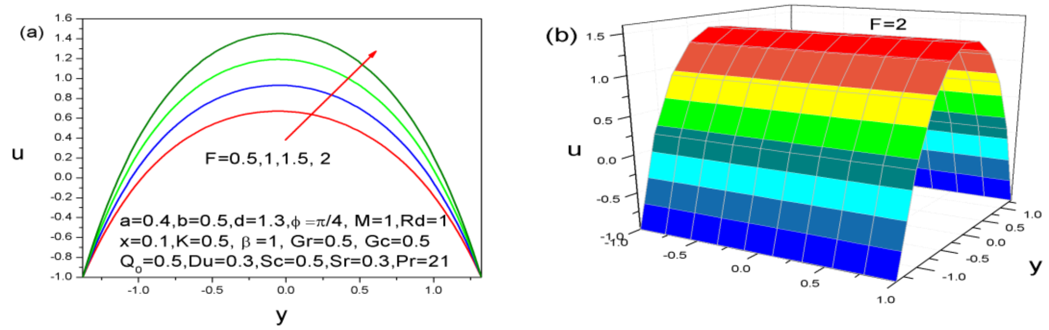

Fig.5. Velocity profiles for different F (a) 2D and (b) 3D

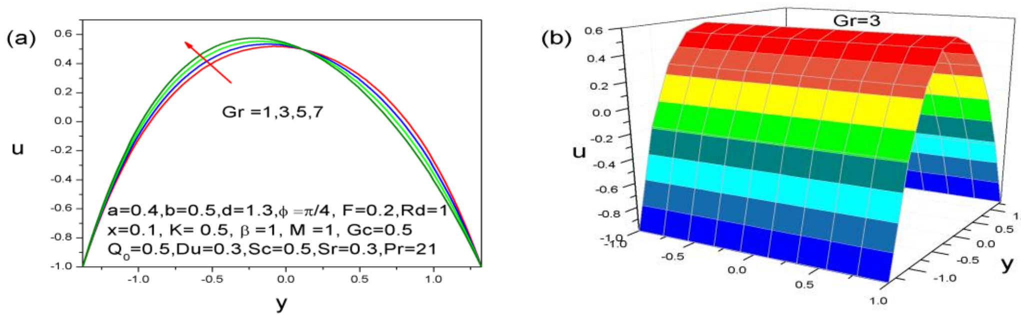

Fig.6. Velocity profiles for different Gr (a) 2D and (b) 3D

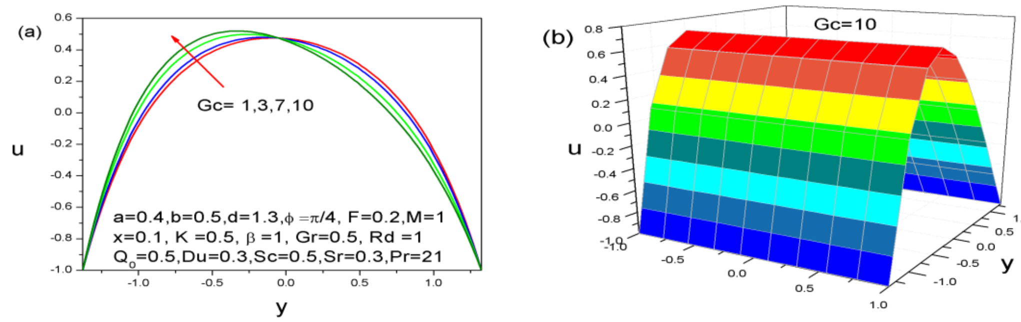

Fig.7. Velocity profiles for different Gc (a) 2D and (b) 3D

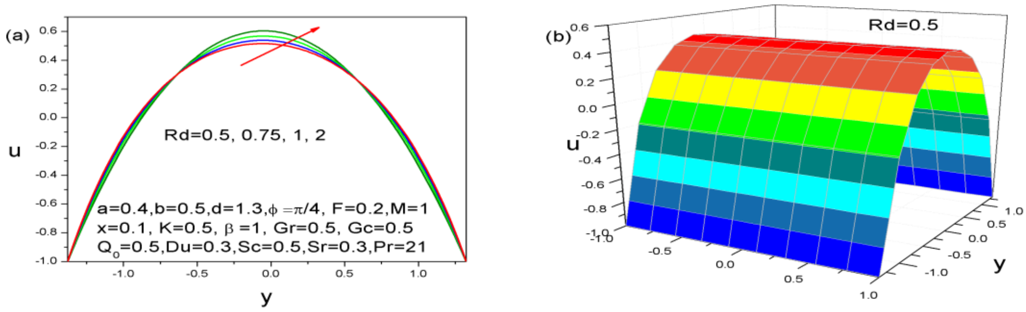

Fig.8. Velocity profiles for different Rd (a) 2D and (b) 3D

5. Results and Discussions:

The behavior of magnetic field parameter M on velocity component u is plotted in Fig.2. Here we see that the velocity profile is parabolic in nature. It is clear that when M is increased, u decreases near the centre of the channel. The applications of applied magnetic field to an electrically conduction flow give rise to a resistive force (called Lorenz force). This force has the tendency to slow down the motion of the fluid. For this reason there is a decreasing impact on velocity field. Fig.3 shows that velocity is increased for large permeability parameter K. The effect of Casson fluid parameter β on velocity is displayed in Fig.4. Large β means a decrease in yield stress. This effectively accelerates the fluid flow. Fig.5 shows that the velocity profile increased with an increase in flow rate F. The effect of temperature Grashof number Gr is observed in Fig.6. It is clear that the axial velocity increases at the left wall while a reserve behavior is seen at the right wall when Gr increases. In this case the fact is that the buoyancy force gives rise to flow. The force has a tendency to increase the flow of the fluid which results in increasing the velocity profiles at the left wall. Similar property follows for concentration Grashof number Gc as seen in Fig.7. The effect of radiation parameter Rd on velocity is sketched in Fig. 8. It viewed that the magnitude of velocity profile enlarges at the central part of the channel with large Rd.

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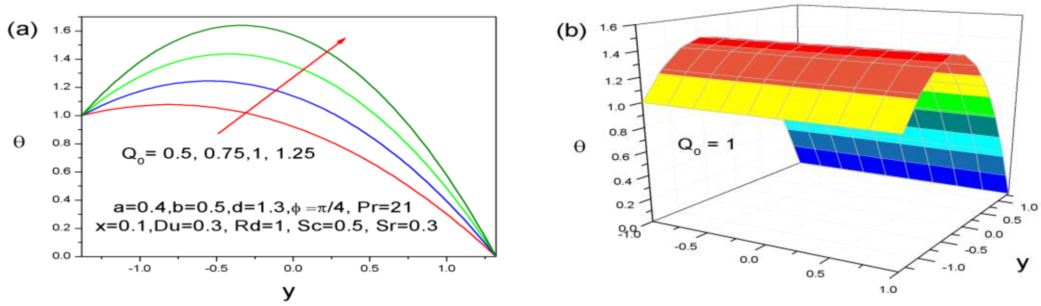

Fig.9. Temperature profiles for different Q_0 (a) 2D and (b) 3D

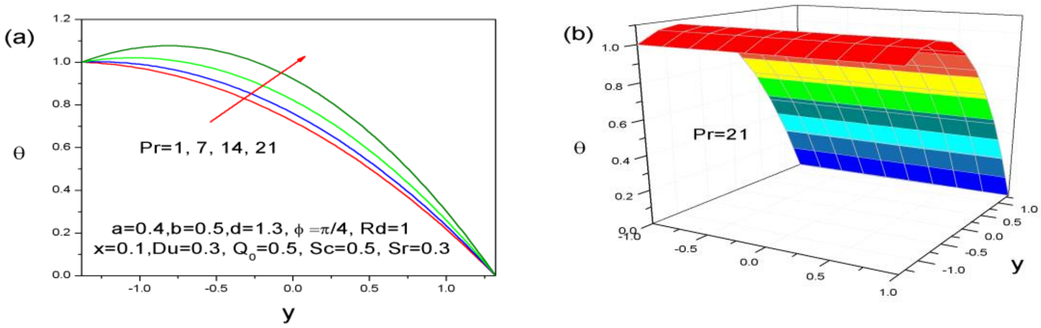

Fig.10. Temperature profiles for different Pr (a) 2D and (b) 3D

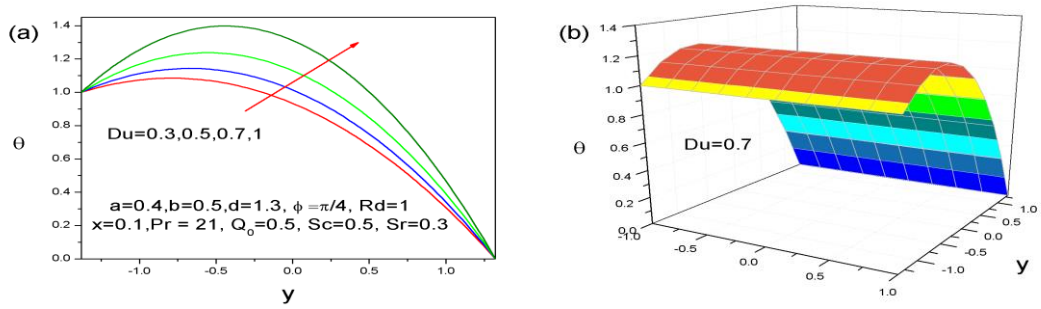

Fig.11. Temperature profiles for different Du (a) 2D and (b) 3D

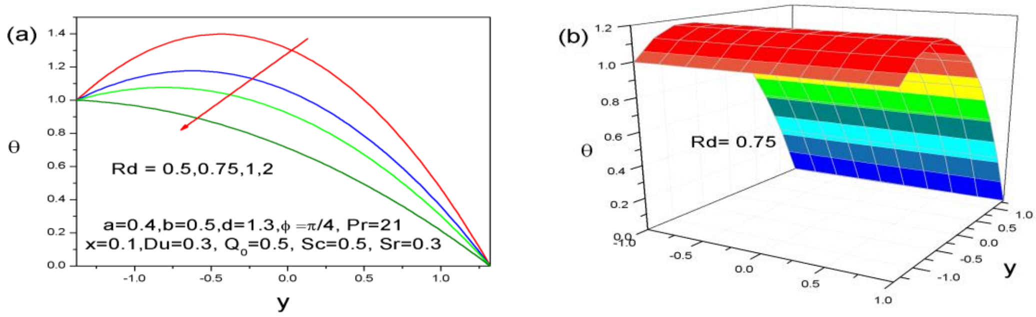

Fig.12. Temperature profiles for different Rd (a) 2D and (b) 3D

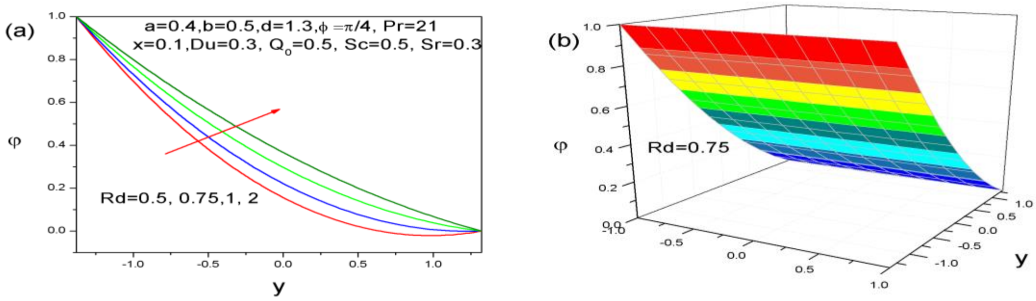

Fig.13. Concentration profiles for different Rd (a) 2D and (b) 3D

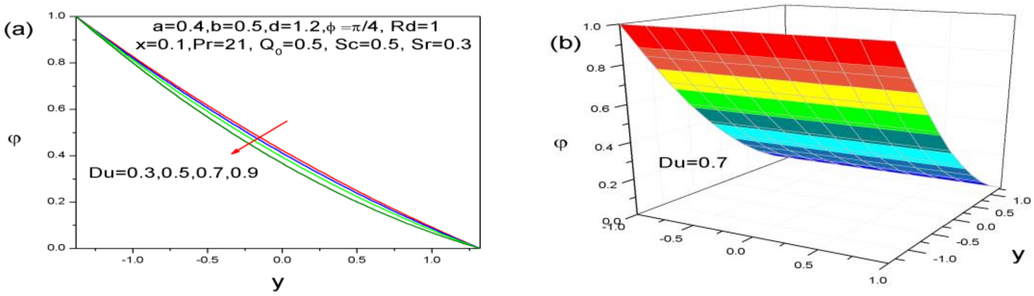

Fig.14. Concentration profiles for different Du (a) 2D and (b) 3D

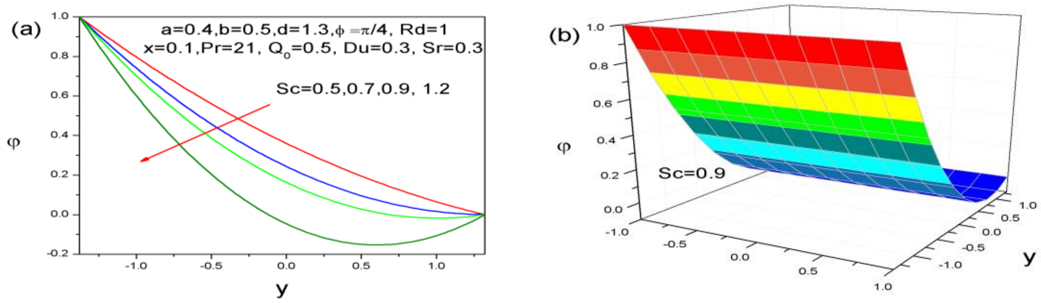

Fig.15. Concentration profiles for different Sc (a) 2D and (b) 3D

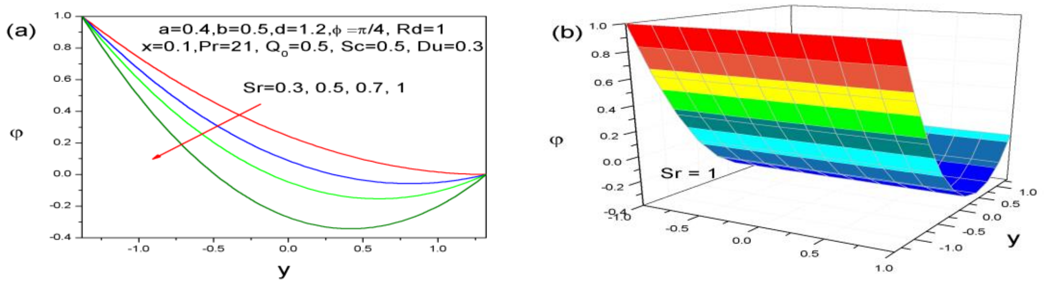

Fig.16. Concentration profiles for different Sr (a) 2D and (b) 3D

The effect of heat generation parameter Q_0 is illustrated in Fig.9. We know that heat generation is nothing but the conversion of one form of energy into thermal energy inside the fluid. The temperature of the fluid increases during this process. The behavior of Prandtl number Pr on dimensionless temperature profiles has been presented in the Fig.10. The temperature field enhances when we increase Pr. We know that Prandtl number is the ratio of momentum diffusivity to thermal diffusivity and it is the function of fluid properties not the flow situations. The thickness of both velocity and thermal boundary layer is same when $Pr = 1$. Again the temperature increases due to increase in Dufour number Du as seen in Fig.11. The influence of radiation parameter Rd on temperature field is given in Fig.12. The higher values of Rd imply higher surface heat flux and thus it decreases the temperature within the channel.

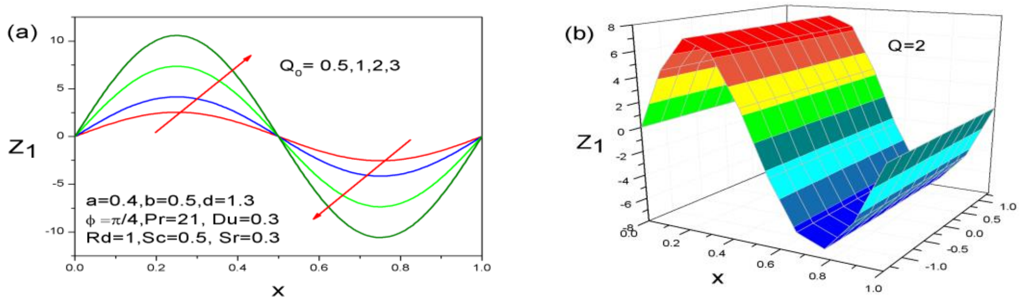

Fig.17. Heat transfer coefficient for different Q_0 (a) 2D and (b) 3D

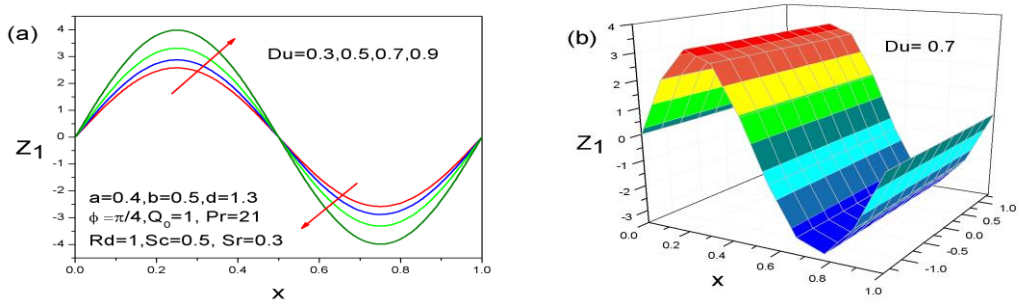

Fig.18. Heat transfer coefficient for different Du (a) 2D and (b) 3D

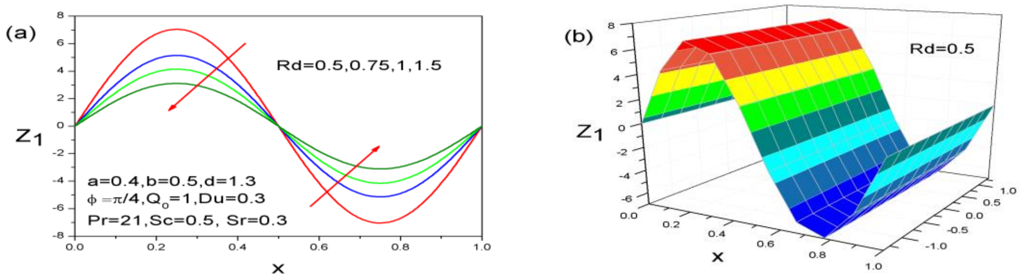

Fig.19. Heat transfer coefficient for different Rd (a) 2D and (b) 3D

Fig.13 shows that radiation parameter Rd has increasing effect on concentration field. On the other hand concentration decreases for increasing Du . In this situation increase in concentration is slow and less as seen in Fig.14. The concentration profiles for different value of Schmidt number Sc is sketched in Fig.15. Sc is the ratio of momentum diffusivity to mass diffusivity. Here the profile decreases with an increase in Sc. The effect of Soret number Sr on concentration profiles is displayed in Fig.16. It is mentioned that concentration profile decreases with increasing Sr.

The effects of on heat transfer coefficient Z_1 at upper wall ($y = h_1$) are plotted in Fig. 17-Fig.19. Here Z_1 has oscillatory property due to peristaltic. It is noticed that the absolute values of Z_1 increases with increase of Q_0 and Du. On the other hand, increasing the radiation parameter Rd decreases the heat transfer coefficient.

In order to verify the accuracy of numerical results, the present study is compared with the previous study of Srinivas & Gayathri [15]. For the purpose of comparison, both the studies have been brought to the same stage by considering equal parameters (Newtonian case). These comparisons are given in Fig.20, which are found in very good agreement.

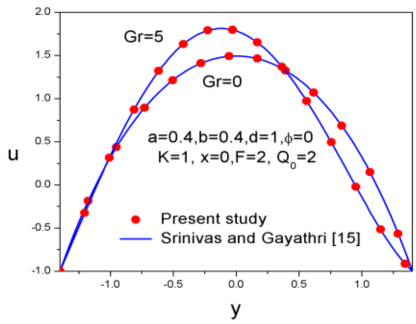

Fig.20. Comparison of velocity profiles (Newtonian Case)

6. Concluding Remarks:

Soret and Dufour effects on the peristaltic transport of Casson fluid in a vertical channel are discussed in this study. Both analytic and numerical solutions are obtained. The impacts of different parameters on flow characteristics are shown graphically. The main findings of the study are

1. Velocity field increases for K, β, F, Gr, Rd and Gc but decreases for M.
2. An increase in Pr, Du and Q_0result an increase in temperature profiles. But opposite property is noticed for Rd.
3. Soret number Sr plays a role on concentration.
4. Absolute value of heat transfer coefficient increases for large Du and Q_0.

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Appendix:

$$\alpha = \sqrt{\frac{M^2}{(1+1/\beta)} + \frac{1}{K}}$$

$$A = Q_0/(1 + Rd - PrDuScSr)$$

$$A_1 = (h_1 + h_2)A/2 + 1/(h_2 - h_1)$$

$$A_2 = h_1^2 A/2 - A_1 h_1$$

$$B_1 = -(h_1 + h_2)ScSrA/2 + 1/(h_2 - h_1)$$

$$B_2 = -ScSrh_1^2 A/2 - B_1 h_1$$

$$C_5 = \frac{\beta}{2\alpha^2(1+\beta)}(GrA_1 + GcB_1)$$

$$C_6 = \frac{\beta}{6\alpha^2(1+\beta)}(-GrA + GcAScSr)$$

$$P_1 = \alpha e^{\alpha h_1}/(e^{\alpha h_1} - e^{\alpha h_2})$$

$$P_2 = e^{-\alpha h_1} - e^{-\alpha h_2}$$

$$P_3 = h_1^2 + h_1 h_2 + h_2^2 - 3(h_1 + h_2)/\alpha$$

$$P_4 = h_1 + h_2 - 2/\alpha$$

$$P_5 = \alpha e^{-\alpha h_1}$$

$$P_6 = -1 - F/(h_1 - h_2)$$

$$P_7 = -P_5 + P_1 P_2 - 2P_2/(h_1 - h_2)$$

$$P_8 = 2h_1 - P_4 - 2(h_1 - h_2)/\alpha$$

$$P_9 = 3h_1^2 - P_3 - 3P_1(h_1^2 - h_2^2)/\alpha$$

$$P_{10} = P_6 - C_5 P_8 - C_5 P_9$$

$$P_{11} = e^{\alpha h_1} - e^{\alpha h_2}$$

$$P_{12} = 2C_5(h_1 - h_2)/\alpha$$

$$P_{13} = 3C_6(h_1^2 - h_2^2)/\alpha$$

$$P_{14} = F/2 - C_5 h_1^2 - C_6 h_1^3$$

$$C_4 = P_{10}/P_7$$

$$C_3 = (C_4 P_2 - P_{12} - P_{13})/P_{11}$$

$$C_2 = -C_5 P_4 - C_6 P_3 + (F - 2C_4 P_2)/(h_1 - h_2)$$

$$C_1 = P_{14} - C_2 h_1 - C_3 e^{\alpha h_1} - C_4 e^{-\alpha h_1}$$

On ρ-Statistical Convergence of Difference Sequences of Fractional Order

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Abstract

In this article, by using definition of ρ-statistical convergence which was defined by Çakallı [11] we introduce the concepts of $\Delta_{\rho}^{\alpha}-$statistical convergence with the fractional order and strongly $\Delta_{\rho}^{\alpha}-$ summable sequences with the fractional order. Also, we establish some inclusion relations between the concepts of $\Delta_{\rho}^{\alpha}-$statistical convergence and strong $\Delta_{\rho}^{\alpha}-$ summability.

Key words and phrases: Difference sequences, Statistical convergence, Cesàro summability.

2010 Mathematics Subject Classification: 40A05, 40C05, 46A45.

1. INTRODUCTION

In 1951, Steinhaus [34] and Fast [19] introduced the notion of statistical convergence and later in 1959, Schoenberg [33] reintroduced independently. Altinok et al. [2], Aral and Günal [5], Caserta et al. [8], Çakallı ([9],[10]), Connor [14], Çolak [13], Et et al. ([12],[17]), Fridy [20], Fridy and Orhan [21], Gadjiev and Orhan [22], Isık et al. ([23],[24],[25]), Kolk [28], Mursaleen [29], Salat [31] and many others investigated some arguments related to this notion.

The opinion of statistical convergence depends on the density of subsets of the natural set ℕ. We say that the $\delta(E)$ is the density of a subset E of ℕ if the following limit exists such that

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of ℕ has zero natural density and $\delta(E^c)=1-\delta(E)$. We say that the sequence $x=(x_k)$ is statistically convergent to ℓ if for every $\varepsilon>0$,

$$\delta(\{k\in\mathbb{N}: |x_k-l|\geq\varepsilon\})=0$$

By S we will denote the set of all statistically convergent sequences.

Difference sequence spaces was defined by Kızmaz [27] and the concept was generalized by Et et al. ([15],[18]) as follows:

$$\Delta^m(X)=\left\{x=(x_k) : (\Delta^m x_k)\in X\right\},$$

where X is any sequence space, $m\in$ ℕ, $\Delta^0 x=(x_k)$, $\Delta x=(x_k-x_{k+1})$, $\Delta^m x=(\Delta^m x_k)=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})$ and so $\Delta^m x_k=\sum_{v=0}^{m}(-1)^v\binom{m}{v}x_{k+v}$.

If $x\in\Delta^m(X)$, then there exists one and only one sequence $y=(y_k)\in X$ such that $y_k=\Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \begin{pmatrix} k-v-1 \\ m-1 \end{pmatrix} y_v = \sum_{v=1}^{k} (-1)^m \begin{pmatrix} k+m-v-1 \\ m-1 \end{pmatrix} y_{v-m}, \qquad (1)$$

$$y_{1-m} = y_{2-m} = \cdots y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. After then some properties of difference sequence spaces have been studied in ([1],[3],[4],[16],[18],[26],[32],[35],[36],[37]).

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r \notin \{0,-1,-2,-3,...\}$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

From the definition, it is observed that:

(*i*) For any natural number n, $\Gamma(n+1) = n!$,

(*ii*) For any real number n and $n \notin \{0,-1,-2,-3,...\}$, $\Gamma(n+1) = n\Gamma(n)$,

(*iii*) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!,....$

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha : w \to w$ defined by

$$\Delta^\alpha (x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}. \qquad (2)$$

In particular, we have $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2} x_{k+1} - \frac{1}{8} x_{k+2} - \frac{1}{16} x_{k+3} - \frac{5}{128} x_{k+4} - \frac{7}{256} x_{k+5} - \frac{21}{1024} x_{k+6} \cdots$

$$\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2} x_{k+1} + \frac{3}{8} x_{k+2} + \frac{5}{16} x_{k+3} + \frac{35}{128} x_{k+4} + \frac{63}{256} x_{k+5} + \frac{231}{1024} x_{k+6} \cdots$$

$$\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3} x_{k+1} - \frac{1}{9} x_{k+2} - \frac{5}{81} x_{k+3} - \frac{10}{243} x_{k+4} - \frac{22}{729} x_{k+5} - \frac{154}{6561} x_{k+6} \cdots$$

$$\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3} x_{k+1} - \frac{1}{9} x_{k+2} - \frac{4}{81} x_{k+3} - \frac{7}{243} x_{k+4} - \frac{14}{729} x_{k+5} - \frac{91}{6561} x_{k+6} \cdots.$$

Without loss of generality, we assume throughout that the series defined in (2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (2) reduces to a finite sum i.e.,

$$\sum_{i=0}^{\alpha}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}.$$

In fact, this operator generalized the difference operator introduced by Et and Çolak [15] and in this case we write $\Delta^m(X)$ $m \in \mathbb{R}$ instead of $\Delta^\alpha(X)$ $(\alpha \in \mathbb{R})$.

Recently, using fractional operator Δ^α (fractional order of α, $\alpha \in \mathbb{R}$) Baliarsingh et al. ([6],[7],[30]) defined the sequence space $\Delta^\alpha(X)$ such as $\Delta^\alpha(X) = \{x = (x_k) : (\Delta^\alpha x_k) \in X\}$, where X is any sequence space.

The concept of $\rho-$statistical convergence was defined by Çakallı [11]. A sequence $x = (x_k)$ is called $\rho-$statistically convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \le n : |x_k - \ell| \ge \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for each positive integer n.

2. MAIN RESULTS

In this section we give the main results of this article. Now we begin two new definitions.

Definition 2.1 Let α be a proper fraction and $0<p<\infty$, then a sequence $x=(x_k)$ is said to be strongly $\Delta_{\rho}^{\alpha}(p)-$summable to ℓ if

$$\lim_{n\to\infty}\frac{1}{\rho_n}\sum_{k=1}^{n}|\Delta^{\alpha}x_k-\ell|^p=0,$$

for each $\varepsilon>0,$ where and afterwards $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n}<\infty,$ $\Delta\rho_n=O(1)$ and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer $n.$ We denote the set of all strongly $\Delta_{\rho}^{\alpha}(p)-$summable sequences by $\Delta_{\rho}^{\alpha}(p)$ and write $\lim_{k\to\infty}x_k=\ell\left(\Delta_{\rho}^{\alpha}(p)\right)$ or $\Delta_{\rho}^{\alpha}(p)-\lim x_k=\ell$. If $p=1,$ then the sequence space $\Delta_{\rho}^{\alpha}(p)$ reduces to the sequence space Δ_{ρ}^{α}. If $\alpha=1,$ then we write $N_{\rho}(p)$ instead of $\Delta_{\rho}^{\alpha}(p)$. In case of $\ell=0,$ we write $N_{\rho}^{0}(p)$ instead of $N_{\rho}(p).$

Definition 2.2 Let α be a proper fraction, then a sequence $x=(x_k)$ is called $\Delta_{\rho}^{\alpha}-$statistically convergent to ℓ if

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\le n:|\Delta^{\alpha}x_k-\ell|\ge\varepsilon\}|=0$$

for each $\varepsilon>0$. In this case we write $\Delta_{\rho}^{\alpha}(S)-\lim x_k=\ell$ or $x_k\to\ell\left(\Delta_{\rho}^{\alpha}\right)$. We denote the set of all $\Delta_{\rho}^{\alpha}-$statistically convergent sequences by $\Delta_{\rho}^{\alpha}(S)$. If $\rho_n=n$ for all $n\in\mathbb{N},$ $\Delta_{\rho}^{\alpha}-$statistical convergence is coincides with $\Delta^{\alpha}-$statistical convergence which was defined and studied Baliarsingh et al. [7].

We give the following theorem without proof.

Theorem 2.3 Let $x=(x_k)$ and $y=(y_k)$ be two sequences of real numbers, then

(*i*) $x_k \to x_0\left(\Delta_\rho^\alpha(S)\right)$ and, $c \notin \mathbb{C}$ implies $(cx_k) \to cx_0\left(\Delta_\rho^\alpha(S)\right)$,

(ii) $x_k \to x_0\left(\Delta_\rho^\alpha(S)\right)$ and $y_k \to y_0\left(\Delta_\rho^\alpha(S)\right)$, implies $(x_k+y_k) \to (x_0+y_0)\left(\Delta_\rho^\alpha(S)\right)$.

Theorem 2.4 Let p be a positive real number and $m \in \mathbb{N}$ then the sequence space $\Delta_\rho^m(p)$ is a Banach space for $1 \le p < \infty$ normed by

$$\|x\|_{\Delta_1} = \sum_{i=1}^{m}|x_i| + \sup_n\left(\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m x_k\right|^p\right)^{\frac{1}{p}} \tag{3}$$

and a complete $p-$ normed space for $0<p<1$ by

$$\|x\|_{\Delta_2} = \sum_{i=1}^{m}|x_i| + \sup_n\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m x_k\right|^p. \tag{4}$$

Proof. It is trivial that $\Delta_\rho^m(p)$ is a normed space normed by (3). Let (x^s) be a Cauchy sequence in $\Delta_\rho^m(p)$, where $x^s=\left(x_i^s\right)_{i=1}^{\infty}=\left(x_1^s,x_2^s,\ldots\right)\in\Delta_\rho^m(p)$, for each $s \in \mathbb{N}$. Then

$$\left\|x^s - x^t\right\|_{\Delta_1} \to 0, \text{ as } s,t \to \infty.$$

Let $\varepsilon>0$ be given, then there exists a positive integer $n_0(\varepsilon)$ such that $\left\|x^s-x^t\right\|_{\Delta_1} \le \varepsilon$, for all $s,t>n_0$. So we have

$$\sum_{i=1}^{m}\left|x_i^s - x_i^t\right| + \sup_n\left(\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m\left(x_k^s - x_k^t\right)\right|^p\right)^{\frac{1}{p}} \le \varepsilon, \quad \text{for all } s,t > n_0.$$

Hence $\left|x_i^s - x_i^t\right| \le \varepsilon$ for all $i = 1,2,\ldots,m$ and

$$\sup_n\left(\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m\left(x_k^s - x_k^t\right)\right|^p\right)^{\frac{1}{p}} \le \varepsilon, \quad \text{for all } s,t > n_0$$

and so

$$\left|\Delta^m\left(x_k^s - x_k^t\right)\right| \le \varepsilon, \quad \text{for all } s,t > n_0.$$

On the other hand we have

$$\left|x_{k+m}^s - x_{k+m}^t\right| \le \left|\sum_{v=0}^{m}(-1)^v\binom{m}{v}\left(x_{k+v}^s - x_{k+v}^t\right)\right| + \left|\binom{m}{0}\left(x_k^s - x_k^t\right)\right| \ldots$$
$$+\left|(-1)^v\binom{m}{m-1}\left(x_{k+(m-1)}^s - x_{k+(n-1)}^t\right)\right|.$$

From the last inequality, we get

$$\left|x_k^s - x_k^t\right| < \varepsilon \quad \text{for all } s,t > n_0, \text{ for each } k \in \mathbb{N}$$

Therefore $\left(x_k^s\right) = \left(x_k^1, x_k^2, \ldots\right)$ is a Cauchy sequence in C. Since C is complete, it is convergent

$$\lim_s x_k^s = x_k$$

say, for each $k \in \mathbb{N}$ Since $\left\|x_k^s - x_k^t\right\|_{\Delta_1} \le \varepsilon$, for each $k \in \mathbb{N}$ and for all $s,t > n_0$, we get

$$\sum_{i=1}^{m}\left|x_i^s - x_i^t\right| \le \varepsilon$$

and

$$\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m\left(x_k^s - x_k^t\right)\right|^p \le \varepsilon^p.$$

Hence,

$$\lim_t \sum_{i=1}^{m}\left|x^s - x_i^t\right| = \sum_{i=1}^{m}\left|x_i^s - x_i\right| \le \varepsilon$$

and

$$\lim_t \frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m\left(x_k^s - x_k^t\right)\right|^p = \frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m\left(x_k^s - x_k\right)\right|^p \le \varepsilon^p,$$

for each $k \in \mathbb{N}$ and for all $s > n_0$. This implies that $\left\|x^s - x\right\|_{\Delta_1} \le 2\varepsilon$ for all $s > n_0$, that is $x^s \to x$ as $n \to \infty$, where $x = \left(x_k\right)$.

Since,

$$\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m x_k - L\right|^p \le 2^p\left\{\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m x_k^{n_0} - L\right|^p + \frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m x_k^{n_0} - \Delta^m x_k\right|^p\right\}$$

we get $x \in \Delta_\rho^m\left(p\right)$. Therefore $\Delta_\rho^m\left(p\right)$ is complete.

It can be shown that $\Delta_\rho^m\left(p\right)$ is a complete $p-$normed space for $0 < p < 1$ by (4).

Theorem 2.5 The space $\Delta_{\rho}^{m}(p)$ is a $BK-$space.

Proof. Let $\left\|x^{s}-x\right\|_{\Delta_1} \to 0$ $(s \to \infty)$. Then given an $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\left\|x^{s}-x\right\|_{\Delta_1} < \varepsilon, \quad \text{for all } s > n_0.$$

Hence, we have

$$\sup_n \left(\frac{1}{\rho_n} \sum_{k=1}^{n} \left| \Delta^m \left(x_k^s - x_k \right) \right|^p \right)^{\frac{1}{p}} < \varepsilon, \text{ for all } s > n_0$$

and so

$$\left| x_k^s - x_k \right| < \varepsilon \rho_1, \quad \text{for all } s > n_0 \text{ and for all } k \in \mathbb{N}$$

Consequently, we have

$$\left| x_k^s - x_k \right| < \varepsilon, \text{ for all } s > n_0 \text{ and for all } k \in \mathbb{N}$$

and this completes the proof.

Theorem 2.6 (*i*) $\Delta_{\rho}^{m-1}(S) \subset \Delta_{\rho}^{m}(S)$ and the inclusion is strict,

(*ii*) $\Delta_{\rho}^{m-1}(p) \subset \Delta_{\rho}^{m}(p)$ and the inclusion is strict.

Proof. We only give the proof for ii), the other can be given similarly. Let $x \in \Delta_{\rho}^{m-1}(p)$, then

$$\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^m x_k - 2L\right|^p \le 2^p\left\{\frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^{m-1} x_k - L\right|^p + \frac{1}{\rho_n}\sum_{k=1}^{n}\left|\Delta^{m-1} x_{k+1} - L\right|^p\right\}$$

and so we have $x \in \Delta_\rho^m(p)$. To see that the inclusion is strict, let $p=1$ and consider a sequence defined by $x=(i^m)$, then $x \in \Delta_\rho^m(p)$, but $x \notin \Delta_\rho^{m-1}(p)$. Actually, if $x=(i^m)$, then $\Delta^m(i^m)=\left((-1)^m m!\right)$ and $\Delta^{m-1}(i^m)=\left((-1)^{m+1} m!\left(i+(m-1)/2\right)\right)$.

We give the following theorem without proof.

Theorem 2.7 (i) If $(x_k) \to \ell\left(\Delta_\rho^\alpha(w)\right)$, then $(x_k) \to \ell\left(\Delta_\rho^\alpha(S)\right)$,

(ii) If $x_k \in \Delta^\alpha(\ell_\infty)$ and $(x_k) \to \ell\left(\Delta_\rho^\alpha(S)\right)$, then $(x_k) \to \ell\left(\Delta_\rho^\alpha(w)\right)$.

Remark 2.8 In Theorem 2.7 (ii), the boundedness condition couldn't omitted. For this, consider a sequence $x=(x_k)$ such as

$$\Delta^m x_k = \begin{cases} k^2, & \text{if } k=m^2 \\ 0, & \text{if } k \ne m^2 \end{cases}, \text{ where } m \in \mathbb{N}$$

and let $\rho_n = n$ for all $n \in \mathbb{N}$, then we get

$$\frac{1}{\rho_n}\left|\{k \le n : |\Delta^m x_k - 0| \ge \varepsilon\}\right| = \frac{\sqrt{n}}{n} \to 0 (n \to \infty).$$

On the other hand,

$$\frac{1}{\rho_n}\sum_{k=1}^{n}|\Delta^m x_k - 0| = \frac{1}{n}\sum_{k=1}^{n}k^2 \nrightarrow 0 \ (n\to\infty),$$

i.e. $\Delta^m x_k$ is not strongly convergent to zero.

Theorem 2.9 Let $\rho=(\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n}<\infty$, $\Delta\rho_n = O(1)$, then $\Delta_\rho^\alpha(S)\subset\Delta^\alpha(S)$.

Proof. Let $\limsup_n \frac{\rho_n}{n}<\infty$, then there exists a constant $M>0$ such that $\frac{\rho_n}{n}\le M$, for all $n\ge 1$. Then for a given $\varepsilon>0$, we have

$$\frac{1}{n}|\{k\le n : |\Delta^\alpha x_k - \ell|\ge\varepsilon\}| = \frac{\rho_n}{n}\frac{1}{\rho_n}|\{k\le n : |\Delta^\alpha x_k - \ell|\ge\varepsilon\}|$$

$$\le \frac{M}{\rho_n}|\{k\le n : |\Delta^\alpha x_k - \ell|\ge\varepsilon\}|.$$

Hence, we get $\Delta_\rho^\alpha(S)\subset\Delta^\alpha(S)$.

Theorem 2.10 Let $\rho=(\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n}<\infty$, $\Delta\rho_n = O(1)$. If $\rho_n\ge n$ for all $n\in\mathbb{N}$ then $\Delta^\alpha(S)\subset\Delta_\rho^\alpha(S)$.

Proof. If $x_k \to \ell\left(\Delta^{\alpha}\left(S\right)\right)$, then for every $\varepsilon > 0$ we have

$$\frac{1}{n}|\{k \le n : |\Delta^{\alpha} x_k - \ell| \ge \varepsilon\}| = \frac{\rho_n}{n}\frac{1}{\rho_n}|\{k \le n : |\Delta^{\alpha} x_k - \ell| \ge \varepsilon\}|$$

$$\ge \frac{1}{\rho_n}|\{k \le n : |\Delta^{\alpha} x_k - \ell| \ge \varepsilon\}|.$$

Theorem 2.11 Although the space $N_{\rho}^{0}\left(p\right)$ is normal and monotone, the sequence space $\Delta_{\rho}^{m}\left(p\right)$ is not solid, is not monotone, is not sequence algebra and is not symmetric, for $m \ge 1$ and $p > 0$.

Proof. Let $x \in N_{\rho}^{0}\left(p\right)$ and $y = \left(y_n\right)$ be a sequence such that $\left|x_n\right| \le \left|y_n\right|$ for each $n \in \mathbb{N}$ Then we get

$$\sup \frac{1}{\rho_n}\sum_{k=1}^{n}\left|x_n\right|^{p} \le \sup \frac{1}{\rho_n}\sum_{k=1}^{n}\left|y_n\right|^{p}.$$

Hence $N_{\rho}^{0}\left(p\right)$ is solid and hence monotone.

Example 1 It is obvious that, if $x = \left(k^{m-2}\right)$, $y = \left(k^{m-2}\right)$, then $x, y \in \Delta_{\rho}^{m}\left(p\right)$, but $xy \notin \Delta_{\rho}^{m}\left(p\right)$. Hence $\Delta_{\rho}^{m}\left(p\right)$ is not a sequence algebra.

Example 2 It is obvious that, if $x=\left(k^{m-1}\right)$, then $x\in\Delta_{\rho}^{m}\left(p\right)$, but $\left(\beta_k x_k\right)\notin\Delta_{\rho}^{m}\left(p\right)$ for $\left(\beta_k\right)=\left(\left(-1\right)^{k}\right)$. Hence $\Delta_{\rho}^{m}\left(p\right)$ is not solid.

Example 3 We have that $x=\left(x_k\right)\in\Delta_{\rho}^{m}\left(p\right)$ if $\left(x_k\right)=\left(k^{m-1}\right)$. Let $\left(y_k\right)$ be a rearrangement of $\left(x_k\right)$ which is defined as follows:

$$\left(y_k\right)=\left(x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10},\ldots\right).$$

Then $y\notin\Delta_{\rho}^{m}\left(p\right)$. Hence $\Delta_{\rho}^{m}\left(p\right)$ is not symmetric.

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ON TOPOLOGIZED POLYGROUPS

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Abstract

A polygroup endowed with some topology is called a topologized polygroup. In this paper, we introduce different topological polygroups and study some of their properties.

Keywords: Topological polygroup, Semi-topological polygroup, *S*-topological polygroup.

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1. Introduction and Preliminaries

The notion of Polygroups, introduced as *Quasi-canonical hypergroups* by Bonansinga et al. [12], are special subclass of hypergroups [6]. Polygroup has large application in many area of Mathematics, such as geometry, lattice theory, combinatorics and color scheme, etc.. Heidari et al. introduced the concept of topological hypergroup [3] and topological polygroup [4]. Singha et al. introduced topological complete hypergroup [7], topological regular hypergroup [7] and investigated some of their properties. Different uniform structures for polygroups have been studied in [8]. Till date, very few papers [13, 3, 4, 9, 7, 8, 10] treated the topological notion of hypergroups and its subclasses. In this new setting, we introduce and study different topological polygroups finding inter-relations between them.

We begin with some basic definitions and results which will be used as ready references. A *hyperoperation* on a nonempty set H is a function $\circ: H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the family of nonempty subsets of H. The ordered couple $(H,\circ)$ is called a *hypergroupoid*. If A and B are two nonempty subsets of a hypergroupoid $(H,\circ)$ and $x \in H$, then

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A *polygroup* [14, 1] is a system $P = \langle P, \circ, e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\circ: P \times P \to \mathcal{P}^*(P)$ satisfying the following axioms for all $x, y, z \in P$:

- $(x \circ y) \circ z = x \circ (y \circ z)$;
- $e \circ x = x \circ e = \{x\}$;
- $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts hold readily from the above axioms:

$$e \in x \circ x^{-1} \cap x^{-1} \circ x,\ e^{-1} = e,\ (x^{-1})^{-1} = x,\ \text{and}\ (x \circ y)^{-1} = y^{-1} \circ x^{-1}.$$

For a nonempty subset A of P, $A^{-1} := \{x^{-1} : x \in A\}$ and A is called symmetric if $A^{-1} = A$. A nonempty subset K of a polygroup P is called *subpolygroup* of P if (i) $a, b \in K$ implies $a \circ b \subseteq K$ and (ii) $a \in K$ implies $a^{-1} \in K$.

For more details about polygroup, one may refer to [1, 14].

Levine [11] introduced semi-open sets in a topological space in 1963. A set A in a topological space X is said to be *semi-open* if and only if there exits an open set O such that $O \subseteq A \subseteq \overline{O}$. Collection of all semi-open sets in X is denoted by $SO(X)$. Arbitrary union of semi-open sets is semi-open but intersection of two semi-open sets not necessarily semi-open. The problem is, "whether the intersection of a semi-open set with an open set is semi-open?". A subset M of X is a *semi-neighborhood* of a point $x \in X$, if there exists $O \in SO(X)$ such that $x \in O \subseteq M$. M is semi-open if it is a semi-neighborhood of each of its points. If $(X \times Y, \tau_1 \times \tau_2)$ is the product space of the topological spaces (X, τ_1), (Y, τ_2) and $A_1 \in SO(X)$, $A_2 \in SO(Y)$, then $A_1 \times A_2 \in SO(X \times Y)$. The complement of a semi-open set is called *semi-closed*; for $A \subseteq X$, the *semi-closure* of A is denoted by $sCl(A)$, is the intersection of all semi-closed sets of X containing A [15, 16]. $x \in sCl(A)$ if and only if for any semi-open neighborhood U of x, $U \cap A \neq \emptyset$.

Let X and Y be topological spaces. A mapping $f: X \to Y$ is *semi-continuous* if for each open set V in Y, $f^{-1}(V) \in SO(X)$. Equivalently, a mapping $f: X \to Y$ is semi-continuous if and only if for each $x \in X$ and each open neighborhood V of $f(x)$ there is a semi-open neighborhood U of x such that $f(U) \subseteq V$.

Separation axioms semi-T_0, semi-T_1, semi-T_2 and s-regular are defined as the classical axioms T_0, T_1, T_2 and regular, replacing everywhere 'open neighborhoods' by 'semi-open neighborhoods' [2].

Let (H, τ) be a topological space and $\mathcal{P}^*(H)$ denotes the collection of all nonempty subsets of H. Then, the family $\mathcal{L} = \{I_V := \{U \in \mathcal{P}^*(H): U \cap V \neq \emptyset\} : V \in \tau\}$ is a subbase for a topology $\tau_{\mathcal{L}}$ on $\mathcal{P}^*(H)$ [17]. Throughout this paper, 'neighborhood' stands for 'open neighborhood'.

2. Different topological polygroups and their inter-relations

In this section, we introduce semi-topological polygroup, $S-$topological polygroup and study some of their properties.

Definition 2.1. Let $(P,\circ, e,^{-1})$ be a polygroup and (P, τ) be a topological space. Then, the system $(P,\circ, e,^{-1}, \tau)$ is called

(*i*) an $\tau_{\mathcal{L}}-$*topological polygroup* if the mappings $\circ: P \times P \to \mathcal{P}^*(P)$ and $^{-1}: P \to P$ are continuous while considering the product topology on $P \times P$ and the topology $\tau_{\mathcal{L}}$ on $\mathcal{P}^*(P)$.
(*ii*) a *semi-topological polygroup* if for each $x, y \in P$ and $W \in \tau$ with $x \circ y^{-1} \cap W \neq \emptyset$, there exist semi-neighborhoods U, V of x, y respectively, such that $u \circ v^{-1} \cap W \neq \emptyset$, for all $u \in U, v \in V$.
(*iii*) an $S-$*topological polygroup* if with respect to the product topology on $P \times P$ and the topology $\tau_{\mathcal{L}}$ on $\mathcal{P}^*(P)$ the mappings $(x, y) \mapsto x \circ y$ from $P \times P \to \mathcal{P}^*(P)$ and $x \mapsto x^{-1}$ from $P \to P$ are semi-continuous.

Example 2.2. Consider the polygroup $P = \{1, 2, 3\}$ together with the hyperoperation $\circ$ defined as follows:

$\circ$	1	2	3
1	$\{1\}$	$\{2\}$	$\{3\}$
2	$\{2\}$	$\{1, 3\}$	$\{2\}$
3	$\{3\}$	$\{2\}$	$\{1, 3\}$

and a topology $\tau = \{\emptyset, \{2\}, \{1, 3\}, P\}$ on it. Then, P is an $\tau_{\mathcal{L}}-$topological polygroup.

It is clear from the above definitions that every $\tau_{\mathcal{L}}$ −topological polygroup is both semi-topological and S −topological polygroup. We present some other examples.

Example 2.3. Consider the polygroup $P = \{1, 2, 3\}$ with the hyperoperation $\circ$ defined as follows:

$\circ$	1	2	3
1	$\{1\}$	$\{2\}$	$\{3\}$
2	$\{2\}$	$\{1, 3\}$	$\{2\}$
3	$\{3\}$	$\{2\}$	$\{1, 3\}$

and a topology $\tau = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, P\}$ on P. Then, P is a semi-topological polygroup but not an $\tau_{\mathcal{L}}$ −topological polygroup.

Theorem 2.4. *For a semi-topological polygroup P, if $A \in SO(P)$, then $A^{-1} \in SO(P)$.*

Proof. The proof is same as of Theorem 5 [5].

Unless otherwise mentioned, always consider the topology $\tau_{\mathcal{L}}$ on $\mathcal{P}^*(P)$ in the sequel.

Theorem 2.5. *If $(P, \circ, e, ^{-1}, \tau)$ is a semi-topological polygroup, then the map $f: P \times P \to \mathcal{P}^*(P)$ defined by $f(x, y) = x \circ y^{-1}$ is semi-continuous with respect to the product topology on $P \times P$.*

Proof. Consider $(x, y) \in P \times P$ and $W \in \tau$ such that $x \circ y^{-1} \in I_W$. Then there exist semi-neighborhoods U and V containing x, y respectively, such that $u \circ v^{-1} \cap W \neq \emptyset$, for all $u \in U$, $v \in V$. This implies $U \times V$ is a semi-neighborhood of (x, y) such that $f(U \times V) \subseteq I_W$.

Theorem 2.6. *If $(P, \circ, e, ^{-1}, \tau)$ is a semi-topological polygroup, then the map $m: P \times P \to \mathcal{P}^*(P)$ defined by $m(x, y) = x \circ y$ and the map $i: P \to P$ defined by $i(x) = x^{-1}$ are semi-continuous.*

Proof. Let $x \in P$ and $N \in \tau$ such that $x^{-1} \in N$. Then, $e \circ x^{-1} \cap N \neq \emptyset$ and there exist semi-neighborhoods U, V of e, x respectively, such that $u \circ v^{-1} \cap N \neq \emptyset$, for all $u \in U, v \in V$. In particular, $e \circ v^{-1} \cap N \neq \emptyset$, for all $v \in V$ and hence $i(V) \subseteq N$.

Now let $(x, y) \in P \times P$ and $M \in \tau$ such that $x \circ y \cap M \neq \emptyset$, i.e., $x \circ (y^{-1})^{-1} \cap M \neq \emptyset$. Then, there exist semi-neighborhoods U, V of x, y^{-1} respectively, such that $u \circ v^{-1} \cap M \neq \emptyset$, for all $u \in U, v \in V$. Then, $W = V^{-1}$ is a semi-neighborhood (by Theorem 2.4) of y and $u \circ w \cap M \neq \emptyset$, for all $u \in U, w \in W$. Therefore, $U \times W$ is a semi-neighborhood of (x, y) in $P \times P$ such that $m(U \times W) \subseteq I_M$.

Corollary 2.7. *Every semi-topological polygroup is an S −topological polygroup.*

Theorem 2.8. *If $A \in \tau$ and $B \subseteq P$, then $A \circ B$, $B \circ A \in SO(P)$.*

Proof. Let $x \in B$ and $y \in A \circ x$. Then, $y \in a \circ x$ for some $a \in A$ and hence $a \in y \circ x^{-1}$, i.e., $y \circ x^{-1} \cap A \neq \emptyset$. There exist semi-neighborhoods M, N of y, x respectively, such that $m \circ n^{-1} \cap A \neq \emptyset$, for all $m \in M$, $n \in N$. In particular, for $n = x$ and for all $m \in M$, $m \circ x^{-1} \cap A \neq \emptyset$, i.e., for each $m \in M$, there exists $a \in A$ such that $m \in a \circ x$. Therefore, $y \in M \subseteq A \circ x$ and hence $A \circ x \in SO(P)$. So, $\bigcup_{b \in B} A \circ b = A \circ B \in SO(P)$.

Proposition 2.9. *In a semi-topological polygroup for a given neighborhood U of the scalar identity e there exists a symmetric semi-neighborhood V of e such that $V \subseteq U$.*

Proof. If U is a neighborhood of e, then U^{-1} is a semi-neighborhood of e (by Theorem 2.4). Take $V = U \cap U^{-1}$, then V is a symmetric semi-neighborhood of e such that $V \subseteq U$.

Next examples illustrate that the converse of the Corollary 2.7 is not true in general.

Example 2.10. Consider a polygroup $P = \{1,2\}$ with hyperoperation $\circ$ on P defined as follows:

$\circ$	1	2
1	$\{1\}$	$\{2\}$
2	$\{2\}$	$\{1, 2\}$

and a topology $\tau = \{\phi, \{1\}, P\}$ on P. Then,

$$SO(P \times P) = \{\phi, \{(1,1)\}, \{(1,1), (1,2)\}, \{(1,1), (2,1)\},$$

$\{(1,1),(1,2),(2,1)\},\{(1,1),(1,2),(2,1),(2,2)\},$

$\{(1,1),(2,2)\},\{(1,1),(1,2),(2,2)\},\{(1,1),(2,1),(2,2)\}\}.$

Here, the map $m: P \times P \to \mathcal{P}^*(P)$ defined by $m(x,y) = x \circ y$ is continuous at $(1,1),(1,2),(2,1)$ and semi-continuous at $(2,2)$. The inverse mapping $i: P \to P$ defined by $i(x) = x^{-1}$ is continuous and hence semi-continuous. Thus, P is an $S-$ topological polygroup. But, P is not a semi-topological polygroup. For, $\{1\} \in \tau$ and $\{1\} \circ 2 = \{2\}$ is not semi-open in P.

Example 2.11. Consider $P = \{0,1,2\}$ with the hyperoperation $\circ$ defined as follows

$\circ$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	P
2	$\{2\}$	P	$\{2\}$

Then $(P,\circ)$ is a polygroup. Equip P with the topology $\tau = \{\emptyset, \{0\}, P\}$.

Here, the map $m: P \times P \to \mathcal{P}^*(P)$ defined by $m(x,y) = x \circ y$ is continuous at $(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(2,2)$. However, it is semi-continuous at $(1,2),(2,1)$. The inverse mapping $i: P \to P$ defined by $i(x) = x^{-1}$ is continuous and hence semi-continuous. Thus, P is an $S-$topological polygroup. Here also P is not a semi-topological polygroup. For, $\{0\} \in \tau$ and $\{0\} \circ 2 = \{2\}$ is not semi-open in P.

Example 2.12. Consider the polygroup $P = \{1, 3, 5, 7\}$ together with the hyperoperation $\circ$ defined as follows:

$\circ$	1	3	5	7
1	{1}	{3}	{5}	{7}
3	{3}	{1}	{5}	{7}
5	{5}	{5}	{1, 3, 7}	{5, 7}
7	{7}	{7}	{5, 7}	{1, 3, 5}

Let P be equipped with the topology $\tau = \{\emptyset, P, \{1\}, \{1, 3, 5\}\}$. Then, P is an S −topological polygroup and not a semi-topological polygroup. As the map $m: P \times P \to \mathcal{P}^*(P)$ defined by $m(x, y) = x \circ y$ is continuous at (1,1), (1,3), (3,1), (1,5), (5,1), (1,7), (7,1), (3,5), (5,3), (3,7), (7,3) and semi-continuous at (3,3), (5,5), (5,7), (7,5), (7,7); inverse mapping $i: P \to P$ defined by $i(x) = x^{-1}$ is continuous and hence semi-continuous. Therefore, P is an S −topological polygroup and not a semi-topological polygroup.

It is clear from the above examples that the class of S −topological polygroups is wider than the class of semi-topological polygroups.

Theorem 2.13. *Let $(P, \circ, e, ^{-1}, \tau)$ be a semi-topological polygroup. Then, for $p \in P$, the maps $l_p: P \to \mathcal{P}^*(P)$ by $x \mapsto p \circ x$ (left translation) and $r_p: P \to \mathcal{P}^*(P)$ by $x \mapsto x \circ p$ (right translation) are semi-continuous.*

Proof. For $x \in P$ and $W \in \tau$, let $l_p(x) \in I_W$, i.e., $p \circ x \cap W \neq \emptyset$. Then, there exist semi-neighborhoods U and V of p and x^{-1} respectively, such that $u \circ v^{-1} \cap W \neq \emptyset$, for all $u \in U$, $v \in V$. By Theorem 2.4, the set V^{-1} is a semi-neighborhood of x. In particular, for $u = p$ and for all $z \in V^{-1}$, $p \circ z \cap W \neq \emptyset$. This shows that for $p \in P$, l_p is semi-continuous on P.

Theorem 2.14. *Let H be a subpolygroup of a semi-topological polygroup P. Then*

(1) *H is semi-open if it contains a nonempty open subset;*

(2) *H is a semi-closed semi-topological polygroup if it is open.*

Proof. (1) Let U be a nonempty open subset of P contained in H. For $h \in H$, $h \circ U$ is semi-open in P (by Theorem 2.8). Therefore, $H = \bigcup_{h \in H} h \circ U$ is semi-open in P.

(2) Suppose H be open in P. Let $x, y \in H$ and W be an open subset of H such that $x \circ y^{-1} \cap W \neq \emptyset$. Since W is open in P, there exist semi-neighborhoods A, B in P of x, y respectively, such that $a \circ b^{-1} \cap W \neq \emptyset$, for all $a \in A, b \in B$. The sets $U = A \cap H$ and $V = B \cap H$ are semi-neighborhoods in H of x, y respectively, such that $u \circ v^{-1} \cap W \neq \emptyset$, for all $u \in U, v \in V$. Thus, H is a semi-topological polygroup.

For $x \in P$, $x \circ H$ is semi-open in P (by Theorem 2.8). Now, $P = \bigcup_{x \in P} x \circ H = (\bigcup_{x \in H} x \circ H) \cup (\bigcup_{x \in P \setminus H} x \circ H) = H \cup (\bigcup_{x \in P \setminus H} x \circ H)$. This implies $H = P \setminus (\bigcup_{x \in P \setminus H} x \circ H)$ and hence H is semi-closed.

It is clear from (2) of Theorem 2.14 that every open subpolygroup of a semi-topological polygroup is also a semi-topological polygroup, such subpolygroups are called *semi-topological subpolygroups*.

Remark 2.15. Consider the Example 2.12 of S −topological polygroup. The collection of semi-open sets in P is

$$SO(P) = \{\emptyset, P, \{1\}, \{1,3\}, \{1,5\}, \{1,7\}, \{1,3,5\}, \{1,3,7\}, \{1,5,7\}\}.$$

Here, P is a semi-T_0 space, but not semi-T_1. Also it is not a s-regular space, since for the closed set $C = \{3,5,7\}$ and $1 \notin C$, there exist no disjoint semi-open sets containing them.

Also, the polygroup in Example 2.10 has the same properties.

Theorem 2.16. *If in a semi-topological polygroup P, for every neighborhood U of the identity e there exists symmetric neighborhood V of e such that $V \circ V \subseteq U$, then P is s-regular at e.*

Proof. Let U be a neighborhood of the identity e. Then, there exists a symmetric neighborhood V of e such that $V \circ V \subseteq U$. To show $sCl(V) \subseteq U$, let $x \in sCl(V)$. $x \circ V$ is a semi-neighborhood of x, so $x \circ V \cap V \neq \emptyset$. Therefore, there exist $v_1, v_2 \in V$ such that $v_2 \in x \circ v_1$. This implies $x \in v_2 \circ v_1^{-1} \subseteq V \circ V^{-1} = V \circ V \subseteq U$.

Theorem 2.17. *Let P be a semi-topological polygroup and A, B are subsets of P. Then, the following results hold.*

(1) $sCl(A) \circ sCl(B) \subseteq \overline{A \circ B}$;

(2) $(sCl(A))^{-1} \subseteq \overline{(A^{-1})}$.

Proof. (1) Suppose, $t \in sCl(A) \circ sCl(B)$. Then, $t \in x \circ y$, for some $x \in sCl(A), y \in sCl(B)$. Let W be a neighborhood of t. Then, $x \circ y \cap W \neq \emptyset$ and there exist semi-neighborhoods U, V of x, y respectively such that for all $u \in U, v \in V$, $u \circ v \cap W \neq \emptyset$. Since $x \in sCl(A), y \in sCl(B)$, there exist $a \in A \cap U$ and $b \in B \cap V$. Now, $\emptyset \neq a \circ b \subseteq A \circ B$ and $a \circ b \cap W \neq \emptyset$. Thus, $t \in \overline{A \circ B}$.

(2) Let $x \in (sCl(A))^{-1}$ and U be a neighborhood of x. Then, U^{-1} is a semi-neighborhood of x^{-1} (by Theorem 2.4). Since $x^{-1} \in sCl(A)$, $A \cap U^{-1} \neq \emptyset$, which implies $A^{-1} \cap U \neq \emptyset$. Thus, $x \in \overline{(A^{-1})}$.

We conclude this section with the following remark.

Remark 2.18. The inclusion in (1) of Theorem 2.17 may not hold for an $S-$topological polygroup. For, consider the Example 2.12 of $S-$topological polygroup and take $A = \{1, 3\}$ and $B = \{5, 7\}$. Then, $sCl(A) = P$, $sCl(B) = \{5,7\}$, $sCl(A) \circ sCl(B) = P$, $A \circ B = \{5, 7\}$ and $\overline{A \circ B} = \{3, 5, 7\}$.

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On the Domain of the Difference Matrix $\widehat{D}(\hat{r}, 0, 0, \hat{s})$ on the Sequence Space $\ell(\widehat{D}, p)$

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Abstract

In this paper, we introduce the new paranormed sequence space $\ell(\widehat{D}, p)$ which is the domain of the difference matrix $\widehat{D}(\hat{r}, 0,0, \hat{s})$ in the sequence space $\ell(p)$ and study some topological properties.

Key Words: *Paranorm sequence space*; *complete linear metric space*; *AK space*; *BK space*.

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1. Introduction

The idea to construct a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Altay and Basar[3, 4], Basar et al.[7], Kirisci and Basar[8], Ng and Lee[12], Sonmez[14] and many more. Moreover, Altay and Basar[1, 2], Malkowsky[10] and Aydin and Basar[6] have employed on to construct new paranormed sequence spaces by means of the domain of some infinite matrices. The domain of generalized difference matrix $B(r, s)$ on some Maddox's spaces was studied by Aydin and Altay [5]. More recently, domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$on some Maddox's spaces was studied by Ozger and Basar[13] and Nergiz and Basar[11].

2. Preliminaries

Throughout the paper we denote w, ℓ_∞, c, c_0 and ℓ_p be the space of all, bounded, convergent, null and p-absolutely summable sequences respectively.

Let X and Y be two sequence spaces and $B = (b_{nk})$ be an infinite matrix of real or complex numbers b_{nk}, where $n, k \in N = \{0,1,2, \ldots\}$. Then, we say that B defines a matrix mapping from X into Y, denoted by $B: X \rightarrow Y$, if for every sequence $x = (x_n) \in X$, the sequence $Bx = (Bx)_n$ is in Y where,

$$(Bx)n = \sum_{k=1}^{\infty} b_{nk} x_k, \ (n \in N \text{ and } x \in X), \tag{2.1}$$

Provided the right hand side converges for every $n \in N$ and $x \in X$.

If μ is a normed sequence space, we write $D\mu(B)$ for $x \in w$ for which the sum in (2.1) converges in the norm of μ. We write $(\lambda, \mu) = \{B: \lambda \subset D_{\mu}(B)\}$ for the space of those matrices which send the whole of the sequence space λ into the sequence space μ in this sense.

The sequence space $\lambda_B = \{x = (x_k) \in w: Bx \in \lambda\}$ is called the domain of an infinite matrix B in a sequence space λ. One can easily verify that the sequence spaces λ_B and λ are linearly isomorphic when B is triangle. A paranormed space (X, g) is a topological linear space in which the topology is given by paranorm g, a real sub-additive function on X such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous means that $\lambda_n \rightarrow \lambda$, $x_n \rightarrow x$ imply $\lambda_n x_n \rightarrow \lambda x$, for scalars λ and vectors x.

We consider (p_k) is a bounded sequence of positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Throughout we assume $p_k^{-1} + (p_k')^{-1} = 1$ provided $0 < \inf p_k \leq H < \infty$.

The linear space $\boldsymbol{\ell(p)}$ were defined by Maddox [9] as follows:

$\boldsymbol{\ell(p)} = \{x = (x_k) \in w: \sum_k |x_k|^{p_k} < \infty\}$, $(0 < p_k \leq H < \infty)$,

which is the complete space paranormed by $h(x) = (\sum_k |x_k|^{p_k})^{\frac{1}{M}}$. Throughout C denotes the complex field.

Let, $\hat{r} = (r_k)$ and $\hat{s} = (s_k)$ are convergent sequences whose entries either constants or distinct non-zero numbers then we define the matrix $\widehat{D}(\hat{r}, 0, 0, \hat{s})$ as follows: $\widehat{D}(\hat{r}, 0, 0, \hat{s}) = [d_{nk}(r, s)]$ where,

$$d_{nk}(r, s) = \begin{cases} r_k, (k = n) \\ s_k, (k = n - 3) \\ 0, otherwise, \end{cases}$$

for all $k, n \in N$.

3. Some new paranormed sequence spaces and their topological properties

We define the sequence space $\ell(\widehat{D}, p)$ as the set of sequences whose transform is in the spaces $\boldsymbol{\ell(p)}$ respectively, that is

$\ell(\widehat{D}, p) = \{x = (x_k): \sum_k |r_k x_k + s_{k-3} x_{k-3}|^{p_k} < \infty\}$, $(0<p_k \leq H <\infty)$.

Theorem 3.1 *The sequence space $\ell(\widehat{D}, p)$ is a the complete linear metric space paranormed by g, defined by* $g(x) = (\sum_k |r_k x_k + s_{k-3} x_{k-3}|^{p_k})^{\frac{1}{M}}$.

Proof: One can easily prove that $\ell(\widehat{D}, p)$ is a linear space with co-ordinate wise addition and scalar multiplication since $\widehat{D}(\hat{r}, 0,0, \hat{s})$ is a triangle matrix and $\boldsymbol{\ell(p)}$ is a linear space.

It is clear that $g(\theta) = 0$, $g(x) \geq 0$ for all $x \in \ell(\hat{D}, p)$ and $g(-x) = g(x)$.

Let $x, y \in \ell(\hat{D}, p)$; then by Minkowski's inequality we have

$$g(x+y)=\{\sum_k |s_{k-3}(x_{k-3} + y_{k-3}) + r_k(x_k + y_k)|^{p_k}\}^{\frac{1}{M}}$$

$$=\{\sum_k (|s_{k-3}(x_{k-3} + y_{k-3}) + r_k(x_k + y_k)|^{\frac{p_k}{M}})^M\}^{\frac{1}{M}}$$

$$\leq(\sum_k |s_{k-3} x_{k-3} + r_k x_k|)^{\frac{1}{M}} + (\sum_k |s_{k-3} y_{k-3} + r_k y_k|)^{\frac{1}{M}}$$

$$=g(x) + g(y)$$

Consider a sequence $\{x^n\}$ of elements of $\ell(\widehat{D}, p)$, such that $g(x^n - x) \to 0$, as $n \to \infty$ and (β_n) is a sequence of scalars with $\beta_n \to \beta$, as $n \to \infty$. Now, we observe that

$$g(\beta_n x^n - \beta x) \leq g[(\beta_n - \beta)(x^n - x)] + g[\beta(x^n - x)] + g[(\beta_n - \beta)x]. \quad (3.1)$$

It follows from $\beta_n \to \beta$ $(n \to \infty)$ that $|\beta_n - \beta| < 1$ for all sufficiently large n; hence
$$lim_{n\to\infty} g[(\beta_n - \beta)(x^n - x)] \leq lim_{n\to\infty} g(x^n - x) = 0. \quad (3.2)$$

Furthermore, we have $lim_{n\to\infty} g[\beta(x^n - x)] \leq max\{1, |\beta|^M\} lim_{n\to\infty} g(x^n - x) = 0.$ (3.3)

Also, we have $lim_{n\to\infty} g[(\beta_n - \beta)x] \leq lim_{n\to\infty} |\beta_n - \beta| g(x) = 0.$ (3.4)

Then, we obtain from (3.1),(3.2),(3.3) and (3.4) that $g(\beta_n x_n - \beta x) \to 0$, as $n \to \infty$.

This shows that g is a paranorm on $\ell(\widehat{D}, p)$.

Furthermore, if $g(x) = 0$, then $(\sum_k |r_k x_k + s_{k-3} x_{k-3}|^{p_k})^{\frac{1}{M}} = 0$.

Therefore, $|r_k x_k + s_{k-3} x_{k-3}|^{p_k} = 0$ for each $k \in N$. If we put $k= 0$, since $s_{-3}= 0$ and $r_0 \neq 0$, we have $x_0= 0$. Similarly, for $k= 1, 2$, since $s_{-2}=s_{-1}= 0$ and $r_1 \neq 0$ and $r_2 \neq 0$, we have $x_1=x_2= 0$. For $k= 3$, since $x_0= 0$, we have $x_4= 0$. Continuing in this way, we obtain $x_k= 0$ for all $k \in N$. That is, $x=\theta$. This concludes that g is a total paranorm.

Now, we show $\ell(\widehat{D}, p)$ is complete. Let $\{x^i\}$be a Cauchy sequence in $\ell(\widehat{D}, p)$, where $x^i=\{x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, - - -\}$. Then, by the definition of Cauchy sequence, for a given $\varepsilon > 0$, there exists a positive integer n_0 depending on ε such that $g(x^i - x^j) < \varepsilon$ for all $i, j \geq 0$.

Now, using the definition of g for each fixed $k \in N$,

$$\left|\left\{(\widehat{D}(\hat{r}, 0,0, \hat{s}) x^i\right\}_k - \left\{(\widehat{D}(\hat{r}, 0,0, \hat{s}) x^j\right\}_k\right| \leq \left\{\sum_k \left|\left\{\widehat{D}(\hat{r}, 0,0, \hat{s}) x^i\right\}_k - \left\{\widehat{D}(\hat{r}, 0,0, \hat{s}) x^j\right\}_k\right|^{p_k}\right\}^{\frac{1}{M}}$$

$$= g(x^i - x^j) < \varepsilon, \qquad (3.5)$$

for every $i, j > n_0(\varepsilon)$, which leads us to the fact that

$\{\{\widehat{D}(\hat{r}, 0,0, \hat{s}) x^1\}_k, \{\widehat{D}(\hat{r}, 0,0, \hat{s}) x^2\}_k, \{\widehat{D}(\hat{r}, 0,0, \hat{s}) x^3\}_k, - - - \}$ is a Cauchy sequence of complex numbers and is convergent for each $k \in N$. Since C is complete, it converges. Suppose, for each fixed k, $\{\widehat{D}(\hat{r}, 0,0, \hat{s}) x^i\}_k \to \{\widehat{D}(\hat{r}, 0,0, \hat{s}) x\}_k$, as $i \to \infty$.

Consider the sequence $\{(\widehat{D}(\hat{r}, 0,0, \hat{s}) x)_1, (\widehat{D}(\hat{r}, 0,0, \hat{s}) x)_2, (\widehat{D}(\hat{r}, 0,0, \hat{s}) x)_3 - - - \}$. Then for each $K \in N$ and $i, j > n_0(\varepsilon)$ we have

$$\left[\sum_{k=0}^{K} \left|\left((\widehat{D}(\hat{r}, 0,0, \hat{s}) x^i\right)_k - \left((\widehat{D}(\hat{r}, 0,0, \hat{s}) x^j\right)_k\right|^{p_k}\right]^{\frac{1}{M}} \leq g(x^i - x^j) < \varepsilon \qquad (3.6)$$

By taking $m, K \to \infty$, we have for $i > n_0(\varepsilon)$ that

$$g(x^i - x)=\left[\sum_k \left|\left((\widehat{D}(\hat{r},0,0,\hat{s})x^i\right)_k - \left((\widehat{D}(\hat{r},0,0,\hat{s})x\right)_k\right|^{p_k}\right]^{\frac{1}{M}} < \varepsilon \tag{3.7}$$

This shows that x^i-$x \in \ell(\widehat{D},p)$. Since $\ell(\widehat{D},p)$ is a linear space.

We conclude that $x \in \ell(\widehat{D},p)$.

It follows that $x^i \to x$, as $i \to \infty$ in $\ell(\hat{D},p)$. Hence, $\ell(\hat{D},p)$ is complete.

Theorem 3.2 *Convergence in $\ell(\widehat{D},p)$ is stronger than coordinator-wise convergence.*

Proof: First we show that $g(x^n - x) \to 0$, as $n \to \infty$ implies $x_k^n \to x_k$,, as $n \to \infty$, for every $k \in N$.

Fix k, then we have

$$\lim_{n\to\infty}\left|s_{k-3}x_{k-3}^{(n)} + r_k x_k^{(n)} - s_{k-3}x_{k-3} - r_k x_k\right|^{p_k}$$

$$\leq \lim_{n\to\infty}\sum_k \left|s_{k-3}x_{k-3}^{(n)} + r_k x_k^{(n)} - s_{k-3}x_{k-3} - r_k x_k\right|^{p_k}$$

$$=\lim_{n\to\infty}(g(x^n - x))^M = 0$$

Hence, we have for $k = 0$ that

$$\lim_{n\to\infty}\left|s_3 x_3^{(n)} + r_0 x_0^{(n)} - s_3 x_3 - r_0 x_0\right| = 0,$$

which gives $|x_0^{(n)} - x_0| \to 0$, as $n \to \infty$.

Similarly, for each $k \in N$, we have $|x_k^{(n)} - x_k| \to 0$, as $n \to \infty$.

A sequence space λ with a linear topology is called a *K*-space provided each of the maps p_i : $\lambda \to C$ defined by $p_i(x) = x_i$ is continuous for all $i \in N$. A *K*-space λ is called an *FK*-space provided λ is complete linear metric space. An *FK*-space whose topology is normable is called *BK*-space. Given a *BK*-space $\lambda \supset \varnothing$, we denote the n^{th} section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)}$, and we say that $x = (x_k)$ has the property *AK* if $\lim_{n\to\infty} ||x - x^{[n]}||_\lambda = 0$. If *AK*-property holds for every $x \in \lambda$, then we say that the space λ is called *AK*-space.

Theorem 3.3 *The space* $\ell(\widehat{D},p)$ *has AK property*.

Proof: For each $x = (x_k) \in \ell(\widehat{D},p)$, we put $x^{[n]}:=\sum_{k=0}^{n} x_k e^{(k)}$, for all $n\in\{1, 2, 3,- - -\}$.

Let $\varepsilon > 0$ and $x \in \ell(\widehat{D},p)$ be given. Then, there exist $n_0 = n_0(\varepsilon)\in N$ such that

$$\sum_{k=n_0}^{\infty} |s_{k-3}x_{k-3} + r_k x_k|^{p_k} < \varepsilon^M$$

Then, we have for all $n\geq n_0$,

$$g(x - x^{[n]}) = g(x-\sum_{k=0}^{n} x_k e^{(k)})=(\sum_{k=n+1}^{\infty}|s_{k-3}x_{k-3} + r_k x_k|^{p_k})^{\frac{1}{M}}$$

$$\leq\left(\sum_{k=n_0}^{\infty}|s_{k-3}x_{k-3} + r_k x_k|^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

This shows that $x =\sum_k x_k e^{(k)}$.

Now we have to show that this representation is unique. We assume that $x =\sum_k \lambda_k e^{(k)}$.

Then for each k;

$$(|s_{k-3}\lambda_{k-3} + r_k\lambda_k - s_{k-3}x_{k-3} - r_k x_k|^{p_k})^{\frac{1}{M}}$$

$$\leq(\sum_k|s_{k-3}\lambda_{k-3} + r_k\lambda_k - s_{k-3}x_{k-3} - r_k x_k|^{p_k})^{\frac{1}{M}}=g(x - x)=0$$

Hence, $s_{k-3}\lambda_{k-3} + r_k\lambda_k = s_{k-3}x_{k-3} + r_k x_k$, for each $k\in N$.

For $k = 0$, $r_0\lambda_0 = r_0x_0$. Since $r_0\neq0$, we have $\lambda_0 = x_0$. Continuing in this way, we obtain $\lambda_k = x_k$, for each $k\in N$. This shows that the representation is unique.

This completes the proof.

4 Conclusion

The difference operator Δ_3 as a particular case of Δ_m studied by Tripathy and Esi[15] can be obtain as a special case of the operator $\widehat{D}(\hat{r},0,0,\hat{s})$, if we consider $\hat{r} = e$ and $\hat{s} = -e$.

Further, $\widehat{D}(\hat{r}, 0,0, \hat{s})$ reduces to the difference operator $D(r, 0, 0, s)$ as the special case when $\hat{r} = re$ and $\hat{s} = se$, investigated by Tripthy and Paul [16]. Therefore, the results related to the domain of the matrix $\widehat{D}(\hat{r}, 0,0, \hat{s})$ investigated in this paper are more general and more comprehensive than the corresponding consequences of the domain of the matrix $D(r, 0, 0, s)$.

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APPLICATIONS OF FRACTIONAL ORDER OPERATORS FOR STRONGLY CONVERGENT SEQUENCES AND THEIR TOEPLITZ DUALS

KULDIP RAJ, SURUCHI PANDOH AND MANISHA DEVI

Abstract

In this paper we define new classes of strongly convergent difference sequences $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ by using the fractional difference operator $\Delta^{(\alpha)}$ and a modulus function. Also, we investigate some topological properties and establish the $\beta-$duals of the spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$. Furthermore, the matrix transformations among these spaces are characterized.

Keywords: Sequence space, Fractional difference operator, modulus function, $\beta-$duals, Matrix transformation.

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1. Introduction and Preliminaries

Let $\Gamma(m)$ be a Gamma function of the real number m and $m \notin \{0, -1, -2, -3, ...\}$. By the definition, it can be expressed as an improper integral, i.e.

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt. \quad (1.1)$$

It is clear from (1.1) that if m lies in the set of nonnegative integers $\mathbb{N}$, then $\Gamma(m + 1) = m!$. Now we discuss some essential properties of Gamma function as follows

(1). If $m \in \mathbb{N}$, then we have $\Gamma(m + 1) = m!$.

(2). If $m \in \mathbb{R} \backslash \{0, -1, -2, -3, ...\}$, then we have $\Gamma(m + 1) = m\Gamma(m)$.

(3). For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, ...$

Let w be the class of all real valued sequences. Any subclass of w is termed as a sequence space. By c, l_∞ and c_0, we denote the class of all convergent, bounded, and null convergent sequences, respectively, with norm $\| x \|_\infty = \sup_k |x|$. The spaces of all p-absolutely and absolutely summable series, normed by $(\sum_k |x_k|^p)^{\frac{1}{p}}$ for $1 < p < \infty$ and $\sum_k |x_k|$, are denoted by l_p and l_1, respectively.

Take two sequence spaces X, Y and $A = (a_{nk})$ be an infinite matrix of a_{nk}, where a_{nk} are real or complex numbers with $n, k \in \mathbb{N}$. A matrix mapping $A : X \to Y$ is called as A-transform of x in Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{1.2}$$

Throughout the paper, the summation notation without limits goes from 0 to ∞. Let the collection of all matrices A such that $A : X \to Y$ is A-transform is denoted by $(X : Y)$. Thus, $A \in (X : Y)$ if and only if the series on the right side of (1.2) converges for all $x \in X$ and each $n \in \mathbb{N}$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for every $x \in X$. A sequence x is A-summable to l if Ax converges to l where l is the A-limit of x.

If a sequence (b_n) is contained in X, a normed sequence space, and has the property that for all $x \in X$, there exists a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \| x - (a_0 b_0 + a_1 b_1 + \cdots + a_n b_n) \| = 0,$$

having the sum x. Then we call it as the expansion of x with respect to sequence (b_n), which can be written as $x = \sum a_k b_k$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for triangle matrices A, B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is again a triangle matrix. Then, $x = U(Vx) = V(U)x$ holds for all $x \in w$. We write additionally $\mathcal{U}$ for the class of all sequences u with the property that $u_k \neq 0$ for all $k \in \mathbb{N}$.

For an infinite matrix A and a sequence space X, the matrix domain X_A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}, \tag{1.3}$$

which forms sequence space. Recently, a number of authors construct new sequence spaces on various normed spaces by considering the matrix domains with particular limitation method, e.g., Wang [26], Ng and Lee [19], Aydın and Başar [4], Altay [1], Altay and Başar [2, 3], Başarır and Kara [5, 6], Demiriz and Çakan [10], Duyar and Demiriz [11], Polat and Başar [20].

Kızmaz [13] introduced the notion of new difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ which was further expanded by Et and Çolak [12] by introducing the spaces $\ell_\infty(\Delta^m)$,

$c(\Delta^m)$ and $c_0(\Delta^m)$. For a non-negative integer m, we have sequence spaces defined as follows

$$\Delta^m(X) = \{x = (x_k) \in w : \Delta^m x_k \in X\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is further equivalent to the following expressions

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}.$$

Taking $m = 1$, we get the spaces studied by Et and Çolak [12]. Furthermore, Malkowsky et al. [14] have been introduced the spaces

$$\Delta_u^{(m)}(X) = \{x \in w: \Delta_u^{(m)} x \in X\},$$

where $\Delta_u^{(m)} x = u\Delta^{(m)} x$ for all $x \in w$. In this study, the operator $\Delta_u^{(m)}: w \to w$ is defined as follows:

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}.$$

The operator $\Delta^{(\alpha)}$ generalizes the operator $\Delta^{(m)}$ introduced by Malkowsky and Parashar [15], Polat and Başar [20], Malkowsky et al. [14] if $\alpha = m$, where m is an integer.

Baliarsingh and Dutta, defined a fractional difference operators $\Delta^\alpha: w \to w$ (for a proper fraction α) and their inverse in [7] as follows:

(1.4) $$\Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha+1-i)} x_{k+i}$$

(1.5) $$\Delta^{(\alpha)} x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha+1-i)} x_{k-i}$$

(1.6) $$\Delta^{-\alpha} x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} x_{k+i}$$

(1.7) $$\Delta^{(-\alpha)}x_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} x_{k-i}.$$

Throughout the text we consider that the series defined in $(1.4) - (1.7)$ are convergent. In particular, for $\alpha = \frac{1}{2}$,

- $\Delta^{\frac{1}{2}}x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \cdots$
- $\Delta^{\frac{-1}{2}}x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \cdots$
- $\Delta^{(\frac{1}{2})}x_k = x_k - \frac{1}{2}x_{k-1} - \frac{1}{8}x_{k-2} - \frac{1}{16}x_{k-3} - \frac{5}{128}x_{k-4} - \frac{7}{256}x_{k-5} - \cdots$
- $\Delta^{(\frac{-1}{2})}x_k = x_k + \frac{1}{2}x_{k-1} + \frac{3}{8}x_{k-2} + \frac{5}{16}x_{k-3} + \frac{35}{128}x_{k-4} + \frac{63}{256}x_{k-5} + \cdots$

Baliarsingh [8] have been defined the spaces $X(\Gamma, \Delta^{\alpha}, u)$ for $X \in \{\ell_\infty, c, c_0\}$ by introducing the fractional difference operator Δ^{α} and a positive fraction α in [13].

2. The sequence spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$

Here, we introduce new sequence spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$.
A modulus function is a function $f: [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) f is increasing,
(3) $f(x+y) \le f(x) + f(y)$, for all $x, y \ge 0$,
(4) f is continuous from the right at 0,

which implies that on $[0, \infty)$, f must be continuous everywhere. The modulus function may or may not be bounded.. For example, for $f(x) = \frac{x}{x+1}$, f is bounded and if $f(x) = x^p$, $0 < p < 1$ then f is unbounded. This function has been studied in ([18], [21], [22], [23], [25]) and references therein.

In [16] and [17] Maddox introduced the concept of strongly almost convergence. Further, a sequence $x = (x_k)$ is called a strongly almost convergent if there occurs a number L such that
$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_{k+s} - L| = 0,$$
uniformly in s.

Let $\mathcal{U}$ be the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in \mathcal{U}$, let $\frac{1}{u}$=$(\frac{1}{u_n})$. For a proper fraction α, let $p = (p_k)$ be a bounded sequence of positive real numbers and f be a modulus function. Then we define the following sequence spaces as:

$$c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right] = \left\{ x = (x_k) \in w: \left(\frac{1}{k} \left[\sum_{j=0}^{k} f\left(u_j \Delta^{(\alpha)} x_j\right) \right]^{p_k} \right) \in c_0 \right\}$$

and

$$c\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right] = \left\{ x = (x_k) \in w: \left(\frac{1}{k} \left[\sum_{j=0}^{k} f\left(u_j \Delta^{(\alpha)} x_j\right) \right]^{p_k} \right) \in c \right\}.$$

If $f(x) = x$. Then above sequence spaces reduce to $c_0\left[\Gamma, \Delta^{(\alpha)}, u, p\right]$ and $c\left[\Gamma, \Delta^{(\alpha)}, u, p\right]$. By taking $p = (p_k) = 1$, for all $k \in \mathbb{N}$, then we get the sequence spaces $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u\right]$ and $c\left[\Gamma, f, \Delta^{(\alpha)}, u\right]$.

Now, we introduce the triangle matrix $\Gamma_u^{(\alpha)} = (\tau_{nk})$,

$$\tau_{nk} = \begin{cases} \sum_{i=0}^{n-k} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha+1-i)} u_{i+k,}, & 0 \leq k \leq n\,; \\ 0, & k > n. \end{cases}$$

for all $k, n \in \mathbb{N}$. Further, for any sequence $x = (x_k)$ we have the sequence $y = (y_k)$ to be used as the $\Gamma_u^{(\alpha)}$-transform of the x, that is

$$y_k = \sum_{j=0}^{k} u_j \Delta^{(\alpha)} x_j = \sum_{j=0}^{k} u_j \left(x_j - \alpha x_{j-1} + \frac{\alpha(\alpha-1)}{2!} x_{j-3} + \cdots \right) \quad (2.1)$$

$$= \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha+1-i)} u_{i+j} \right) x_j$$

for all $k \in \mathbb{N}$. With the expression of (1.4), the sequence spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u,\ p]$ and $c\left[\Gamma, f, \Delta^{(\alpha)}, u,\ p\right]$ may be restated as follows

$$c_0\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big] = \{c_0(f,p)\}_{\Gamma_u^{(\alpha)}} \text{ and } c\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big] = \{c(f,p)\}_{\Gamma_u^{(\alpha)}}. \tag{2.2}$$

Lemma 2.1 ([7], Theorem 2.2).

$$\Delta^{(\alpha)} \circ \Delta^{(\beta)} = \Delta^{(\beta)} \circ \Delta^{(\alpha)} = \Delta^{(\alpha+\beta)}.$$

Lemma 2.2 ([7], Theorem 2.3).

$$\Delta^{(\alpha)} \circ \Delta^{(-\alpha)} = \Delta^{(-\alpha)} \circ \Delta^{(\alpha)} = Id$$

Where Id denotes the identity operator on w.

Theorem 2.3. *Let $p = (p_k)$ be a bounded sequence of positive real numbers and f be a modulus function. Then the sequence spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big]$ forms linear metric spaces with the paranorm by g, defined by*

$$g(x) = \sup_k \left(\frac{1}{k}\left|\sum_{j=0}^{k} f\big(u_j\Delta^{(\alpha)}x_j\big)\right|^{p_k}\right)^{\frac{1}{H}}, \tag{2.3}$$

where $H = \max(1, \sup_k p_k)$.

Proof. We consider the theorem for the space $c_0\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big]$. It is clear that $g(0) = 0$ and $g(-x) = g(x)$ for all $x \in c_0\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big]$. Take any two sequences $s, t \in c_0\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big]$ and $\alpha', \beta' \in \mathbb{R}$ to check the linearity of $c_0\big[\Gamma, f, \Delta^{(\alpha)}, u, p\big]$ corresponding to the coordinate wise addition and scalar multiplication. By the linearity of operator $\Delta^{(\alpha)}$ and Maddox [29], we find that

$$\begin{aligned} g(\alpha' s, +\beta' t) &= \sup_k \left(\frac{1}{k}\left|\sum_{j=0}^{k} f\big(u_j\Delta^{(\alpha)}(\alpha' s_j + \beta' t_j)\big)\right|^{p_k}\right)^{\frac{1}{H}} \\ &\leq \max\{1, |\alpha'|\} \sup_k \left(\frac{1}{k}\left|\sum_{j=0}^{k} f\big(u_j\Delta^{(\alpha)}s_j\big)\right|^{p_k}\right)^{\frac{1}{H}} \\ &\quad + \max\{1, |\beta'|\} \sup_k \left(\frac{1}{k}\left|\sum_{j=0}^{k} f\big(u_j\Delta^{(\alpha)}t_j\big)\right|^{p_k}\right)^{\frac{1}{H}} \\ &= \max\{1, |\alpha'|\}\, g(s) + \max\{1, |\beta'|\} g(t) \end{aligned}$$

This gives the subadditivity of g, that is,

$$g(s+t) \leq g(s) + g(t). \tag{2.4}$$

Let $\{x^n\}$ be any sequence of the points in $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$ such that $g(x^n - x) \to 0$ and (α'_n) be any sequence of real scalars with $\alpha'_n \to \alpha'$. Then by inequality (2.4) we get

$$g(x^n) \leq g(x) + \ g(x^n - x).$$

Since, $\{g(x^n)\}$ is bounded, thus, it implies the scalar multiplication for g to be scontinuous which can be followed from the inequality

$$g(\alpha'_n x^n - \alpha' x) = \sup_k \left(\frac{1}{k} \left| \sum_{j=0}^{k} f\big(u_j \Delta^{(\alpha)} (\alpha'_n x_j^n + \alpha' x_j)\big) \right|^{p_k} \right)^{\frac{1}{H}}$$

$$\leq \left(|\alpha'_n - \alpha'|^{\frac{p_k}{H}} g(x^n) + |\alpha'|^{\frac{p_k}{H}} g(x^n - x) \right)$$

$$< \infty,. \text{ for all } n \in \mathbb{N}.$$

Therefore, the sequence space $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$ is paranormed by g. This completes the proof.

Theorem 2.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers and f be a modulus function. Then the sequence spaces $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$ and $c\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$ forms complete metric spaces with a paranorm by g, defined in* (2.3).

Proof. We consider this theorem for the space $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$. Suppose that $\{x^n\}$ is a Cauchy sequence in the space $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$, where $x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots\}$. Since $\{x^n\}$ is a Cauchy sequence, thus for given $\epsilon > 0$, there will be a positive integer $N_0(\varepsilon)$ such that

$$g(x^n - x^m) < \varepsilon, \qquad \text{for all } m, n \geq N_0(\varepsilon).$$

By using the definition of g for each fixed $k \in \mathbb{N}_0(\epsilon)$, we have

$$\frac{1}{k}\left|f\left(\{\Gamma_u^{(\alpha)}x^n\}_k-\{\Gamma_u^{(\alpha)}x^m\}_k\right)\right|^{\frac{p_k}{H}}\leq \sup_k\frac{1}{k}\left|f\left(\{\Gamma_u^{(\alpha)}x^n\}_k-\{\Gamma_u^{(\alpha)}x^m\}_k\right)\right|^{\frac{p_k}{H}}<\varepsilon,$$

for all $m,n\geq N_0(\varepsilon)$ and by continuity of f, $\left\{(\Gamma_u^{(\alpha)}x^0)_k,(\Gamma_u^{(\alpha)}x^1)_k,\ldots\right\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $k\in\mathbb{N}_0$. By completeness of $\mathbb{R}$, the sequence $\{\Gamma_u^{(\alpha)}x^n\}_k$ converges and suppose that

$$\{\Gamma_u^{(\alpha)}x^n\}_k\to\{\Gamma_u^{(\alpha)}x\}_k \quad \text{as } n\to\infty.$$

For each fixed $k\in\mathbb{N}_0$, $m\to\infty$ and $n\geq N_0(\epsilon)$, it is clear that

$$\frac{1}{k}\left|f\left(\{\Gamma_u^{(\alpha)}x^n\}_k-\{\Gamma_u^{(\alpha)}x\}_k\right)\right|^{\frac{p_k}{H}}\leq\varepsilon. \tag{2.5}$$

Since $x^n=\{x_k^{(n)}\}$ is a sequence in $c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]$, we have

$$\frac{1}{k}\left|f\{\Gamma_u^{(\alpha)}x^n\}_k\right|^{\frac{p_k}{H}}\leq\mathcal{H},\ \text{for each positive integer } k\in\mathbb{N} \tag{2.6}$$

Then by combining (2.5) and (2.6), we obtain that

$$\frac{1}{k}\left|f\{\Gamma_u^{(\alpha)}x\}_k\right|^{\frac{p_k}{H}}\leq\frac{1}{k}\left|f\left(\{\Gamma_u^{(\alpha)}x^n\}_k-\{\Gamma_u^{(\alpha)}x\}_k\right)\right|^{\frac{p_k}{H}}+\frac{1}{k}\left|f\{\Gamma_u^{(\alpha)}x^n\}_k\right|^{\frac{p_k}{H}}$$

$$\leq \quad \mathcal{H}\ \text{ for all } n\geq N_0(\varepsilon).$$

This shows that the sequence $\{\Gamma_u^{(\alpha)}x\}$ lies in the space $c_0(p)$. Since $\{x^n\}$ is an arbitrary Cauchy sequence, the space $c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]$ is complete, which completes the proof.

Theorem 2.5. *The sequence spaces* $c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]$ *and* $c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]$ *are BK-spaces under the norm*

$$\|x\|_{c_0[\Gamma,f,\Delta^{(\alpha)},u,p]}=\|x\|_{c[\Gamma,f,\Delta^{(\alpha)},u,p]}=\sup_k\frac{1}{k}\left|\sum_{j=0}^{k}f\left(u_j\Delta^{(\alpha)}x_j\right)\right|^{p_k}.$$

Proof. Since (2.2) holds, and c_0, c form *BK*-spaces under their natural norms (see [9], pp. 16-17) and the matrix $\Gamma_u^{(\alpha)} = (\tau_{nk})$ is a triangle (Theorem 4.3.12 of Wilansky [27], pp. 63) resuls the fact that the spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ form BK-spaces with the given norms. It completes the proof.

Now, we consider the following theorem concerning the isomorphism among the spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$, $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and c_0, c respectively.

Theorem 2.6. *The sequence space* $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ *and* $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ *are isometrically isomorphic to the spaces c and* c_0 *respectively, i.e,*

$$c_0[\Gamma, f, \Delta^{(\alpha)}, u, p] \cong c_0 \text{ and } c[\Gamma, f, \Delta^{(\alpha)}, u, p] \cong c.$$

Proof. We prove the theorem for the space $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$. To prove this, we only need to show that there exist a linear bijection among the spaces $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and c. Consider the transformation T from $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ to c defined by using the notation of (2.1), as $x \to y = Tx = \Gamma_u^{(\alpha)} x$. Then, clearly T is linear. Moreover, $x = 0$ whenever $Tx = 0$ is trivial. Hence T is injective.
We assume that $y = (y_k) \in c$ and define the sequence $x = (x_k) \in c[\Gamma, f, \Delta^{(\alpha)}, u, p]$ by

$$x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} \frac{y_{k-i} - y_{k-i-1}}{u_{k-i}}$$

Then by lemma 2.2, we deduce that

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{k} \left[\sum_{j=0}^{k} f\left(u_j \Delta^{(\alpha)} x_j\right) \right]^{p_k} \\ &= \lim_{k\to\infty} \frac{1}{k} \left[\sum_{j=0}^{k} f\left(u_j \Delta^{(\alpha)} \left(\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} \frac{y_{j-i} - y_{j-i-1}}{u_{j-i}} \right) \right) \right]^{p_k} \\ &= \lim_{k\to\infty} \frac{1}{k} \left[\sum_{j=0}^{k} f\left(u_j \Delta^{(\alpha)} \left(\Delta^{(-\alpha)} \left(\frac{y_j - y_{j-1}}{u_j} \right) \right) \right) \right]^{p_k} \end{aligned}$$

$$= \lim_{k\to\infty} \frac{1}{k}\left[\sum_{j=0}^{k} f\left(y_j - y_{j-1}\right)\right]^{p_k}$$
$$= y_k.$$

Hence, $x \in c\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$, so T is surjective. Furthermore one can easily show that T is norm preserving. With this the proof is completed.

3. The $\beta -$ Dual of The Spaces $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$ and $c\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$

Here, we determine the β −dual of the spaces $c_0\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$ and $c\left[\Gamma, f, \Delta^{(\alpha)}, u, p\right]$. For the sequence spaces X and Y, define the set $S(X, Y)$ by

(3.1) $$S(X, Y) = \{z = (z_k) \in w : xz = (x_k z_k) \in Y \text{ for all } x \in X\}.$$

With the notation of (3.1), β −dual of a sequence space X is defined by

$$X^{\beta} = S(X, cs).$$

Stieglitz and Tietz [24] gave the following lemmas which are used to arrive at the main results.

Lemma 3.1. $A \in (c_0 : c)$ if and only if

(3.2) $$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N},$$

(3.3) $$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty.$$

Lemma 3.2. $A \in (c_0 : c)$ if and only if (3.2) and (3.3) hold, and (3.4)

$$\lim_{n\to\infty} \sum_k a_{nk} \text{ exists.}$$

Theorem 3.1. *Define the sets Γ_1, Γ_2 and Γ_3 by*

$$\Gamma_1 = \left\{a = (a_n) \in w : \left(\frac{1}{k}\left[f\left(\lim_{n\to\infty} t_{nk}\right)\right]^{p_k}\right) = \alpha_k \; exists \; for \; each \; k \in \mathbb{N}\right\}$$

$$\Gamma_2 = \left\{a = (a_n) \in w : \left(\frac{1}{k}\left[f\left(\sup_{n\in\mathbb{N}} \sum_k |t_{nk}|\right)\right]^{p_k}\right) < \infty\right\}$$

$$\Gamma_3 = \left\{a = (a_n) \in w : \left(\frac{1}{k}\left[f\left(\sup_{n\in\mathbb{N}} \lim_{n\to\infty} \sum_k t_{nk}\right)\right]^{p_k}\right) \; exists \right\}$$

and define a matrix $T = (t_{nk})$ *by*

$$t_{nk} = \begin{cases} t_k - t_{k+1}, & (k < n) \\ t_n, & (k = n) \\ 0, & (k > n) \end{cases}$$

$$for \; all \; k, n \in \mathbb{N} \; where \; t_k = a_k \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} \frac{1}{u_{k-i}}.$$

Then, $\{c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]\}^\beta = \Gamma_1 \cap \Gamma_2$ *and* $\{c[\Gamma, f, \Delta^{(\alpha)}, u, p]\}^\beta = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$.

Proof. We prove the theorem for the space $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$. Let $a = (a_n) \in w$ and $x = (x_k) \in c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$. Then, we obtain the equality

$$\frac{1}{k}\left[f \sum_{k=0}^{n} a_k x_k\right]^{p_k} = \frac{1}{k}\left[f \sum_{k=0}^{n} \left(a_k \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} \frac{1}{u_{k-i}}\right)(y_{k-i} - y_{k-i-1})\right]^{p_k}$$

$$= \frac{1}{k}\left[f \sum_{k=0}^{n} \left(a_k \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} \frac{1}{u_{k-i}}\right)(y_{k-i} - y_{k-i-1})\right]^{p_k}$$

. $$= \frac{1}{k}\Big[f \textstyle\sum_{k=0}^{n-1} \left(a_k \sum_{i=0}^{k} (-1)^i \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)} \frac{1}{u_{k-i}} - a_{k+1} \sum_{i=0}^{k+1} (-1)^i \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)} \frac{1}{u_{k+1-i}}\right) y_k + \left(a_n \sum_{i=0}^{n} (-1)^i \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)} \frac{1}{u_{n-i}}\right) y_n\Big]^{p_k}$$

$$= (Ty)_n.$$

Then, we deduce that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ if and only if $Ty \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \{c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]\}^\beta$ if and only if $T \in (c_0 : c)$. Therefore, by using Lemma 3.1, we obtain;

$$\left(\frac{1}{k}\left[f\left(\lim_{n\to\infty} t_{nk}\right)\right]^{p_k}\right) = \alpha_k \text{ exists for each } k \in \mathbb{N}, \left(\frac{1}{k}\left[f\left(\sup_{n\in\mathbb{N}} \sum_k |t_{nk}|\right)\right]^{p_k}\right) < \infty.$$

Hence, we conclude that $\{c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]\}^\beta = \Gamma_1 \cap \Gamma_2$.

4. Some Matrix Transformations Related to the Sequence Spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$

In this section, we discuss results on various matrix mappings on the spaces $c_0[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and $c[\Gamma, f, \Delta^{(\alpha)}, u, p]$. Simply, we can write

$$\tilde{a}_{nk} = z_{nk} - z_{n,k+1} \text{ and } b_{nk} = \left(\frac{1}{k}\left[\sum_{j=0}^{n} f\left(\sum_{i=0}^{n-j}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha+1-i)} u_{i+j}\right) a_{jk}\right]^{p_k}\right), \qquad (4.1)$$

for all $k, n \in \mathbb{N}$, where

$$z_{nk} = \left(\frac{1}{k}\left[f\left(\sum_{i=0}^{k}(-1)^i \frac{\Gamma(1-\alpha)}{i!\,\Gamma(1-\alpha-i)} \frac{1}{u_{k-i}}\right) a_{nk}\right]^{p_k}\right).$$

Theorem 4.1. *Let λ be any given sequence space and $\mu \in \{c_0, c\}$. Then, $A = (a_{nk}) \in (\mu[\Gamma, f, \Delta^{(\alpha)}, u, p] : \lambda)$ if and only if $C \in (\mu : \lambda)$ and*

$$C^{(n)} \in (\mu : c) \qquad (4.2)$$

for every fixed $n \in \mathbb{N}$, where $c_{nk} = \tilde{a}_{nk}$ and $C^{(n)} = (c_{mk}^{(n)})$ with

$$(c_{mk}^{(n)}) = \begin{cases} z_{nk} - z_{n,k+1}, & (k < m) \\ z_{nm}, & (k = m) \\ 0, & (k > m) \end{cases}$$

Proof. Let λ be any given sequence space. Suppose that (4.1) holds between the entries of $A = (a_{nk})$ and $C = (c_{nk})$, and consider that the spaces $\mu[\Gamma, f, \Delta^{(\alpha)}, u, p]$ and μ are linearly isomorphic.

Let $A = (a_{nk}) \in (\mu[\Gamma, f, \Delta^{(\alpha)}, u, p] : \lambda)$ and take any $y = (y_k) \in \mu$. Then $C\Gamma_u^{(\alpha)}$ exits and $\{a_{nk}\}_{k\in\mathbb{N}} \in (\mu[\Gamma, f, \Delta^{(\alpha)}, u, p])^{\beta}$ which yields that (4.2) is necessary and $\{c_{nk}\}_{k\in\mathbb{N}} \in \mu^{\beta}$ for each $n \in \mathbb{N}$. Hence, Cy exists for each $y \in \mu$ and thus by letting $m \to \infty$ in the equality

$$\frac{1}{k}\left[f\left(\sum_{k=0}^{m} a_{nk}x_k\right)\right]^{p_k} = \frac{1}{k}\left[f\left(\sum_{k=0}^{m-1}(z_{nk} - z_{n,k+1})y_k + z_{nm}y_m\right)\right]^{p_k} ; (m, n \in \mathbb{N})$$

we have that $Cy = Ax$ and so we have that $C \in (\mu : \lambda)$.

Conversely, suppose that $C \in (\mu : \lambda)$ and (4.2) hold, and let $x = (x_k) \in \mu[\Gamma, f, \Delta^{(\alpha)}, u, p]$. Then, we get $\{c_{nk}\}_{k\in\mathbb{N}} \in \mu^{\beta}$, which together with (4.2) gives $\{a_{nk}\}_{k\in\mathbb{N}} \in (\mu[\Gamma, f, \Delta^{(\alpha)}, u, p])^{\beta}$ for each $n \in \mathbb{N}$. Thus, Ax exists. Therefore, from the equality we obtain

$$\frac{1}{k}\left[f\left(\sum_{k=0}^{m} c_{nk}y_k\right)\right]^{p_k} = \frac{1}{k}\left[f\sum_{k=0}^{m}\left(\sum_{j=k}^{m}\left(\sum_{i=0}^{j-k}(-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha+1-i)} u_{i+k}\right)c_{nj}\right)x_k\right]^{p_k}$$

for all $n \in \mathbb{N}$, as $m \to \infty$ that $Ax = Cy$ and this proves that $A \in (\mu[\Gamma, f, \Delta^{(\alpha)}, u, p] : \lambda)$. This completes the proof.

Theorem 4.2. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation* (4.1) *and λ be given sequence space and $\mu = \{c_0, c\}$. Then $A = (a_{nk}) \in (\lambda : \mu[\Gamma, f, \Delta^{(\alpha)}, u, p])$ if and only if $B = (b_{nk}) \in (\lambda : \mu)$.*

Proof. Let $z = (z_k) \in \lambda$ and consider the following equality with (4.1)

$$\frac{1}{k}[f(\textstyle\sum_{k=0}^{m} b_{nk}z_k)]^{p_k} = \frac{1}{k}\left[f\sum_{j=0}^{n}\left(\sum_{k=0}^{m}\left(\sum_{i=0}^{n-j}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha+1-i)}u_{i+j}\right)a_{jk}\right)z_k\right]^{p_k} (m, n \in \mathbb{N}),$$

which yields as $m \to \infty$ that $(Bz)_n = [\Gamma_u^{(\alpha)}(Az)]_n$. Hence, we obtain that $Az \in \mu[\Gamma, f, \Delta^{(\alpha)}, u, p]$ whenever $z \in \lambda$ if and only if $Bz \in \mu$ whenever $z \in \lambda$.

We will have several consequences by using Theorem 4.1 and Theorem 4.2. But we must give firstly some relations which are important for consequences:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \qquad (4.3)$$

$$\lim_{n\to\infty} a_{nk} = a_k \text{ exists for each fixed } k \in \mathbb{N}. \qquad (4.4)$$

$$\lim_{k\to\infty} a_{nk} = 0 \text{ exists for each fixed } n \in \mathbb{N}. \qquad (4.5)$$

$$\lim_{n\to\infty} \sum_k a_{nk} \text{ exists}. \qquad (4.6)$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0. \qquad (4.7)$$

$$\sup_{k\in\mathcal{F}} \sum_n \left|\sum_{k\in K} a_{nk}\right| < \infty. \qquad (4.8)$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0. \qquad (4.9)$$

$$\sup_{n,k} |a_{nk}| < \infty. \qquad (4.10)$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |a_k|. \qquad (4.11)$$

Now, we can give the corollaries.

Corollary 4.3. *The following statements hold:*
(i) $A = (a_{nk}) \in (c_0[\Gamma, f, \Delta^{(\alpha)}, u, p] : \ell_\infty) = (c[\Gamma, f, \Delta^{(\alpha)}, u, p] : \ell_\infty)$ *if and only if* (4.3) *holds with* $\tilde{a}_{nk}$ *instead of* a_{nk} *and* (4.2) *also holds.*
(ii) $A = (a_{nk}) \in (c_0[\Gamma, f, \Delta^{(\alpha)}, u, p] : c)$ *if and only if* (4.3) *and* (4.4) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *and* (4.2) *also holds.*
(iii) $A = (a_{nk}) \in (c_0[\Gamma, f, \Delta^{(\alpha)}, u, p] : c_0)$ *if and only if* (4.3) *and* (4.5) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *and* (4.2) *also holds.*
(iv) $A = (a_{nk}) \in (c[\Gamma, f, \Delta^{(\alpha)}, u, p] : c)$ *if and only if* (4.3), (4.4) *and* (4.6) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *and* (4.2) *also holds.*

(v) $A=(a_{nk})\in\left(c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]:\,c_0\right)$ *if and only if* (4.3), (4.5) *and* (4.7) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *and (4.2) also holds.*
(vi) $A=(a_{nk})\in\left(c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]:\,\ell\right)=\left(c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]:\,\ell\right)$ *if and only if* (4.8) *holds with* $\tilde{a}_{nk}$ *instead of* a_{nk} *and* (4.2) *also holds.*

Corollary 4.4. *The following statements hold:*
(i) $A=(a_{nk})\in\left(\ell_\infty:\,c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.9) *hold with* b_{nk} *instead of* a_{nk}.
(ii) $A=(a_{nk})\in\left(c:\,c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.3), (4.5) *and* (4.7) *hold with* b_{nk} *instead of* a_{nk}.
(iii) $A=(a_{nk})\in\left(c_0:\,c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.3) *and* (4.5) *hold with* b_{nk} *instead of* a_{nk} .
(iv) $A=(a_{nk})\in\left(\ell:c_0\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.5) *and* (4.10) *hold with* b_{nk} *instead of* a_{nk} .
(v) $A=(a_{nk})\in\left(\ell_\infty:c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.4) *and* (4.11) *hold with* b_{nk} *instead of* a_{nk} .
(vi) $A=(a_{nk})\in\left(c:c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.3), (4.4) *and* (4.6) *hold with* b_{nk} *instead of* a_{nk} .
(vii) $A=(a_{nk})\in\left(c_0:c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.3) *and* (4.4) *hold with* b_{nk} *instead of* a_{nk} .
$(viii)$ $A=(a_{nk})\in\left(\ell:\,c\left[\Gamma,f,\Delta^{(\alpha)},u,p\right]\right)$ *if and only if* (4.4) *and* (4.10) *hold with* b_{nk} *instead of* a_{nk} .

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Compactness in Neutrosophic Minimal Spaces

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Abstract

The aim of this paper is to generalize the concept of compactness in neutrosophic minimal spaces. We shall introduce neutrosophic *m*-compactness in neutrosophic minimal spaces and some related basic results in this new setting would be investigated. Further, the concept of neutrosophic countably *m*-compactness would be introduced in neutrosophic minimal spaces and some of its basic properties would be studied.

Keywords: Neutrosophic set, Neutrosophic topology, Minimal Structure, compactness.

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1. Introduction:

Zadeh [18] introduced the notion of fuzzy set in the year 1965. Uncertainty plays an important role in our everyday life problems. Zadeh associated the membership value with the elements to control the uncertainty. It was not sufficient to control uncertainty, so Atanaosv [4] added non-membership value along with the membership value and introduced the notion of intuitionistic set. Still it was difficult to handle all types of problems under uncertainty, in particular for problems on decision making. In order to overcome this difficulty, Smarandache [15] considered the elements with membership, nonmembership and indeterministic values and introduced the notion of neutrosophic set. The concept of neutrosophic set has been applied in many branches of science and technology. Das et al. [6] have studied algebraic operations neutrosophic fuzzy matrices, Das and Tripathy [5] have investigated different properties of neutrosophic multi set topological space. Salama et al. [14] have studied neneutrosophic crisp topological space.

The notion of neutrosophic topological space was first introduced by Salama and Alblowi [12], followed by Salama and Alblowi [13]. The notion of minimal structure in topological space was introduced by Makai et al. [9]. It is found to have useful applications and the notion was investigated by Madok [10]. The notion of minimal structure in a fuzzy topological space was introduce by Alimohammady and Roohi [1] and further investigated by Tripathy and Debnath [16] and others [7, 8, 11, 17]. Arokiaranil et al [3] extended minimal structures to fuzzy minimal structures and established some results in this setting. Alimohammady and Roohi [2] introduced compactness in fuzzy minimal spaces.

2. Preliminaries:

Definition 2.1. Let X be an universal set. A *neutrosophicsetA* in X is a set contains triplet having truthness, falseness and indeterminacy membership values that can be characterized independently, denoted by T_A, F_A, I_A in [0,1]. The neutrosophic set is denoted as follows:

$A = \{(x, T_A(x), F_A(x), I_A(x)): x\in X, \text{and} T_A(x), F_A(x), I_A(x)\in[0,1] \}$.

There is no restriction on the sum of $T_A(x)$, $F_A(x)$ and $I_A(x)$, so

$$0 \le T_A(x) + F_A(x) + I_A(x) \le 3.$$

Throughout, we denote a*neutrosophic set A* by$A=\{(x, T_A(x), F_A(x), I_A(x)): x\in X, \text{and} T_A (x), F_A(x), I_A(x)\in [0,1]\}$.

The null and full NSs on a nonempty set X are denoted by0_N and 1_N, defined as follows:

Definition 2.2. The neutrosophic sets 0_N and 1_N in X are represented as follows:
(*i*) $0_N = \{<x, 0, 0, 1>: x\in X\}$.
(*ii*) $0_N = \{<x, 0, 1, 1 >: x\in X \}$.
(*iii*) $0_N = \{<x, 0, 1, 0 >: x\in X \}$.
(*iv*) $0_N = \{<x, 0, 0, 0 >: x\in X \}$.
(*v*) $1_N = \{<x, 1, 0, 0 >: x\in X \}$.
(*vi*) $1_N = \{<x, 1, 0, 1 >: x\in X \}$.
(*vii*) $1_N = \{<x, 1, 1, 0 >: x\in X \}$.
(*viii*) $1_N = \{<x, 1, 1, 1 >: \in X \}$.

Clearly**,** $0_N\subseteq 1_N$.We have, for any neutrosophic set A,$0_N\subseteq A\subseteq 1_N$.

Now we procure the basic operations on neutrosophic sets, those will be used throughout the article.

Definition 2.3. Let $A = (x, T_A, F_A, I_A)$ be a *NS* over *X*, then the complement of *A* is defined by $A^c=\{(x,1\text{-}T_A(x),1\text{-}F_A(x),1\text{-}I_A(x)): x\in X\}$.

Definition 2.4. A *neutrosophic set* $A = (x, T_A, F_A, I_A)$ is contained in the other *neutrosophic set* $B = (x, T_B, F_B, I_B)$ (i.e. $A\subseteq B$) if and only if $T_A(x)\leq T_B(x)$, $F_A(x)\geq F_B(x)$, $I_A(x)\geq I_B(x)$, for each $x\in X$.

Definition 2.5. If $A = (x, T_A, F_A, I_A)$ and $B=(x, T_B, F_B, I_B)$ are any two *NSs* over *X*, then $A\cup B$ and $A\cap B$ is defined by
$A\cup B =\{(x,T_A(x)\vee T_B(x), F_A(x)\wedge F_B(x), I_A(x)\wedge I_B(x)): x\in X\}$.
$A\cap B =\{(x, T_A(x)\wedge T_B(x), F_A(x)\vee F_B(x), I_A(x)\vee I_B(x)): x\in X\}$.

The neutrosophic topological space is defined as follows:

Definition 2.6. Let *X* be a non-empty set and τ be the collection of neutrosophic subsets of *X* then τ is said to be a *neutrosophic topology* (in short *NT*) on *X* if the following properties holds:
(*i*) $0_N,1_N\in\tau$.
(*ii*) $U_1,U_2\in\tau\Rightarrow U_1\cap U_2\in\tau$.
(*iii*) $\cup_{i\in\Delta}u_i\in\tau$, for every $\{u_i: i\in\Delta\}\subseteq\tau$.

Then (X,τ) is called a *neutrosophic topological space* (in short *NTS*) over *X*. The members of τ are called neutrosophic open sets (in short *NOS*). A neutrosophic set *D* is calledneutrosophic closed set (in short *NCS*) if and only if D^c is a neutrosophic open set.

The neutrosophic interior and neutrosophic closure of a neutrosophic set is defined as follows:

Definition 2.7. Let (X,τ) be a *NTS* and *U* be a *NS* in *X*. Then the neutrosophic interior (in short N_{int}) and neutrosophic closure (in short N_{cl}) of *U* are defined by

$N_{int}(U) = \cup\{E : E$ is a *NOS* in X and $E\subseteq U\}$,
$N_{cl}(U) = \cap\{F : F$ is a *NCS* in X and $U\subseteq F\}$.

Remark 2.8. Clearly $N_{int}(U)$ is the largest neutrosophic open set over *X* which is contained in *U* and $N_{cl}(U)$ is the smallest neutrosophic closed set over *X* which contains *U*.

Proposition 2.9. For any *NSB* in (X,τ) we have

(i) $N_{int}(B^c) = (N_{cl}(B))^c$.
(ii) $N_{cl}(B^c) = (N_{int}(B))^c$.

Definition 2.10. Let (X,τ) be a neutrosophic topological space and G be a neutrosophic set over X. Then G is called,
(i) Neutrosophic semi-open (in short *NSO*) set if and only if $G \subseteq N_{cl}(N_{int}(G))$;
(ii) Neutrosophic pre-open (in short *NPO*) set if and only if $G \subseteq N_{int}(N_{cl}(G))$.

3. Neutrosophic m-compact space

Definition 3.1. Suppose (X, $\mathscr{M}$) is a neutrosophic minimal space and A = { A_j : j∈J } is a family of neutrosophic sets in X. A is called a neutrosophic cover of X if $\bigvee_j A_j = 1_N$. Also A is called a neutrosophic set B in X if B ≤ $\bigvee_j A_j$. It is a neutrosophic m-open cover if eachA_j is neutrosophic m-open. A neutrosophic subcover of A is a subfamily of it which is a neutrosophic cover too.

Definition 3.2. Suppose (X, $\mathscr{M}$) is a neutrosophic minimal space. A neutrosophic set B in X is said to be neutrosophic m-compact if every neutrosophic m-open cover A = { A_j : j∈J } of B has a finite fuzzy subcover. Also Y ⊆ X is called neutrosophic m-compact if 1_{N_Y} is a neutrosophic m-compact set.

Definition 3.3. A family A = {A_j : j∈J} of neutrosophic sets in X has finite intersection property if each finite subfamily of {A_j : j∈J} has non-empty intersection.

Theorem 3.4. A neutrosophic minimal space (X, $\mathscr{M}$) is neutrosophic m-compact iff $\bigwedge_j A_j \neq 0_N$ for any family {A_j : j∈J} of neutrosophic m-closed sets in X which has the finite intersection property.

Proof. Suppose { A_j : j∈J } is neutrosophic m-open cover of X. If $\bigvee_{i=1}^{n} A_{j_i} \neq 1_N$ for any choice $j_1, j_2, \ldots\ldots\ldots, j_n$∈J, then $\bigwedge_{i=1}^{n} {A^c}_{j_i} \neq 0_N$. Now from the assumption $\bigwedge_j {A^c}_j \neq 0_N$, that is

$\bigvee_j A_j \neq 1_N$, which is a contradiction.

Conversely, suppose {A_j : j∈J} is a family of neutrosophic m-closed sets which satisfy in finite intersection property. If $\bigwedge_j {A^c}_j \neq 0_N$, then $\bigvee_j {A^c}_j = 1_N$, by the assumption there exist

$j_1, j_2, \ldots\ldots\ldots, j_n$∈J such that $\bigvee_{j_i}^{n} {A^c}_{j_i} = 1_N$;i.e. $\bigvee_{i=1}^{n} A_{j_i} = 0_N$ which is impossible.

Theorem 3.5. A neutrosophic space (X, $\mathscr{M}$) is neutrosophic B-compact if and only if every collection $\{A_j : j \in J\}$ of neutrosophic B-closed sets of X having the finite intersection property, $\bigwedge_{j\in J} A_j \neq 01_N$.

Proof. Since the family of all neutrosophic B-open sets forms a neutrosophic minimal structure on X, so it follows from Theorem 3.4.

Definition 3.6. A family B of neutrosophic sets in a neutrosophic minimal space (X, $\mathscr{M}$) is called a neutrosophic filter base if $\bigvee_{B\in F} B \neq 0_N$ for any finite subfamily $F \subseteq B$.

Theorem 3.7. A neutrosophic minimal space (X, $\mathscr{M}$) is neutrosophic m-compact if and only if $\wedge_{B\in B}$ m-cl(B) $\neq 0_N$ for every neutrosophic filter base B in X.

Proof. Suppose $\{A_j : j \in J\}$ is a family of neutrosophic m-closed sets which satisfy in finite intersection property. It is easy to see that $\{A_j : j \in J\}$ is neutrosophic filter base for X, now by the assumption $\bigwedge_j m-cl(A_j) = \bigwedge_j (A_j) \neq 0_N$. That is X is neutrosophic m-compact follows from theorem Theorem 3.4.

Conversely, suppose there is a neutrosophic filter base $\{B_j : j \in J\}$ such that $\bigwedge_j m-cl(B_j) = 0_N$.

From definition of m-closure, we can write $m-cl(B_j) = \bigwedge_k B_{(j,k)}$, where $B_{(j,k)}$'s are m - closed and $B_j \leq B_{(j,k)}$, k is index set K_j. Then $\bigwedge_j \bigwedge_k B_{(j,k)} = 0_N$, so $\bigwedge_j \bigwedge_k B^c{}_{(j,k)} = 1_N$. From neutrosophic m-compactness of X, we have $\bigvee_{i=1}^n B^c_{(j_i,k_i)} = 1_N$ so, $\bigwedge_{i=1}^n B_{(j_i,k_i)} = 0_N$. Since $m-cl(B_{ji}) = \bigwedge_k B_{(j_i,k_i)} \leq B_{(j_i,k_i)}$, so $\bigwedge_{i=1}^n m-cl(B_{ji}) = 0_N$. Therefore, $\{B_j : j \in J\}$ has non-empty finite intersection property.

Theorem 3.8. Suppose f : (X, $\mathscr{M}$) $\longrightarrow$(X, N) is neutrosophic m-continuous. If A is neutrosophic m-compact set, then f(A) is neutrosophic m-compact set too.

Proof. Let $\{B_j : j \in J\}$ be a family of neutrosophic m-open sets in Y which satisfy in $f(A) \leq \bigvee_j B_j$. Since, $A \leq f^{-1}(f(A)) \leq f^{-1}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} f^{-1}(B_j)$, f is neutrosophic m-continuous and A is neutrosophic m-compact set, there exist $j_1, j_2, \ldots\ldots, j_n \in$J such that A $\leq \bigvee_{i=1}^n f^{-1}(B_{j_i})$, i.e. $A \leq f^{-1}(\bigvee_{i=1}^n B_{j_i})$, Consequently, $f(A) \leq f(f^{-1}(\bigvee_{i=1}^n B_{j_i})) \leq \bigvee_{i=1}^n B_{j_i}$, which it completes the proof.

Definition 3.9. A function $f : X \longrightarrow Y$ is said to be neutrosophic MB-continuous if the inverse image of every neutrosophic B-open set in Y is neutrosophic B-open set in X.

Theorem 3.10. Suppose (X, $\mathscr{M}$) is a neutrosophic m-compact and f : (X, $\mathscr{M}$) ⟶(Y, N) is a surjective neutrosophic m-compact. Then Y is neutrosophic m-compact.

Proof. Since f is onto we have $f(1_N) = 1_N$, now applying Theorem 3.8.

Definition 3.11. A function f : (X, $\mathscr{M}$) ⟶(Y, N) is called neutrosophic m-open if f(A)∈ N for each A∈ $\mathscr{M}$.

Corollary 3.12. Suppose f : (X, $\mathscr{M}$) ⟶(X, N) is bijective and nutrosophic m-open. Then Y is neutrosophic m-compact whenever X is neutrosophic m-compact.

Proof. It is easy to see that f^{-1} is neutrosophic m-continuous and onto. Now apply Theorem 3.10.

Theorem 3.13. Suppose neutrosophic minimal space (X, $\mathscr{M}$) is neutrosophic *m*-compact if and only if every neutrosophic m-open cover has a finite o-partition.

Proof. Suppose, A = { A_j : j∈J } is a neutrosophic m-open cover of *X*. Since, *X* is neutrosophic *m*-compact, *A* has a finite subcover, say A_0 = {A_i : I = 1,2, , n} and so $\vee_{i=1}^{n} A_i = 1_N$. Therefore, A_0 is a o-partition of X and since A_0 is a subfamily of *A*, then A_0 is o-partition of *X* by *A*.

Conversely, suppose that every neutrosophic m-open cover of X has a finite o-partition. Hence any neutrosophic m-open cover *A* = { A_j : j∈J } has a finite o-partition {$\Gamma_{i,0}$: i = 1,2, … n}. Let A_i be a neutrosophic set in correspondence of $\Gamma_{i,0}$. Now it is easy to see that {A_i : i = 1,2, . . ., n} is a finite subfamily of *A* which is also a neutrosophic cover of *X*.

Corollary 3.14. Suppose (X, $\mathscr{M}$) is a neutrosophic minimal space. X is not neutrosophic m-compact if there exist a neotrosophic m-open cover A of X and a point x∈X with $A_i(x) < 1$ for all A_i∈ A .

Proof. Obvious, so left.

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Some Results on Sub-manifolds of Warped Product Manifolds

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Abstract: We shall study $R^m(k, f)$ which is the warped product of an open interval I and a Riemannian manifold S of constant sectional curvature k.

1. Introduction

Suppose B and F are semi Riemannian manifolds and let $f > 0$ be a smooth function of B. The Warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with metric tensor,

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F) \quad (1.1)$$

where π and σ are the projections of $B \times F$ onto B and F, respectively. Here f is called warping function. Explicitly, if X is a tangent to $B \times F$ at (p, q) then,

$$< x, x > = < d\pi(x), d\pi(x) > + f^2(p)\big(d\sigma(x), d\sigma(x)\big). \quad (1.2)$$

An important example in this class is the Roberson-Walker space-times:

$$L_1^4(k, f) = \left(I \times S, \overline{g}_f^k\right), \quad (1.3)$$

Where, $\overline{g}_f^k = -dt^2 + f^2(t) g_k$.

An n −dimensional generalized Robertson-Walker spacetime with $n \geq 3$ is a Lorentzian manifold which is a warped product manifold $I \times_f F$ of an open interval I of the real line $\mathfrak{R}$ and a Riemannian $(n-1)$ −manifold (F, g_F) endowed with the Lorentzian metric,

$$g = -\pi_I^*(dt^2) + f(t)^2 \pi_F^*(g_F) \quad (1.4)$$

where π_I and π_F denote the projections onto I and F, respectively, and f is a positive smooth function on I. In a classical Robertson-Walker(RW) spacetime, the fiber is three dimensional and of constant sectional curvature, and the warping function f is arbitrary.

On the Warped product $R^m(k,f) = I \times_f S$, let t be an arclength parameter of I. Denote by ∂_t the lift t of the standard vector field d/dt on I to $I \times_f S$. So we have $\partial_t \in L(I)$. $L(S)$ is similarly defined that is $X \in L(S)$ means X is the lift of some vector say X_s in S to $I \times_f S$.

2. Preliminaries

We assume the following two lemmas from [1].

Lemma 2.1. For $X, Y \in L(S)$, we have

(1) $\tilde{\nabla}_{\partial_t}\partial_t = 0$
(2) $\tilde{\nabla}_{\partial_t}X = \tilde{\nabla}_X\partial_t = (\ln f)'X$
(3) $< \tilde{\nabla}_X Y, \partial_t > = -< X, Y > (\ln f)'$

Lemma 2.2. For $X, Y, Z \in L(S)$, the curvature tensor $\tilde{R}$ of $R^m(k,f)$ satisfies

$$\tilde{R}(\partial_t, X)\partial_t = \frac{f''}{f}X,\ \tilde{R}(X, \partial_t)Y = < X, Y > \frac{f''}{f}\partial_t\ ,\ \tilde{R}(X, Y)\partial_t = 0$$

and $\tilde{R}(X,Y)Z = \frac{k-f'^2}{f^2}\{< Y, Z > X - < X, Z > Y\}$ (2.1)

3. Some Results for Submanifolds in $R^m(k,f)$

For a vector field V on $R^m(k,f)$, we decompose V into a sum

$$V = \phi_V\partial_t + \hat{V} \tag{3.1}$$

where $\phi_V = < V, \partial_t >$ and $\hat{V}$ is the vertical component of V that is orthogonal to ∂_t.

Let $e_1, e_2, \dots, e_n$ be a local orthonormal frame field on M. From (3.1), we have,

$$e_j = \phi_j \partial_t + \hat{e}_j \tag{3.2}$$

where $\phi_j = < e_j, \partial_t >, 1 \leq j \leq n$.

Then using (3.2) in lemma (2.1) we get

$$\tilde{R}(e_i, e_j)e_k = (\delta_{ik}\phi_j - \delta_{jk}\phi_i)\frac{f''}{f}\partial_t + \{\phi_i\phi_k\frac{f''}{f} + \frac{(f'^2 - k)}{f^2}(\delta_{ik} - \phi_i\phi_k)\}\hat{e}_j$$

$$-\left\{\phi_j\phi_k\frac{f''}{f} + \frac{(f'^2-k)}{f^2}(\delta_{jk} - \phi_j\phi_k)\right\}\hat{e}_i \tag{3.3}$$

Theorem 3.1 The Ricci tensor of $R^m(k,f)$ is given by

$$R_{jk} = -(n-2)\left((lnf)'' + \frac{k}{f^2}\right)\phi_j\phi_k - \left((n-1)\left(\frac{f'^2 - k}{f^2}\right) - \left(\sum \phi_i^2\right)(\ln f)'' + \frac{k}{f^2}\right)\delta_{jk} \tag{3.4}$$

Proof: Contracting (3.3) with respect to e_i, we have

$$g(R(e_i, e_j)e_k, e_i) = (\delta_{ik}\phi_j - \delta_{jk}\phi_i)\frac{f''}{f}g(\partial_t, e_i) + \{\phi_i\phi_k\frac{f''}{f}$$

$$+ \frac{(f'^2 - k)}{f^2}(\delta_{ik} - \phi_i\phi_k)g(\hat{e}_j, e_i)$$

$$-\left\{\phi_j\phi_k\frac{f''}{f} + \frac{(f'^2 - k)}{f^2}(\delta_{jk} - \phi_j\phi_k)\right\}g(\hat{e}_i, e_i)$$

that is

$$R_{jk} = (\delta_{ik}\phi_j - \delta_{jk}\phi_i)\frac{f''}{f}g(\partial_t, e_i)$$

$$+\{\phi_i\phi_k\frac{f''}{f} + \frac{(f'^2 - k)}{f^2}(\delta_{ik} - \phi_i\phi_k)\}g(\hat{e}_j, e_i)$$

$$-\left\{\phi_j\phi_k\frac{f''}{f} + \frac{(f'^2 - k)}{f^2}(\delta_{jk} - \phi_j\phi_k)\right\}g(\hat{e}_i, e_i)$$

and simplifying we get the required result.

Theorem 3.2. For $X, Y, Z \in L(S)$ and $\partial_t \in L(I)$,

$$\left(\nabla_{\partial_t}R\right)(X,Y)Z = -2(lnf)'R(X,Y)Z \tag{3.5}$$

Proof: We have,

$$\left(\nabla_{\partial_t}R\right)(X,Y)Z = \nabla_{\partial_t}R(X,Y)Z - R(\nabla_{\partial_t}X,Y)Z$$
$$-R\left(X,\nabla_{\partial_t}Y\right)Z - R(X,Y)\nabla_{\partial_t}Z \tag{3.6}$$

that is,

$$\left(\nabla_{\partial_t}R\right)(X,Y)Z = (\ln f)'R(X,Y)Z - R((\ln f)'X,Y)Z$$
$$-R(X,(\ln f)'Y)Z - R(X,Y)(\ln f)'Z \tag{3.7}$$

which on simplifying we get the required result.

Let M be a submanifold of $R^m(k,f)$. If $\widetilde{\nabla}_X$ and ∇ are Levi Cevita connections on $R^m(k,f)$ and M respectively, then the second fundamental form h, on M is defined as

$$h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y \tag{3.8}$$

Theorem 3.3. If $\nabla_Y Z,\ \widetilde{\nabla}_Y Z \in L(S)$ then

$$\tilde{R}\left(\partial_t, h(Y,Z)\right)\partial_t = \frac{f''}{f}\ h(Y,Z) \tag{3.9}$$

Proof: For any $X \in L(S)$ we have

$$\tilde{R}(\partial_t, X)\partial_t = \frac{f''}{f} X. \tag{3.10}$$

Hence,

$$\tilde{R}\big(\partial_t, \tilde{\nabla}_Y Z\big)\partial_t = \frac{f''}{f} \tilde{\nabla}_Y Z \tag{3.11}$$

and

$$\tilde{R}(\partial_t, \nabla_Y Z)\partial_t = \frac{f''}{f} \nabla_Y Z \tag{3.12}$$

From the linearity of $\tilde{R}$ taking the difference of (3.11),(3.12) gives the required result.

Theorem 3.4. If $R^m(k,f)$ is symmetric, and f' non constant, then for any submanifold M, the second fundamental form maps $\big(T_pM \cap L(S)\big) \times \big(T_pM \cap L(S)\big)$ to $T_pM^{\perp} \cap L(S)$

Proof: We have from theorem (3.3),

For $\nabla_Y Z, \tilde{\nabla}_Y Z \in L(S)$,

$$\tilde{R}\big(\partial_t, h(Y,Z)\big)\partial_t = \frac{f''}{f} h(Y,Z) . \tag{3.13}$$

Differentiating covariantly with respect to X, we get

$$\nabla_X \tilde{R}\big(\partial_t, h(Y,Z)\big)\partial_t = \frac{f''}{f} \nabla_X h(Y,Z).$$

That is, $(\nabla_X \tilde{R})\big(\partial_t, h(Y,Z)\big)\partial_t + \tilde{R}\big(\nabla_X \partial_t, h(Y,Z)\big)\partial_t$

$$+ \tilde{R}\big(\partial_t, \nabla_X h(Y,Z)\big)\partial_t + \tilde{R}\big(\partial_t, h(Y,Z)\big)\nabla_X \partial_t = \frac{f''}{f} \nabla_X h(Y,Z).$$

Using lemma (2.1) we get

$$(\nabla_X \tilde{R})\big(\partial_t, h(Y,Z)\big)\partial_t + \tilde{R}\big((lnf)'X, h(Y,Z)\big)\partial_t$$

$$+ \tilde{R}\big(\partial_t, \nabla_X h(Y,Z)\big)\partial_t + \tilde{R}\big(\partial_t, h(Y,Z)\big)(lnf)'X = \frac{f''}{f} \nabla_X h(Y,Z).$$

Now using lemma (2.2) we get

$$(\nabla_X \tilde{R})\big(\partial_t, h(Y,Z)\big)\partial_t + \tilde{R}\big((lnf)'X, h(Y,Z)\big)\partial_t$$

$$+\frac{f''}{f}\nabla_X h(Y,Z) + \tilde{R}\big(\partial_t, h(Y,Z)\big)(lnf)'X = \frac{f''}{f}\nabla_X h(Y,Z).$$

which on simplifying we get

$$(\nabla_X \tilde{R})\big(\partial_t, h(Y,Z)\big)\partial_t + \tilde{R}\big((lnf)'X, h(Y,Z)\big)\partial_t + \tilde{R}\big(\partial_t, h(Y,Z)\big)(lnf)'X = 0.$$

Further from lemma (2.2) we have $\tilde{R}(X,Y)\partial_t = 0$ which implies

$$(\nabla_X \tilde{R})\big(\partial_t, h(Y,Z)\big)\partial_t + \tilde{R}\big(\partial_t, h(Y,Z)\big)(lnf)'X = 0$$

This gives,

$$\big(\tilde{\nabla}_X \tilde{R}\big)\big(\partial_t, h(Y,Z)\big)\partial_t = (lnf)' < h(Y,Z), X > \frac{f''}{f}\,\partial_t \tag{3.14}$$

If $R^m(k,f)$ is symmetric, then $\big(\tilde{\nabla}_X \tilde{R}\big) = 0$ and hence

$$< h(Y,Z), X > = 0$$

Which implies h maps $\big(T_pM \cap L(S)\big) \times \big(T_pM \cap L(S)\big)$ to $T_pM^{\perp} \cap L(S)$.

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Some δ – open sets in intuitionistic fuzzy setting

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Abstract

The aim of this paper is to introduce the concept of intuitionistic fuzzy δ –semiopen sets. Some of its basic properties would be investigated in intuitionistic fuzzy setting. Different know properties of fuzzy sets would be generalized with the help of this new concepts. Some characterization theorems would also be obtained in intuitionistic fuzzy topological spaces.

Key words: Intuitionistic fuzzy open sets, intuitionistic fuzzy topology, intuitionistic fuzzy δ – semiopen sets.

1. Introduction

Zadeh [10] introduced the notion of fuzzy sets. There after several researches contributed themselves on the generalizations of the notion of fuzzy sets. Chang [3] introduced the concept of fuzzy topological spaces. Azad [2] introduced the concept of fuzzy semi continuity in fuzzy setting. Atanassov [1] introduced the notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [4] introduced the notion of intuitionistic fuzzy topological spaces using the notion of intuitionistic fuzzy sets. After that many researchers [5, 6] introduced different types of intuitionistic fuzzy sets. In section 2, some known definitions and results would be mentioned as ready reference. In section 3, the concept of a new class of sets, called - intuitionistic fuzzy δ – semiopen sets is to be introduced and some of their basic properties are to obtained.

2. Preliminaries

In this section, some known definitions and results are to be mentioned as ready reference.

Definition 2.1. [1] An intuitionistic fuzzy set (IFS, in short) A in X is an object having the form

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$$

where the function $\mu_A(x) : X \to [0, 1]$ and $\nu_A(x)\} : X \to [0, 1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of non-membership (namely, $\nu_A(x)$) of each element $x \in X$ to the set A, respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X.

Definition 2.2. [1] Let A and B be IFSs of the form $A = \{x, \mu_A(x), \nu_A(x)\} : x \in X\}$and $B = \{x, \mu_B(x), \nu_B(x)\} : x \in X\}$. Then

(i) $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,
(ii) A= B if and only if $A \leq B$ and $B \leq A$,
(iii) $A^c = \{(x, \nu_A(x), \mu_A(x))\} : x \in X\}$,
(iv) $A \wedge B = \{x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\} : x \in X\}$,
(v) $A \vee B = \{x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\} : x \in X\}$.

For the sake of simplicity, we shall use the notation $A=(x, \mu_A, \nu_A)$ instead of $A=\{(x, \mu_A(x), \nu_A(x)) : x \in X\}$. Also, for the sake of simplicity, we shall use the notation $A=\{(x, (\mu_A, \mu_B))\}$ instead of $A=\{(x, (\mu_A(x), \nu_A(x)) : x \in X\}$.

The intuitionistic fuzzy sets $0_\sim = \{(X, 0, 1) : x \in X\}$ and $1_\sim = \{(X, 1, 0) : x \in X\}$are respectively called the empty set and the whole set of X.

Definition 2.3. [4] An intuitionistic fuzzy topology (IFT, in short) on X is a family τ of IFSs in X satisfying the following axioms :

(i) $0_\sim, 1_\sim \in \tau$
(ii) $G_1 \wedge G_2 \in \tau$ for any $G_1, G_2 \in \tau$
(iii) $\vee G_i \in \tau$ for any family $\{G_i : i \in J\} \leq \tau$

In this case, the pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, in short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS, in short) in X.
The complement A^c of an IFOS A in IFTS (X, τ) is called an intuitionistic fuzzy closed set (IFCS, in short) in X.

Definition 2.4. [4] Let (X, τ) be an intuitionistic fuzzy topological space (IFTS) and $A=(x, \mu_A, \nu_A)$ be an IFS in X. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure are defined by

$$\text{int}(A) = \vee\{G : G \text{ is an IFOS in X and } G \leq A\}$$
$$\text{cl}(A) = \wedge\{K : K \text{ is an IFCS in X and } A \leq K\}$$

Note that for any IFS A in (X, τ), we have $\text{cl}(A^c) = [(\text{int}(A))]^c$ and $\text{int}(A^c) = [(\text{cl}(A))]^c$.

Definition 2.5. An IFS A= {(x, μ_A, ν_A)} in an IFTS (X, τ) is said to be an

(i) [7] intuitionistic fuzzy semiopen set (IFSOS, in short) if A≤ cl(int(A)),
(ii) [9] intuitionistic fuzzy α-open set (IFαOS, in short) if A≤ int(cl(int(A))),
(iii)]7] intuitionistic fuzzy regular open set (IFROS, in short) if A= int(cl(A)),
(iv) [7] intuitinistic fuzzy preopen set (IFPOS, in short)) if A≤ int(cl(A))),
(v) [8] intuitinistic fuzzy γ - open set (IF γOS, in short)) if A≤ int(cl(A))∨cl(int(A)).

The family of all IFOS (respectively IFSOS, IFαOS, IFROS, IFPOS, IF γOS) of an IFTS (X, τ) is denoted by IFO(X) (respectively IFSO(X), IFαO(X), IFRO(X), IFPOS(X), IF γOS(X)).

Definition 2.6. An IFS A = (x, μ_A, ν_A) in an IFTS (X, τ) is said to be an

(i) [7] intuitionistic fuzzy semiclosed set (IFSCS, in short) if int(cl(A)) ≤ A,
(ii) [9] intuitionistic fuzzy α-closed set (IFαCS, in short) if cl(int(cl(A))) ≤ int(cl(int(A))),
(iii) [7] intuitionistic fuzzy regular closed set (IFRCS, in short) if A=cl(int(A)),
(iv) [7] intuitinistic fuzzy preclosed set (IFPCS, in short)) if cl(int(A)) ≤ A,
(v) [8] intuitinistic fuzzy γ - closed set (IF γCS, in short)) if int(cl(A)) ∩ cl(int(A)) ≤ A.

The family of all IFCS (respectively IFSCS, IFαCS, IFRCS, IFPCS, IF γCS) of an IFTS (X, τ) is denoted by IFC(X) (respectively IFSC(X), IFαC(X), IFRC(X), IFPCS(X), IF γCS(X)).

Definition 2.7. [7] Let A be an IFS in an IFTS (X, τ). Then semi interior of A (sint(A), in short) and semi closure of A (scl(A), in short) are defined as

sint(A) = ∨{G: G is an IFSOS in X and G ≤ A}
and scl(A) = ∧{K: K is an IFSCS in X and A ≤ K}.

3. Intuitionistic fuzzy δ – semiopen sets

In this section, the concept of a new kind of set, known as intuitionistic fuzzy δ – semiopen set is to be introduced. Some of its basic properties are also to be investigated.

Definition 3.1. An IFS A in an IFTS (X, τ) is said to be

(i) an intuitionistic fuzzy δ – semiopen (IFδSO) set of X if A ≤ cl(δint(A)) or equivalently there exists an intuitionistic fuzzy δ – open (IFδO) set B ∈ τ such that B ≤ A ≤ δcl(B),

(ii) an intuitionistic fuzzy δ – semiclosed (IFδSC) set of X if there exists an intuitionistic fuzzy δ – closed (IFδC) set B of (X, τ) such that δint(B) ≤ A ≤ B.

It can be easily shown that δ – closure of an IFO (IFC) set of (X, τ) is IFδSO (IFδSC).

Remark 3.2. It is obvious that every IFδO (IFδC) set in an IFTS (X, τ) is IFδSO (IFδSC) set but the converse is not true.

We denote the family of all intuitionistic fuzzy δ – semiopen sets and intuitionistic fuzzy δ – semiclosed sets by ψ(X) and ψ (X) respectively.

Definition 3.3. An IFS A in an IFTS (X, τ) is said to be an intuitionistic fuzzy regular δ – open set if int(δcl(A)) = A.

The complement of an intuitionistic fuzzy regular δ – open set is called intuitionistic fuzzy regular δ – closed set.

Definition 3.4. Let A be an IFS in an IFTS (X, τ). Then

(i) δsintA = ∨ {B : B ∈ IFδSO, B ≤ A} is called intuitionistic fuzzy δ – semi interior of A and
(ii) δsclA = ∧{B : B ∈ IFδSC, A ≤ B} is called intuitionistic fuzzy δ – semi closure of A.

Remark 3.5. From the above definitions, we have the following relations :

(i) $\delta scl(A^c) = (\delta sintA)^c$
(ii) $(\delta scl(A)^c = \delta sint(A^c)$.

Remark 3.6. For any IFS U, U is an intuitionistic fuzzy δ – semi open set if and only if δsintU = U because U is an intuitionistic fuzzy δ – semi open set if and only if U^c is an intuitionistic fuzzy δ – semi closed set, i.e., if and only if $U^c = \delta scl(U^c)$, i.e., if and only if $U = (\delta scl(U^c))^c = \delta sint(U)$.

Definition 3.7. An IFS A in an IFTS (X, τ) is called an intuitionistic fuzzy δ – semi nbd of a fuzzy point x_p if there exists a fuzzy set U ∈ IFδSO such that $x_p \in U \leq A$.
We denote the set of all intuitionistic fuzzy δ – semi nbds of a fuzzy point x_p by $\xi(x_p)$.

Definition 3.8. An IFS A in an IFTS (X, τ) is called an intuitionistic fuzzy δ – semi q - nbd of a fuzzy point x_p if there exists a fuzzy set U ∈ IFδSO such that x_p q U ≤ A.

We denote the set of all intuitionistic fuzzy δ – semi q - nbds of a fuzzy point x_p by $\eta(x_p)$.

Theorem 3.9. (*a*) The union of any collection of intuitionistic fuzzy δ – semiopen sets in an IFTS (X, τ) is also intuitionistic fuzzy δ – semiopen set.

(*b*) The intersection of any collection of intuitionistic fuzzy δ – semiclosed sets in an IFTS (X, τ) is also intuitionistic fuzzy δ – semiclosed set.

Proof. (*a*) Let $\{A_i : i \in J\}$be a collection of intuitionistic fuzzy δ – semiopen sets in an IFTS (X, τ). Then $A_i \le cl(\delta int(A_i))$ for each i. Now $\vee A_i \le \vee cl(\delta int(A_i))$ $A_i \le cl(\delta int(\vee A_i))$. This shows that $\vee A_i$ is an intuitionistic fuzzy δ – semiopen set.

(*b*) Let $\{A_i : i \in J\}$be a collection of intuitionistic fuzzy δ – semiopen sets in an IFTS (X, τ). Then by (*a*), $\vee A_i$ is an intuitionistic fuzzy δ – semiopen sets. Therefore $(\vee A_i)^c$ is an intuitionistic fuzzy δ – semiclosed set. Thus, $\wedge(A_i)^c$ is an intuitionistic fuzzy δ – semiclosed set. But $(A_i)^c$ is an intuitionistic fuzzy δ – semiclosed set as A_i is a fuzzy δ – semiopen set. Thus, any collection of intuitionistic fuzzy δ – semiclosed sets in an IFTS (X, τ) is also intuitionistic fuzzy δ – semiclosed set.

Theorem 3.10. A fuzzy set A ∈ IFδSO if and only if every intuitionistic fuzzy point $x_p \in A$, there exists a fuzzy set B ∈ IFδSO such that x_p q $B \le A$.

Proof. If A ∈ IFδSO, then we may take B = A, for every $x_p \in A$. Conversely, we have $A = \vee\{x_p\} \le \vee B \le A$, for every $x_p \in A$. The result now follows from the fact that any union of intuitionistic fuzzy δ – semiopen sets is intuitionistic fuzzy δ – semiopen set.

Theorem 3.11. A fuzzy set A ∈ IFδSO if and only if every intuitionistic fuzzy point $x_p \in A$, A is an intuitionistic fuzzy δ – semi prenbd of x_p.

Proof. Obvious.

Theorem 3.12. Let A be an intuitionistic fuzzy set of an IFTS (X, τ). Then an intuitionistic fuzzy point $x_p \in$ intδsclA if and only if every intuitionistic fuzzy δ – semi pre q - nbd of x_p is quasicoincident with A.

Proof. Necessity. Suppose that $x_p \in$ intδsclA and if possible, let there exists a fuzzy δ – semi pre q - nbd B of x_p such that B $\not q$ A. Then there exists a fuzzy set $B_1 \in$ IFδSO such that x_p q

$B_1 \leq B_1 \leq B$, which shows that $B_1 \not{q} A$ and hence $A \leq B_1^c$. As $B_1^c \in IF\delta SC$, intδsclA$\leq B_1^c$. Since $x_p \in B_1^c$, we obtain that $x_p \notin$ intδsclA, which is a contradiction.

Sufficiency. Suppose that every intuitioistic fuzzy δ – semi q – nbd of x_p is quasicoincident with A. If $x_p \notin$ intδsclA, then there exists an intuitionistic fuzzy δ – semiclosed set $B \geq A$ such that $x_p \notin B$. So $B^c \in$ IFδSO such that x_p q B^c and $B^c \not{q}$ A, a contradiction.

Theorem 3.13. Let U and V be two IF sets in an IFTS (X, τ). Then we have the following properties :

(a) $\delta scl(0_\sim) = 0_\sim$
(b) $U \leq \delta scl(U)$
(c) $U \leq V \Rightarrow \delta scl(U) \leq \delta scl(V)$
(d) $\delta scl(U) \vee \delta scl(V) = \delta scl(U \vee V)$
(e) $\delta scl(U \wedge V) \leq \delta scl(U) \wedge \delta scl(V)$.

Proof. (a) Obvious
(b) Since $U \leq cl(U) \leq scl(U) \leq \delta scl(U)$
Therefore U) $\leq \delta scl(U)$.
(c) Let $x_{(\alpha, \beta)}$ be an IF point in X such that $x_{(\alpha, \beta)} \notin \delta scl(V)$. Then there is an IF regular open q – nbd A of $x_{(\alpha, \beta)}$ such that $A \not{q} V$. Since $U \leq V$, we have $A \not{q} U$. Therefore $x_{(\alpha, \beta)} \notin \delta scl(U)$.
(d) Since $U \leq (U \vee V)$, $\delta scl(U) \leq \delta scl(U \vee V)$. Similarly, $\delta scl(V) \leq \delta scl(U \vee V)$. Hence $\delta scl(U) \vee \delta scl(V) \leq \delta scl(U \vee V)$. To show that $\delta scl(U \vee V) \leq \delta scl(U) \vee \delta scl(V)$, take any $x_{(\alpha, \beta)} \in \delta scl(U \vee V)$.Then for any IF regular open q – nbd of A of $x_{(\alpha, \beta)}$, Aq$(U \vee V)$. Hence AqU or AqV. Therefore, $x_{(\alpha, \beta)} \in \delta scl(U)$ or $x_{(\alpha, \beta)} \in \delta scl(V)$. Hence $x_{(\alpha, \beta)} \in \delta scl(U) \vee \delta scl(V)$.
(e) Since $U \wedge V \leq U$, $\delta scl(U \wedge V) \leq \delta scl(U)$. Similarly, $\delta scl(U \wedge V) \leq \delta scl(V)$. Therefore, $\delta scl(U \wedge V) \leq \delta scl(U) \wedge \delta scl(V)$.

Lemma 3.14. (*a*) For any IF set U in an IFT (X, τ), int(δcl(U)) is an IF regular δ - open set.
(*b*) For any IF open set U in an IFT (X, τ) such that $x_{(\alpha, \beta)}$ q U, int(δcl(U)) is an IF regular δ - open q – nbd of $x_{(\alpha, \beta)}$.

Proof. (*a*) It is enough to show that int(δcl(U)) = int(δcl(int(δcl(U))). Since int(δcl(U)) $\leq$ δcl(int(δcl(U))), we have int(int(δcl(U))) $\leq$ int(δcl(int(δcl(U)))). Thus int(δcl(U)) $\leq$ int(δcl(int(δcl(U)))). Conversely, since int(δcl(U)) $\leq$ δcl(U), we have δcl(int(δcl(U)) $\leq$

δcl(δcl(U)) = δcl(U). Thus, int(δcl(int(δcl(U))) ≤ int(δcl(U)). Hence int(δcl(U)) is an IF regular δ - open set.
(*b*) Clearly int(U) ≤ int(δcl(U)). Since U is an IF open set, we have

U = int(U) ≤ int(δcl(U)).

By (*a*), int(δcl(U)) is an IF regular δ - open set. Therefore, int(δcl(U)) is an IF regular δ - open q – nbd of $x_{(\alpha, \beta)}$.

Theorem 3.15. Let U and V be two IF sets in an IFTS (X, τ). Then we have the following properties :

(a) δsint($1_\sim$) = $1_\sim$
(b) δsint(U) ≤ U,
(c) U ≤ V ⇒ δsint(U) ≤ δsint(V),
(d) δsint(U) ∨ δsint(V) ≤ δsint(U ∨ V),
(e) δsint(U ∧ V) = δsint(U) ∧ δsint(V).

Theorem 3.16. If U is an IF δ - semiopen set in an IFT (X, τ), the IF closure and IF δ – closure of U are the same, i.e., cl(U) = δcl(U).

Theorem 3.17. Let (X, τ) be an IFTS. Then the followings are equivalent :

(a) A is IFδSC set
(b) $\bar{A}$ is IFδSO set
(c) intδcl(A) ≤ A
(d) clδint($\bar{A}$) ≥ $\bar{A}$

Proof. (a) ⇒ (b). By definition there exists an intuitionistic fuzzy δ – closed (IFδC) set B of (X, τ) such that int(B) ≤ A ≤ B. This implies, $\overline{intB} \geq \bar{A} \geq \bar{B}$. Using proposition 3.15 of D.Coker [7], we get, $\bar{B} \leq \bar{A} \leq$ cl($\overline{B}$) where $\bar{B}$ is an IFδO set in (X, τ). Therefore, $\bar{A}$ is IFδSO set in (X, τ).

(b) ⇒ (a).Similar.

(a) ⇒ (c). By definition there exists an intuitionistic fuzzy δ – closed (IFδC) set B of (X, τ) such that int(B) ≤ A ≤ B. Using proposition 3.16 of D. Coker [7], we get, int(B) ≤ A ≤ cl(A) ≤ B. Since int(B) is the largest IFδO set contained in B, we have intδcl(A) ≤ int(B) ≤ A, i.e. intδcl(A) ≤ A.

(c) ⇒ (a).It follows by taking $B = \delta cl(A)$ i.e, $int(B) \leq A$. Also we know that $A \leq \delta cl\ A = B$ and hence $(B) \leq A \leq B$, as $B = \delta(A)$ is a closed set, therefore A is IFδSC.

(b) ⇒(d) can similarly be proved.

Theorem 3.18. Union of a finite number of IFδSO sets is a IFδSO set and intersection of a finite number of IFδSC sets is a IFδSC.

Proof. 1st part. Let A_1, A_2,………., be IFδSO sets of (X,τ). Then their exist IFδO sets B_1, B_2,……….., of (X,τ) such that $B_i \leq A_i \leq \delta cl B_i$, i = 1, 2,....,n. Generalizing the idea of proposition 3.16, D.Coker[7], we get, $\cup Bi \leq \cup Ai \leq \cup \delta cl\ Bi = \delta cl(\cup Bi)$. Also $\cup Bi \in \tau$, Hence $\cup Ai$ is IFδSO.

2nd part. Similar.

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Some Algebraic and Topological Properties of Quasi Cauchy Sequences of Interval Numbers

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ABSRACT: In this paper we have introduced the notion of quasi Cauchy sequence of interval numbers. We have introduced the classes of sequence $\ell^I_\infty(\Delta)$, $c^I_0(\Delta)$ and $c^I(\Delta)$. We have studied some algebraic and topological properties like solidness, symmetric, convergence free etc. of different classes of sequence.

Key words: quasi Cauchy; interval number; solid, symmetric; convergence free.

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1. PRILIMIRARIES AND BACKGROUND

Interval arithmetic was first suggested by Dwyer [5] in 1951. Thereafter the concept has been studied by Moore [10] and applied in different fields. Chiao in [3] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [14] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently Dutta and Tripathy [5] introduced the class of p-absolutely summable sequence $\ell^i(p)$ of interval numbers and studied some important properties. Further development on interval arithmetic was done by Moore and Yang [11, 12], Fischer [7] and others ([2], [4], [6], [7], [13], [14], [15]).

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $\mathbb{R}$. Any elements of $\mathbb{R}$ is called closed interval and denoted by $\bar{x}$. That is $\bar{x} = \{x \in \mathbb{R}: a \leq x \leq b\}$. An interval number $\bar{x}$ is a closed subset of real numbers. Let x_l and x_r be first and last points of the interval number $\bar{x}$, respectively. For $x_1, x_2 \in \mathbb{R}$, we have $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$,

$$\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\} \text{ and if } \alpha \geq 0, \text{ then}$$

$$\alpha\, \bar{x} = \{x \in \mathbb{R} : \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$$

and

$$\alpha < 0 \text{ then } \alpha\bar{x} = \{x \in \mathbb{R}: \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\},$$

$$\bar{x}_1\bar{x}_2 = \begin{Bmatrix} x \in R: min\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\} \leq x \leq \\ max\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\} \end{Bmatrix}$$

The set of all interval numbers $\mathbb{R}$ is a complete metric space defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}$$

In the special case $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

2. INTRODUCTION AND DEFINITIONS

Let us define the transformation $f : N \to \mathbb{R}$ by $k \to f(k) = \bar{x}$, $x = (x_k)$. Then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. The $\bar{x}_k$ is called the k^{th} term of the sequence $\bar{x} = (\bar{x}_k)$.

Throughout the article w^i will denote the set of all interval numbers with real terms.

Definition 2.1 A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each □□> 0 there exists a positive integer k_0 such that $d(\bar{x}_k, \bar{x}_0) <$ □□for all $k \geq k_0$. We denote it by $\lim_k \bar{x}_k = \bar{x}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$.

Definition 2.2 An interval valued sequence space $\bar{E}$ is said to be solid if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $|\bar{y}_k| \leq |\bar{x}_k|$, or all $k \in$N and $\bar{x} = (\bar{x}_k) \in \bar{E}$.

Definition 2.3 An interval valued sequence space $\bar{E}$ is said to be monotone if $\bar{E}$ contains the canonical Pre-image of all its step spaces.

Definition 2.4 An interval valued sequence space $\bar{E}$ is said to be convergence free if $\bar{y} = (\bar{y}_k)$ $\in \bar{E}$ whenever $\bar{x} = (\bar{x}_k) \in \bar{E}$ and $\bar{x}_k = \bar{0}$ implies $\bar{y}_k = \bar{0}$.

Definition 2.5 A sequence (x_k) of real numbers is said to be a Cauchy sequence if for a given $\varepsilon > 0$, there exist a positive integer K such that $|x_m - x_n| < \varepsilon$, for all $m, n \geq K$.

Burton and Coleman [2] defined quasi Cauchy sequence as follows.

Definition 2.5 A sequence $x = (x_k)$ of points in R is called quasi Cauchy if given any ε > 0, there exist an integer $K > 0$ such that $n \geq K$ implies $|x_{n+1} - x_n| < \varepsilon$.

The notion of difference sequence space for complex terms was introduced by Kizmaz [8] defined by $Z(\Delta) = \{(x_k) : (\Delta x_k) \in Z\}$ for $Z = \ell_\infty$, c, c_0 where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$ and studied the sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.
We introduce the quasi Cauchy sequence of interval number in terms of the notion of difference sequence as introduced by Kizmas as follows.

A sequence $z(\Delta) = \{(x_k) \in w^i : (\Delta x_k) \in z\}$ is called quasi Cauchy sequence of interval number if (Δx_k) is a null sequence where $\Delta x_k = x_k - x_{k+1}$ for z = ℓ^I_∞, c^I_0, c^I .

3. MAIN RESULTS

Theorem 1. The classes of sequence $\ell^I_\infty(\Delta)$, $c^I_0(\Delta)$ and $c^I(\Delta)$ are solid and hence monotone.

Proof: Let us consider the sequence $x = (x_k) \in \ell^I_\infty(\Delta)$ and $y = (y_k)$ be another sequence such that $|y_k| < |x_k|$ for all $k \in N$.
Then we have

$$\sum_{k=1}^{\infty} d\left(\Delta x_k, \bar{0}\right) < \infty$$

and

$$\sum_{k=1}^{\infty} d\left(\Delta y_k, \bar{0}\right) < \sum_{k=1}^{\infty} d\left(\Delta x_k, \bar{0}\right) < \infty.$$

This implies $y = (y_k) \in \ell^I_\infty(\Delta)$ and hence the sequence space $\ell^F_\infty(\Delta)$ is solid and hence monotone.
This completes the proof.

Theorem 2. The classes of sequence $\ell^I_\infty(\Delta)$, $c^I_0(\Delta)$ and $c^I(\Delta)$ are not symmetric in general.

Proof. The proof follows from the following example.

Example1. Consider the sequence $(x_k) = \{A, B, A, B, A, B \ldots\ldots\}$

Where

$$A = [-1,\ 1] \text{ and}$$
$$B = [-2,\ 2]$$

Then clearly, $(x_k) \in c^I(\Delta)$.

Now, consider the re-arrangement (y_k) of the sequence (x_k) as follows:

$(y_k) = \{A, A, B, B, A, A \ldots\ldots\}$. But $(y_k) \notin c^I(\Delta)$.

Hence $c^I(\Delta)$ is not a symmetric space.

Theorem 2. The classes of sequence $\ell^I_\infty(\Delta)$, $c^I_0(\Delta)$ and $c^I(\Delta)$ are not Convergence free in general.

Proof. The proof follows from the following example.

Example 2. Consider the sequence (x_k)

$$x_k = \left[-\frac{1}{k}, \frac{1}{k}\right] \text{ implies } x_{k+1} = \left[-\frac{1}{k+1}, \frac{1}{k+1}\right].$$

Now,

$$\Delta x_k = \left[-\frac{1}{k} - \frac{1}{k+1}, \frac{1}{k} + \frac{1}{k+1}\right] = \left[-\frac{2k-1}{k(k+1)}, \frac{2k+1}{k(k+1)}\right]$$

Then, we have $\lim_{k\to\infty} \Delta x_k = 0$.

This implies, $(x_k) \in c^I_0(\Delta)$.

Now, consider the sequence given by

$$y_k = [-k, k]$$

and then

$$y_{k+1} = [-(k+1), (k+1)]$$

Now we have

$$\Delta y_k = [-k, k] - [-(k+1), (k+1)]$$
$$= [-(2k+1), (2k+1)]$$

Clearly,

$$(y_k) \notin c_0^I(\Delta).$$

Hence the space is not convergence free.

This complete the proof

Theorem 2. The classes of sequence $\ell_\infty^I(\Delta)$, $c_0^I(\Delta)$ and $c^I(\Delta)$ are sequence algebra.

Proof. We prove the result for the sequence space $\ell_\infty^I(\Delta)$. Consider the sequences (x_k) and (y_k) such that $x = (x_k) \in \ell_\infty^I(\Delta)$ and $y = (y_k) \in \ell_\infty^I(\Delta)$.

Then, by the property of the space, we have that

$$\sum_{k=1}^{\infty} d(\Delta x_k, \bar{0}) < \infty \text{ and } \sum_{k=1}^{\infty} d(\Delta y_k, \bar{0}) < \infty.$$

Now, keeping in mind the algebraic properties of the space we have,

$$\sum_{k=1}^{\infty} d\left(\Delta(x_k y_k), \bar{0}\right) = \sum_{k=1}^{\infty} d\left((\Delta x_k \Delta y_k), \bar{0}\right)$$

$$= \sum_{k=1}^{\infty} [d(\Delta x_k, \bar{0}).d(\Delta y, \bar{0})]$$

$$\leq \sum_{k=1}^{\infty} d(\Delta x_k, \bar{0}) \sum_{k=1}^{\infty} d(\Delta y_k, \bar{0})$$

$$< \infty.$$

Thus, $(x_k y_k) \in \ell_\infty^I(\Delta)$

Hence, the sequence space $\ell_\infty^I(\Delta)$ is sequence algebra.

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On Some Generalisations of Open and Closed Sets in Topological Spaces

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Abstract

In this article we generalise the notions of open sets and closed sets in topological spaces. We also introduce some notations in this connection which are very useful in investigation.

Keywords: Interior, Closure, α-closed, c_n-closed.

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1. Introduction:

Levine [1] introduced the notion of semi-open set in a topological space. Crossley and Hildebrand [2] introduced the notion of semi-closed set. Njastad [3] introduced the notion of α-open sets and Abd El-Monsef has introduced the notion of β-open sets. Later on many researcher have investigated these notions and also contributed in these directions. In this article we generalize these concepts on considering these as operators on the subsets of the topological soace.

2 Definitions and preliminaries:

First, we introduce some of our own notations in context of topological spaces. Interior of a set A in a topological space X is generally denoted by $int\ (A)$ or A^{o}. We write

$$i(A) = int(A).$$

The following results can be found in general topology books:-

$i(A) \subseteq A$, $i(\varnothing) = \varnothing$, $i(X) = X$

$i^2(A) = i(i(A)) = i(A)$. Thus, we can write $i^2 = i$.
$i(A) \cup i(B) \subseteq i(A \cup B)$
$i(A \cap B) = i(A) \cap i(B)$.
$A \subseteq B \Longrightarrow i(A) \subseteq i(B)$.
A set A is open $\Leftrightarrow A = i(A)$.
Complement of a set A is usually denoted by A^c or A'. Here we use notation $c_0(A)$.
So, $A \cap c_0(A) = \emptyset$, $A \cup c_0(A) = X$.

Result 2.1. Let A be a closed set, *then* $c_0{}^2(A) = c_0(c_0(A)) = A$. Thus, $c_0{}^2 = I$, where I stands for identity set function $I(A) = A$, for A a subset of (X, τ).

The following result can be found in literature on topology.

Result 2.2. Relation between closure of a set A is denoted by $\bar{A}$ and *int A* is given by

$$\bar{A} = (int\ \acute{A}).$$

Note 2.1. Let us write $c(A)$ in place of $\bar{A}$. So the above relation gives
$c(A) = c_0(i(c_0(A))) = (c_0 i c_0)(A)$.
So, we can write $c = c_0 i c_0$.
We have the following results from the literature.

Result 2.3. A is closed $\Leftrightarrow c(A) = A$.

For any two sets A and B, we have $c(A \cup B) = c(A) \cup c(B)$, $c(\emptyset) = \emptyset$, $c(X) = X$, $c(c(A)) = c(A)$.*i.e.* $c^2 = c$. Also, $i(A)$ is the largest open set contained in A and $c(A)$ is the smallest closed set containing A. $A \subseteq B \Longrightarrow c(A) \subseteq c(B)$. Exterior of a set A, written as $ext(A)$ is defined as $ext(A) = int(A')$.

Remark 2.1. In our notations $e(A)$ stands for *ext* (A). Then $e(A) = i(c_0(A))$. Thus $e = ic_0$.

Also, $ie = e$ and $ec_0 = I$, $e(\emptyset) = X$, $e(A) \subseteq c_0(A)$, $e(A) = e(c_0(e(A)))$. Thus $e = ec_0e$, $e(A \cup B) = e(A) \cap e(B)$. Further we have, $c = c_0 e$ and $ic = ic_0 ic_0 = (ic_0)^2 = e^2$. Thus $e^2(A) = int\ \bar{A}$.

Hence, $e^2A \subseteq \bar{A} = c(A)$, so we have $A \subseteq \bar{A} \Longrightarrow int\, A \subseteq int\, \bar{A} \subseteq \bar{A}$. Thus $i(A) \subseteq e^2(A)$.

When $A \subseteq int\, \bar{A}$, A is defined to be pre-open, when $\overline{intA} \subseteq A$, A is defined to be pre-closed.

According to our notations, we have

A is pre-open if $A \subseteq ic(A)$, A is pre-closed if $ci\,(A) \subseteq A$.

3. Properties of closed set and open sets.

Definition 3.1 (Levine in [1]). A is semi-open if $A \subseteq ci(\mathrm{A})$.

Definition 3.2 (Crossley and Hildebrand [2]). A is semi-closed if $ic(A) \subseteq A$.

Definition 3.3 (Njastad [3]). A is α-open, if $A \subseteq ici(A)$.

Remark 3.1: A is α-closed if $cic(A) \subseteq A$.

Thus α-closed set is semi-closed.

Definition 3.4 (Abd El-Monsef [4]). A is β-open if $A \subseteq cic(A)$.

Note 3.1. β-open is also called semi pre-open. A is β-closed or semi pre-closed if $ici(A) \subseteq A$. Thus we have several generalisations of open and closed sets.

The main objective of using this notation is that we shall generalize these notions farther in the next section. Let us write $c_1i_1c_1 = c_2$.

Thus, $c_2(A)$ is a closed set . So if $c_2(A) \subseteq A$, we say A is c_2-closed instead of α-closed we investigate some properties of c_2-closed sets, as follows:

Property 3.1. (*i*) $c_2(\varnothing) = c_1i_1c_1(\varnothing) = \varnothing$, $c_2(X) = X$.

(*ii*) $A \subseteq B \Longrightarrow c_1(A) \subseteq c_1(B) \Longrightarrow i_1c_1(A) \subseteq i_1c_1(B)$.
$\Longrightarrow c_1i_1c_1(A) \subseteq c_1i_1c_1\,(B) \Longrightarrow c_2(A) \subseteq c_2(B)$.

(*iii*) $c_2(A) \cup c_2(B) = c_1i_1c_1(A) \cup c_1i_1c_1(B) = c_1(\,i_1c_1\,(A) \cup i_1c_1(B))$.
$\subseteq c_1i_1(c_1\,(A) \cup c_1(B)) = c_1i_1c_1\,(A \cup B) = c_2(A \cup B)$.
So, $c_2(A) \cup c_2(B) \subseteq c_2(A \cup B)$.

$= c_1 i_1 [c_1 (A) \cap c_1(B)] \subseteq c_1 i_1 c_1 (A \cap B) = c_2 (A \cap B)$.

Hence, $c_2(A) \cap c_2 (B) \subseteq c_2 (A \cap B)$.

Theorem 3.1. For any set A, $c_2(A)$ is c_2 – closed. Now we shall show that $c_2(A)$ is the largest c_2- closed set contain in A.

Proof. For this we need to show, $c_2 (c_2(A)) \subseteq c_2(A)$.

Now, $c_1c_2 = c_1c_1i_1c_1 = c_1^2i_1c_1 = c_1i_1c_1 = c_2$.

So, $c_2c_2 = (c_1i_1c_1) (c_1i_1c_1) = c_1i_1c_1^2i_1c_1 = c_1i_1c_1i_1c_1 = c_1i_1c_2$.

Also, $i_1(c_2A) \subseteq c_2(A) \Longrightarrow c_1i_1c_2(A) \subseteq c_1c_2(A) = c_2(A)$.

Hence, $c_2(c_2(A)) = c_1i_1c_2(A) \subseteq c_2(A)$.

Thus, $c_2(A)$ is c_2-closed.

For this, let B be c_2-closed such that $B \subseteq A$.

Then, $c_2(B) \subseteq c_2(A) \subseteq A$.

So, $c_2(A)$ is the largest c_2-closed set contained in A.

Also, $i_1c_1 (A) \subset c_1(A) \Longrightarrow c_1i_1c_1(A) \subset c_1^2(A) = c_1A$

$\Rightarrow c_2(A) \subset c_1(A)$.

Theorem 3.2. If a set A is closed or c_1-closed then it is c_2-closed.But the converse is not true in general.

Proof: If A is closed , then we have $c_1(A)=A$.

So, $c_1c_1(A) = i_1(A) \subseteq A \Rightarrow c_1i_1c_1(A) \subseteq c_1(A) = A$.

Thus, $c_2(A) \subseteq A$ i.e., A is c_2-closed.

But, converse is not true i.e. c_2-closed $\not\Rightarrow c_1$-closed.

The second part follows from the following:

Example 3.1. Let $X=\{a, b, c\}$, $\tau=\{\varnothing, X, \{a\}\}$, $A=\{b\}$.

Here closed sets are $\varnothing$, X and $\{b, c\}$ so A is not closed or c_1-closed.

But, $c_1(A) = \{b, c\}$, $i_1c_1(A) = \varnothing$, $c_1i_1c_1(A) = \varnothing \subseteq A$.

So, $c_2(A) \subseteq A$ *i.e.* A is c_2-closed.

Example 3.2. Let $X=\{a, b, c, d\}$ and $T=\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$.

So, closed sets are $\varnothing$, X, $\{b, c, d\}$, $\{c, d\}$, $\{b, d\}$, $\{d\}$

Let, $A = \{b\}$ then A is neither closed nor c_1-closed.

But, $c_1(A) = \{b, d\}$, $i_1c_1(A) = \varnothing$, $c_1i_1c_1(A) = \varnothing \subseteq A$.

So, $c_2(A) \subseteq A$ i.e. A is c_2-closed.

We shall use the notation, $i_2 = i_1 c_1 i_1$

Then, $i_2(A)$ is an open set we define a set A to be i_2-*open* if $A \subseteq i_2(A)$.

We have seen in definitions and preliminaries that this has been defined as α-open by Njastad [3].

Note 3.2. We have, $i_2(\emptyset)=\emptyset$, $i_2(X)=X$,

$i_1(A) \subseteq A \Rightarrow c_1 i_1(A) \subseteq c_1(A) \Rightarrow i_1 c_1 i_1(A) \subseteq i_1 c_1(A) \subseteq c_1(A)$.

So, $i_2(A) \subseteq c_1(A)$.

Thus, $A \subseteq i_2(A) \subseteq c_1(A)$, i.e. there exists an open set which contains A but is contained in the closure of A.

Also, $A \subseteq B \Rightarrow (i_1 c_1 i_1)(A) \subseteq (i_1 c_1 i_1)(B) \Rightarrow i_2(A) \subseteq i_2(B)$

We establish some results on algebra of subsets of X is (X,τ)

Theorem 3.3. if A and B are i_2-open then so is $A \cup B$.

Proof: Here, $A \subseteq i_2(A)$, $B \subseteq i_2(B)$.

Hence, $A \cup B \subseteq i_2(A) \cup i_2(B)$

$$= (i_1 c_1 i_1)(A) \cup (i_1 c_1 i_1)(B)$$

$$\subseteq i_1(c_1 i_1(A)) \cup c_1 i_1(B)) = i_1 c_1(i_1(A) \cup i_1(B))$$
$$\subseteq (i_1 c_1 i_1)(A \cup B) = i_2(A \cup B).$$

Hence, union of two i_2-open sets is i_2-open.

Theorem 3.4. The intersection of two i_2-open sets is open.

Proof: Let A and B be two i_2-open sets i.e. $A \subseteq i_2(A)$, $B \subseteq i_2(B)$

$$\Rightarrow A \cap B \subseteq i_2(A) \cap i_2(B) = (i_1 c_1 i_1)(A) \cap (i_1 c_1 i_1)(B)$$
$$= i_1[(c_1 i_1)(A) \cap (c_1 i_1)(B))] \subseteq i_1 c_1(i_1(A) \cap i_1 (B))$$

$=(i_1c_1i_1)(A \cap B) = i_2(A \cap B)$

This also shows that $i_2(A) \cap i_2(B) \subseteq i_2(A \cap B)$.

Now we show that for any set A, $i_2(A)$ is i_2-*open* i.e. $i_2(A) \subset i_2(i_2(A))$.

But ,$i_2c_2 = (i_1c_1i_1)(i_1c_1i_1) = i_1c_1i_1^2c_1i_1 = i_1c_1i_1c_1i_1$.

So , $(i_1c_1i_1)(A) \subset (i_1c_1i_1c_1i_1)(A)$

$\Longrightarrow (i_1c_1i_1)(A) \subset (i_1c_1i_1c_1i_1)(A) \Longrightarrow i_2(A) \subset i_2(i_2(A))$.

Thus, $i_2(A)$ is i_2-open.

Now we establish a relation between i_2-open sets and c_2-closed sets.

Theorem 3.5. If a set A is i_2-open, then its complement $c_0(A)$ is c_2-closed and conversely.

Proof: Let A be i_2-open then $A \subseteq i_2(A)$.

Now,

$c_2c_0 = c_1i_1c_1c_0$

$=c_0i_1c_0i_1c_0i_1c_0c_0$

$= c_0i_1c_0i_1c_0i_1$ [as $c_0^2=i$]

$= c_0i_1(c_0i_1c_0)i_1$

$=c_0i_1c_1i_1$

$=c_0i_2$.

Since, $A \subseteq i_2(A) \Longrightarrow c_0i_2(A) \subseteq c_0(A)$.

$\Longrightarrow c_2c_0(A) \subseteq c_0(A)$ i.e. $c_0(A)$ is c_2-closed.

Now, we show to the converse part i.e. to show if A is c_2-closed then $c_0(A)$ is c_2-open. For this it is sufficient to show that if $c_0(A)$ *is* c_2-closed then A is c_2-open. Thus

$c_2c_0(A) \subset c_0(A)$. As seen before $c_2c_0=c_0c_2$. So $c_0i_2(A) \subset c_0(A) \Longrightarrow A \subseteq i_2(A)$ i.e. A is i_2-open.

4. Further generalization of closed sets and open sets.

In this section we farther generalize the notion of closed and open sets and establish some results. We can generalise closed sets further by defining $c_3=c_2i_1c_2$.

Thus, c_3A is a closed set.

We define A to be c_3-closed if $c_3(A) \subseteq A$.

Theorem 4.1. A is c_2-closed $\Longrightarrow$ A is c_3-closed.

Proof: Let a set A be c_2-closed i.e. $c_2(A) \subseteq A$.

Now, $i_1c_2(A) \subset c_2(A) \Longrightarrow c_2i_1c_2(A) \subset c_2(c_2(A)) \subset c_2(A)$
$\Longrightarrow c_3(A) \subset c_2(A) \subset A$.

Hence, A is c_3-closed.

Remark 4.1. The converse of the above results is not true in general i.e if a set A is c_2-closed, it is c_3-closed but the converse may not be true. Example similar to earlier once can be constructed.

Proof: Now, we show that $c_3(A)$ is c_3-closed i.e. $c_3(c_3(A)) \subset c_3(A)$.

Replacing A by $i_1c_2(A)$ in above relation, we have
$c_3(i_1c_2A) \subset c_2(i_1c_2(A)) = c_3(A)$.
Now $c_3^2(A) = c_3(c_3(A)) = c_3(c_2i_1c_2)(A) = (c_2i_1c_2)(c_2i_1c_2)(A)$
$=(c_2i_1c_2^2ic_2)(A) \subset (c_2i_1c_2ic_2)(A) = c_3(ic_2)(A) \subset c_3(A)$.

Thus, c_3A is c_3-closed.

It is clear that $A \subseteq B \Longrightarrow c_3(A) \subseteq c_3(B)$.

Note 4.1. Similarly, we generalise open sets. We define c_3-open sets by defining A to be i_3-open if

$$A \subseteq i_3(A), \text{ where } i_3=i_2c_1i_2.$$

We continue this process and define c_n and i_n by

$$c_n=c_{n-1}i_1c_{n-1} \text{ and } i_n=i_{n-1}c_1i_{n-1}.$$

Define, A to be c_n-closed if $c_n(A)\subseteq A$ and i_n-open if $A \subseteq i_n(A)$.

Clearly, a c_{n-1} closed set is c_n-closed, $c_n(A)$ is c_n-closed, $c_n(A)$ is a closed set and

$$c_n(A) \subset c_{n-1}(A)$$

Result 4.1. The relation between c_n and i_n is given by $c_n=c_0i_nc_0$.

Proof. We establish this result by the method of induction.

When n=1, $c_1=c_0i_1c_0$ which is true.

Let $c_{n-1}=c_0i_{n-1}c_0$.

Then, $c_n=c_{n-1}i_1c_{n-1} =(c_0i_{n-1}c_0)i_1(c_0i_{n-1}c_0) =c_0i_{n-1}(c_0i_1c_0)i_{n-1}c_0 =c_0i_{n-1}c_1i_{n-1}c_0=c_0i_nc_0$.

Thus the proof is complete by induction.

We define i_0 to be c_0 also for then

$$i=i_1=i_0ci_0 \text{ is true, since } i_0ci_0=c_0c_0ic_0c_0=i$$

We also establish relations between c_n, i_n and e.

Theorem 4.2. The relation $c_n=c_0e^{2\wedge n-1}$, holds for all n≥1.

Proof. Let n=1, then we have $c_1=c_0e^{2-1}=c_0e=c_0i$, c_0, which is true.

Assume that $c_n=c_0e^{2\wedge n-1}$.

Then, $c_{n+1}=c_n i c_n=c_0 e^{2^{\wedge} n\text{ -}1}.ic_0 e^{2^{\wedge} n\text{ -}1}=c_0 e^{2^{\wedge}n\text{-}1}.e.e^{2^{\wedge}n\text{-}1}=c_0 e^{2^{\wedge}n\text{-}1+1+2^{\wedge}n\text{-}1}=c_0 e^{2.2^{\wedge}n\text{ -}1}$
$=c_0 e^{2^{\wedge}(n+1)\text{-}1}$.

Hence, for any positive integer n, $c_n=c_0 e^{2^{\wedge}n\text{-}1}$.

Theorem 4.3. The relation $i_n= e^{2^{\wedge}n\text{ -}1} c_0$, holds for all $n\geq 1$.

Proof. Let, $n=1$, $e^{2^{\wedge}n\text{ -}1}c_0=ec_0=ic_0c_0=i$.

So, assume that $i_n= e^{2^{\wedge}n\text{-}1}c_0$ holds.

Then,

$$
\begin{aligned}
i_{n+1} &= i_n c i_n = e^{2^{\wedge}n\text{-}1}c_0 c\, e^{2^{\wedge}n\text{-}1} c_0 \\
&= e^{2^{\wedge}n\text{-}1}c_0 c_0 e e^{2^{\wedge}n\text{-}1}c_0 \\
&= e^{2^{\wedge}n\text{-}1}.e.e^{2^{\wedge}n\text{-}1}c_0 \\
&= e^{2^{\wedge}n\text{-}1+1+2^{\wedge}n\text{-}1}c_0 \\
&= e^{2^{\wedge}(n+1)\text{ -}1}c_0 \text{ , for all } n\geq 1.
\end{aligned}
$$

Hence the proof is complete.

Corollary 4.1. For all $n \geq 1$, $i_n\, c_n = e^{2^{\wedge}(n+1)\text{ -}2}$.

Proof. $i_n\, c_n=e^{2^{\wedge}n\text{-}1}.c_0.c_0\, e^{2^{\wedge}n\text{-}1}=e^{2(2^{\wedge}n\text{-}1)}=e^{2^{\wedge}(n+1)\text{ -}2}$.

Theorem 4.4. If A is a pre-open set , then $\bar{A}=c_1(A)=c_2(A)=c_3(A)=........=c_n(A)=.......$

Proof. If A is pre-open then $A\subset ic(A)=e^2(A)$.

Hence, $c(A)\subset ce^2(A)=c_0e.e^2(A)=c_0e^3(A)=c_2(A)$

Thus, $c_1(A)\subseteq c_2(A)$.

But, we know that $c_2(A)\subseteq c_1(A)$.

Hence, $c_1(A)=c_2(A)$ i.e., $c_1=c_2$. Thus $c_2(A)=\bar{A}$

Then, $c_3=c_2ic_2=c_1ic_1=c_2$. Hence $c_3(A)=c_2(A)=\bar{A}$ and so on.

Hence, $\bar{A}=c_1(A)=c_2(A)=....=c_n(A)=....$

Corollary 4.2. In case of an open set, since it is pre-open also,

$$c_1=c_2=c_3=...c_n=...$$

Theorem 4.5. Let a be a β-open or semi pre-open set, then

$$\bar{A}=c_1(A)=c_2(A)=c_3(A)=...=c_n(A)=...$$

Proof. In case of β-open set, $A\subseteq c_2(A)$.

Hence, $c_1(A)\subseteq c_1c_2(A)=c_2(A)\subseteq c_1(A)$.

Therefore, $c_1(A)=c_2(A)$. Thus $c_1=c_2$, and as above we have,
$\bar{A}=c_1(A)=c_2(A)=c_3(A)=.....=c_n(A)=....$

Now, we express conditions of continuity in terms of interior

Let $f: X\to Y$, where X and Y are topological spaces, and f is a continuous mapping of X into Y. If $G \subseteq Y$ then iG is open in Y. So $f^{-1}(iG)$ is open in X i.e. $if^{-1}(i(G))=f^{-1}(i(G))$.

Since, G is any set, so we have $if^{-1}i = f^{-1}i$.

Conversely, let $if^{-1}i= f^{-1}i$.

Let, $G\subseteq Y$ such that G is open. Hence $iG=G$.

So, $f^{-1}(G)= f^{-1}(iG)= (f^{-1} i)(G)= i f^{-1}i(G)=i f^{-1}(G)$.

Hence, $f^{1}(G)$ is open in X and f is continuous.

Thus, continuity of f is equivalent to the condition that $i f^{-1} i = f^{-1} i$.

Next we consider the case when f is an open mapping. Let $G \subseteq X$, then iG is open.

Hence, $f(iG)$ is open in Y i.e. $if(iG)=f(iG)$.

So, $ifi=fi$.

Conversely, let $ifi=fi$ and G be open in X i.e. $iG=G$.

Hence, $ifi(G)=fi(G) \Longrightarrow if(G)=f(G)$ i.e. $f(G)$ is open

So, f is *open* $\Leftrightarrow$ $ifi = fi$, and f is continuous $\Leftrightarrow$ $i f^{-1} i = f^{-1} i$.

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