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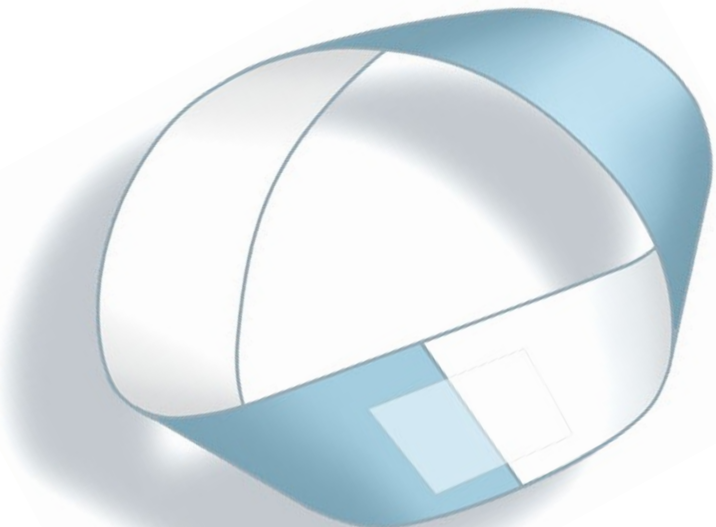

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On A New Difference Sequence Set

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Abstract. The idea of difference sequences first was introduced by Kizmaz [10] in 1981 and then this subject has been studied and generalized by various mathematicians. In this study, we have introduced the difference sequence set

$$m(\phi, p)(\Delta_v^r) = \{x = (x_k) : \Delta_v^r x \in m(\phi, p)\}$$

and noted some topological and some other results with this set and some other related sets. The results obtained in this study generalizes some known results.

Key words: Difference sequence, Solid space, BK-space.

AMS Subject Classification: 40C05, 46A45.

1. Introduction

In this study w will denote the linear space of all complex sequences and ℓ_∞, c and c_o denote the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$||x||_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, 3, \dots\}$.

Let $x = (x_n) \in w$ and $S(x)$ be the set of all sequences $x = \left(x_{\pi(n)}\right)$, i.e $S(x) = \{(x_{\pi(n)}) : \pi$ is a permutation on $\mathbb{N}\}$, where $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation.

A sequence space E is said to be symmetric if $S(x) \subset E$ for every $x \in E$.

A sequence space E is said to be solid if $(y_n) \in E$, whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$.

A sequence space E is said to be sequence algebra if $x.y \in E$, whenever $x, y \in E$.

A sequence space E with a linear topology is called a K - space provided each of the maps $p_i : E \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for each $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A *K*-space *E* is called an *FK*-space provided *E* is a complete linear metric space. An *FK*-space whose topology is normable is called a *BK*-space.

The notion of difference sequence spaces was introduced by Kizmaz [10] and it was generalized by Et and Çolak [4] as follows:

Let m be a non-negative integer, then

$$\Delta^m(X) = \{x = (x_k): (\Delta^m x_k) \in X\}$$

where $\Delta^m x_k = \Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}$ for all $k \in \mathbb{N}$. Then Et and Esi [5] generalized the above sequence spaces to the following sequence spaces.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers, then

$$\Delta_v^m(X) = \{x = (x_k): (\Delta_v^m x_k) \in X\}$$

where $\Delta_v^m x_k = \Delta_v^{m-1}x_k - \Delta_v^{m-1}x_{k+1} = \sum_{i=0}^{m}(-1)^i\binom{m}{i} v_{k+i}x_{k+i}$ for all $k \in \mathbb{N}$.

The sequence spaces $\Delta_v^m(X)$ are Banach spaces normed by

$$||x||_\Delta = \sum_{i=1}^{m} |v_i x_i| + ||\Delta_v^m x||_\infty$$

for $X = \ell_\infty, c$ and c_0. Recently difference sequence spaces have been studied actively in ([1], [2], [6], [7], [8], [9], [11]) and by many others.

We note that $\Delta^m(X)$ and $\Delta_v^m(X)$ overlap but neither one contains the other. For example if we choose $x = (k^m)$ and $v = (k)$, then $x \in \Delta^m(X)$, but $x \notin \Delta_v^m(X)$, conversely if we choose $x = (k^{m+1})$ and $v = (k^{-1})$, then $x \in \Delta_v^m(X)$, but $x \notin \Delta^m(X)$.

2. Main Results

In this section we introduce a new class of sequences and establish some topological properties and some inclusion relations. The obtained results are more general than those of Çolak and Et [3], Sargent [12], and Tripathy and Sen [13].

We use φ_s to denote the class of all subsets of $\mathbb{N}$, those do not contain more than s elements. Let (ϕ_n) be a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n + 1)\, \phi_n$ for all $n \in \mathbb{N}$. The class of all sequences (ϕ_n) is denoted by Φ.

The sequence spaces $m(\phi)$ and $m(\phi, p)$ were introduced by Sargent [12], and Tripathy and Sen [13] as follows for $1 \le p < \infty$, respectively

$$m(\phi) = \{x\in w: ||x||_{m(\phi)} = \sup_{s\ge 1,\sigma\in\varphi_s} \frac{1}{\phi_s}\sum_{n\in\sigma}|x_n| < \infty\}$$

$$m(\phi, p) = \left\{x\in w: ||x||_{m(\phi,p)} = \sup_{s\ge 1,\sigma\in\varphi_s} \frac{1}{\phi_s}\sum_{n\in\sigma}|x_n|^p < \infty \right\}.$$

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers, r be a positive integer and $1\le p < \infty$. Now we define the sequence space $m(\phi, p)$ (Δ_v^r) as

$$m(\phi, p)\ (\Delta_v^r) = \{x\in w: \sup_{s\ge 1,\sigma\in\varphi_s} \frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta_v^r x_n|^p < \infty\}.$$

From the definition it is clear that $m(\phi, p)$ $(\Delta^0) = m(\phi, p)$ and $m(\phi, 1)$ $(\Delta^0) = m(\phi)$, in the special case $v = (1)$. In case of $v = (1)$, we shall write $m(\phi, p)$ (Δ^r) instead of $m(\phi, p)$ (Δ_v^r) and in case of $p = 1$, we shall write $m(\phi)$ (Δ_v^r) instead of $m(\phi, p)$ (Δ_v^r). The sequence space $m(\phi,p)(\Delta_v^r)$ contains some unbounded sequences for $r\ge1$, for example the sequence $x = (k^r)$ is an element of $m(\phi, p)$ (Δ_v^r) for $v = (1)$, but it is not an element of ℓ_∞.

It can easily be shown that $m(\phi, p)$ (Δ_v^r) is a linear space.

Theorem 1. The space $m(\phi, p)$ (Δ_v^r) is a Banach space with the norm

$$||x||_{\Delta_v^r} = \sum_{i=1}^{r} |v_i x_i| + \sup_{s\ge 1,\sigma\in\varphi_s} \frac{1}{\phi_s} \left(\sum_{n\in\sigma}|\Delta_v^r x_n|^p\right)^{\frac{1}{p}} < \infty,\ 1\le p <\infty. \quad (1)$$

where $v_k \neq 0$ for each $k\in \mathbb{N}$.

Proof. Let (x^s) be a Cauchy sequence in $m(\phi, p)$ (Δ_v^r) where $x^s = (x_i^s)_{i=1}^{\infty}$. Then we have

$$||x^s - x^t||_{\Delta_v^r} \to 0, \text{ as s, t} \to \infty.$$

Then there exists a positive integer n_0 such that $||x^s - x^t||_{\Delta_v^r} < \varepsilon$ for all s, t > n_0. Hence (x_i^s) (for $i \le r$) and $(\Delta_v^r(x_i^s))$ for all $s\in \mathbb{N}$, are Cauchy sequences in $\mathbb{C}$. Since $\mathbb{C}$ is complete, these sequences are convergent in $\mathbb{C}$. Suppose that $x_i^s \to x_i$ (for $i \le r$) and $\Delta_v^r(x_i^s)\to y_i$, for each $i\in \mathbb{N}$ as s$\to\infty$. Then we can find a sequence (x_k) such that $y_k = \Delta_v^r x_k$ for each $k\in \mathbb{N}$. These x_k's can be written as

$$x_k = v_k^{-1}\sum_{i=1}^{k-r}(-1)^r\binom{k-i-1}{r-1} y_i = v_k^{-1}\sum_{i=1}^{k}(-1)^r\binom{k-r-i-1}{r-1} y_{i-r},$$

for sufficiently large k, for instance $k > r$, where $y_{1-r} = y_{2-r} = \ldots = y_0 = 0$. Thus $(\Delta_v^r(x_k^s)) = ((\Delta_v^r(x_k^1)), (\Delta_v^r(x_k^2), \ldots)$ converges to $\Delta_v^r x_k$, for each $k \in \mathbb{N}$ in $\mathbb{C}$. Hence $||x^s - x||_{\Delta_v^r} \to 0$ as s $\to \infty$. Since $(x^s - x)$, $(x^s) \in m(\phi, p)(\Delta_v^r)$ and the space $m(\phi, p)(\Delta_v^r)$ is linear we have $x = x^s - (x^s - x) \in m(\phi, p)(\Delta_v^r)$. Hence $m(\phi, p)(\Delta_v^r)$ is complete.

It can be shown that the space $m(\phi, p)(\Delta_v^r)$ is *K*-space and so *BK*-space.

Theorem 2. [13]

i) The space $m(\phi, p)$ is a symmetric space,

ii) The space $m(\phi, p)$ is a normal space.

Theorem 3. The sequence space $m(\phi, p)(\Delta_v^r)$ is not sequence algebra, is not solid and is not symmetric, for $r, p \geq 1$.

Proof. For the proof of the Theorem, consider the following examples:

Example 1. It is obvious that, if $x = (k^{r-2})$, $y = (k^{r-2})$, $v = (1)$, then $x, y \in m(\phi, p)(\Delta_v^r)$, but $x.y \notin m(\phi, p)(\Delta_v^r)$. Hence $m(\phi, p)(\Delta_v^r)$ is not a sequence algebra.

Example 2. It is obvious that, if $x = (k^{r-1})$, $v = (1)$ then $x \in m(\phi, p)(\Delta_v^r)$, but $(\alpha_k x_k) \notin (\phi, p)(\Delta_v^r)$ for $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $m(\phi, p)(\Delta_v^r)$ is not solid.

Example 3. Let us consider the sequence $x = (k^{r-1})$, $v = (1)$, then $x \in m(\phi, p)(\Delta_v^r)$. Let (y_k) be a rearrangement of (x_k) which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots\}.$$

Then $y \notin m(\phi, p)(\Delta_v^r)$. Hence $m(\phi, p)(\Delta_v^r)$ is not symmetric.

Theorem 4. $m(\phi)(\Delta_v^r) \subset m(\phi, p)(\Delta_v^r)$ for each $p \geq 1$.

Proof. Let $x \in m(\phi)(\Delta_v^r)$. Then there is a positive number K such that, for each fixed s

$$\sum_{n \in \sigma} |\Delta_v^r x_n| \leq K\phi_s, \quad \sigma \in \varphi_s.$$

Hence $(\sum_{n \in \sigma} |\Delta_v^r x_n|^p)^{\frac{1}{p}} < K\phi_s$, for each $p \geq 1$ and $\sigma \in \varphi_s$. Thus $x \in m(\phi, p)(\Delta_v^r)$.

Theorem 5. $m(\phi, p)(\Delta_v^r) \subset m(\psi, p)(\Delta_v^r)$ iff $\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty$.

Proof. Suppose that $\sup_{s\geq 1}\frac{\phi_s}{\psi_s} < \infty$. Then there exists a positive number K such that $\phi_s \leq K\psi_s$ for all $s \in \mathbb{N}$. If $x \in m(\phi, p)(\Delta_v^r)$, then

$$\sup_{s\geq 1, \sigma\in\varphi_s}\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta_v^r x_n|^p < \infty.$$

Now we have

$$\sup_{s\geq 1, \sigma\in\varphi_s}\frac{1}{\psi_s}\sum_{n\in\sigma}|\Delta_v^r x_n|^p < \sup_{s\geq 1}\frac{\phi_s}{\psi_s}\sup_{s\geq 1, \sigma\in\varphi_s}\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta_v^r x_n|^p < \infty.$$

Hence $x \in \mathrm{m}(\psi, p)(\Delta_v^r)$.

Conversely let $m(\phi, p)(\Delta_v^r) \subset m(\psi, p)(\Delta_v^r)$ and suppose that $\sup_{s\geq 1}\frac{\phi_s}{\psi_s} = \infty$. Then there exists a sequence (s_i) of natural numbers such that $\lim_i \frac{\phi_{s_i}}{\varphi_{s_i}} = \infty$. Then for $x \in m(\phi, p)(\Delta_v^r)$ we have

$$\sup_{s\geq 1, \sigma\in\varphi_s}\frac{1}{\psi_s}\sum_{n\in\sigma}|\Delta_v^r x_n|^p \geq \sup_{i\geq 1}\frac{\phi_{s_i}}{\psi_{s_i}} . \sup_{s\geq 1, \sigma\in\varphi_{s_i}}\frac{1}{\phi_s}\sum_{n\in\sigma}|\Delta_v^r x_n|^p = \infty.$$

Therefore $x \notin m(\psi, p)(\Delta_v^r)$. This contradicts $m(\phi, p)(\Delta_v^r) \subset m(\psi, p)(\Delta_v^r)$. Hence $\sup\frac{\phi_s}{\psi_s} < \infty$.

From Theorem 5, we get the following result.

Corollary 1. $m(\phi, p)(\Delta_v^r) = m(\psi, p)(\Delta_v^r)$ iff $0 < \inf_{s\geq 1}\frac{\phi_s}{\psi_s} \leq \sup_{s\geq 1}\frac{\phi_s}{\psi_s} < \infty$.

Theorem 6. $m(\phi, p)(\Delta_v^{r-1}) \subset m(\phi, p)(\Delta_v^r)$ and the inclusion is strict.

Proof. Let $x \in m(\phi, p)(\Delta_v^{r-1})$. It is well known that, the inequality $| a + b |^p \leq 2^p (| a |^p + | b |^p)$ is satisfied for $a,b \in \mathbb{C}$, if $1\leq p < \infty$. Hence for $1\leq p < \infty$, we have

$$\frac{1}{\phi_s}(\sum_{n\in\sigma}|\Delta_v^r x_n|^p) \leq 2^p\frac{1}{\phi_s}(\sum_{n\in\sigma}|\Delta_v^{r-1} x_n|^p + \sum_{n\in\sigma}|\Delta_v^{r-1} x_{n+1}|^p).$$

Hence $x \in m(\phi, p)(\Delta_v^r)$.

To show the inclusion is strict consider the following example.

Example 4. Let $\phi_n = 1$ for all $n \in \mathbb{N}$, $v = (1)$ and $x = (k^{r-1})$, then $x \in \ell_p(\Delta_v^r) \setminus \ell_p(\Delta_v^{r-1})$.

Theorem 7. For any ϕ and $1\leq p < \infty$ we have $\ell_p(\Delta_v^r) \subset m(\phi, p)(\Delta_v^r) \subset \ell_\infty(\Delta_v^r)$.

Proof. Since $m(\phi, p)$ $(\Delta_v^r) \subset \ell_p(\Delta_v^r)$ in case ϕ_n=1 for all n$\in \mathbb{N}$, then $\ell_p(\Delta_v^r) \subset m(\phi, p)(\Delta_v^r)$.

Now assume that $x \in m(\phi, p)(\Delta_v^r)$. Then we have

$$\sup_{s\geq 1,\sigma\in\varphi_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta_v^r x_n|^p \quad < \infty$$

and so $|\Delta_v^r x_n| < K\phi_1$ for all $n\in \mathbb{N}$ and for some positive number K. Thus $x\in \ell_\infty(\Delta_v^r)$.

Theorem 8. If $0 < p < q$, then $m(\phi, p)$ $(\Delta_v^r) \subset m(\phi, q)$ (Δ_v^r).

Proof. Proof follows from the following inequality

$$\left(\sum_{k=1}^n |\Delta_v^r x_k|^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^n |\Delta_v^r x_k|^p\right)^{\frac{1}{p}}, \ (0 < p < q).$$

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λ-STATISTICAL CONVERGENCE OF INTERVAL NUMBERS OF ORDER α

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ABSTRACT. In this paper we generalized the concept of λ-statistical convergence of order α given by Çolak and Bektaş [2] to interval numbers. We give some new definitions about them and some inclusion relations between the λ-statistical convergence of interval numbers of order α and strongly λ-summable of interval numbers of order α.

Key words: Sequence space, interval numbers, sequence of interval numbers, λ-statistical convergence, strongly λ-summable.

AMS Subject Classification: 40C05, 40J05, 46A45.

1. Introduction

The idea of statistical convergence for ordinary sequences was introduced by Fast [8] in 1951. Schoenberg [16] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

Interval arithmetic was first suggested by Dwyer [3] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [12] in 1959 and Moore and Yang [14] 1962. Furthermore, Moore and others [3], [4], [9] and [13] have developed applications to differential equations.

Chiao in [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [17] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently, Esi in [7] and [6] defined and studied λ-statistical and lacunary statistical convergence of interval numbers, respectively. A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analytical properties. We denote the set of all real valued closed intervals by IR. Any elements of IR is a closed interval and denoted by $\overline{A}$. That is $\overline{A} = \{x \in \mathbb{R}: a \leq x \leq b\}$. An interval number $\overline{A}$ is a closed subset of real numbers [1]. Let x_l and x_r be first and last points of $\overline{A}$ interval number, respectively. For $\overline{A}, \overline{B} \in \text{I}R$, we have $\overline{A} = \overline{B} \Leftrightarrow x_{1_l}$=$x_{2_l}$, x_{1_r}=x_{2_r}. $\overline{A} + \overline{B} = \{x \in \mathbb{R}: x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{A} = \{x \in \mathbb{R}: \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{A} = \{x \in \mathbb{R}: \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\overline{A}.\overline{B} = \{x \in \mathbb{R}: \min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x$$
$$\leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}\}.$$

The set of all interval numbers IR is a complete metric space defined by

$$d(\overline{A}, \overline{B}) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}10.$$

In the special case $\overline{A} = [a, a]$ and $\overline{B} = [b, b]$, we obtain usual metric of R.

Let us define transformation $f: \mathbb{N} \to R$ by $k \to f(k) = \overline{A}, \overline{A} = (\overline{A}_k)$. Then $\overline{A} = (\overline{A}_k)$ is called sequence of interval numbers. The $\overline{A}_k$ is called k^{th} term of sequence $\overline{A} = (\overline{A}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [11].

Now we give the definition of convergence of interval numbers:

Definition 1 *([1])* A sequence $\overline{A} = (\overline{A}_k)$ of interval numbers is said to be convergent to the interval number $\overline{A}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{A}_k, \overline{A}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $lim_k\overline{A}_k = \overline{A}_o$.

Thus, $\lim_k\overline{A}_k = \overline{A}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

2 Main Results

Definition 2 *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. Let α be any real number such that $0 < \alpha \leq 1$. The sequence $\overline{A} = (\overline{A}_k)$ of interval numbers is said to be strongly λ-summable of order α if there is an interval number $\overline{A}_o$ such that*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} d(\overline{A}_k, \overline{A}_o) = 0.$$

In which case we say that the sequence $\overline{A} = (\overline{A}_k)$ of interval numbers strongly λ-summable to interval numbers $\overline{A}_o$ of order α. The set of all strongly λ-summable to interval sequences of order α will be denoted by $[\overline{w}^\alpha]_\lambda$. If $\lambda_n = n$, then strongly λ-summable of order α reduces to strongly Cesaro summable of order α defined as follows:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k\in I_n} d(\overline{A}_k, \overline{A}_o) = 0.$$

The set of all strongly Cesaro summable to interval sequences of order α will be denoted by $[\overline{w}^\alpha]$. If $\lambda_n = n$ and $\alpha = 1$, then strongly λ-summable reduces to strongly Cesaro summable of interval numbers which was defined by Esi [5].

Definition 3 Let the sequence $\lambda = (\lambda_n)$ of real numbers be defined as Definition 2 and $\alpha \in (0,1]$ be given. A sequence $\overline{A} = (\overline{A}_k)$ of interval numbers is said to be statistically λ-convergent to interval number $\overline{A}_o$ of order α if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left|\left\{k \in I_n : d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\right\}\right| = 0.$$

In this case we write $\overline{s}_\lambda^\alpha - \lim \overline{A}_k = \overline{A}_o$. The set of all statistically λ-convergent interval sequences of order α will be denoted by $\overline{s}_\lambda^\alpha$. We will write $\overline{s}_{\lambda,\overline{0}}^\alpha$ to denote the set of statistically λ-null sequences of interval numbers of order α. If $\lambda_n = n$, then statistically λ-convergence of order α reduces to statistically convergence of order α as follows:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \in I_n : d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\right\}\right| = 0.$$

In this case we write $\overline{s}^{\alpha}-\lim\overline{A}_k=\overline{A}_o$. The set of all statistically convergent interval sequences of order α will be denoted by $\overline{s}^{\alpha}$. If $\alpha=1$, then statistically λ-convergence of order α reduces to statistically λ-convergence which was defined by Esi [5].

It is evident that $\overline{s}^{\alpha}_{\lambda,\overline{0}}\subset\overline{s}^{\alpha}_{\lambda}$ for each $\alpha\in(0,1]$. The statistical λ-convergence of interval numbers of order $\alpha=1$ is same with the statistical λ-convergence, that is $\overline{s}^{\alpha}_{\lambda,\overline{0}}=\overline{s}^{\alpha}_{\lambda}$ for $\alpha=1$. The statistical λ-convergence of interval numbers of order α is well-defined for $0<\alpha\leq 1$. But it is not well-defined for $\alpha>1$ in general.

Example 1 *Let* $\overline{A}=(\overline{A}_k)$ *interval sequence be defined as follows*

$$\overline{A}_k=\begin{cases}\overline{1} & , \quad k=2n, n=1,2,\dots \\ \overline{0} & , \quad \text{otherwise}\end{cases}.$$

Then both

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n: d\left(\overline{A}_k,\overline{1}\right)\geq\varepsilon\right\}\right|\leq\lim_{n\to\infty}\frac{[\lambda_n]+1}{2\lambda_n^{\alpha}}=0$$

and

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n: d\left(\overline{A}_k,\overline{0}\right)\geq\varepsilon\right\}\right|\leq\lim_{n\to\infty}\frac{[\lambda_n]+1}{2\lambda_n^{\alpha}}=0$$

for $\alpha>1$, such that $\overline{A}=(\overline{A}_k)$ statistically λ-convergent of order α both to interval number $\overline{1}$ and $\overline{0}$, but this is impossible.

Theorem 1 *Let* $0<\alpha\leq 1$ *and* $\overline{A}=(\overline{A}_k)$ *and* $\overline{B}=(\overline{B}_k)$ *be sequences of interval numbers.*

(i) If $\overline{s}^{\alpha}_{\lambda}-\lim\overline{A}_k=\overline{A}_o$ and $b\in\mathbb{R}$, then $\overline{s}^{\alpha}_{\lambda}-\lim b\overline{A}_k=b\overline{A}_o$.

(ii) If $\overline{s}^{\alpha}_{\lambda}-\lim\overline{A}_k=A_a$ and $\overline{s}^{\alpha}_{\lambda}-\lim\overline{B}_k=\overline{B}_o$, then $\overline{s}^{\alpha}_{\lambda}-\lim\left(\overline{A}_k+\overline{B}_k\right)=\overline{A}_o+\overline{B}_o$.

Proof. **(i)** Let $b\in\mathbb{R}$. We have $d\left(b\overline{A}_k,b\overline{A}_o\right)=|b|d\left(\overline{A}_k,\overline{A}_o\right)$. For a given $\varepsilon>0$

$$\frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(b\overline{A}_k, b\overline{A}_o\right) \geq \varepsilon\right\}\right| \leq \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(\overline{A}_k, \overline{A}_o\right) \geq \frac{\varepsilon}{|b|}\right\}\right|.$$

Hence $\overline{s}_\lambda^\alpha - \lim b\overline{A}_k = b\overline{A}_o$.

(ii) Suppose that $\overline{s}_\lambda^\alpha - \lim\overline{A}_k = \overline{A}_o$ and $\overline{s}_\lambda^\alpha - \lim\overline{B}_k = \overline{B}_o$. We have

$$d\left(\overline{A}_k + \overline{B}_k, \overline{A}_o + \overline{B}_o\right) \leq d\left(\overline{A}_k, \overline{A}_o\right) + d\left(\overline{B}_k, \overline{B}_o\right).$$

Therefore given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(\overline{A}_k + \overline{B}_k, \overline{A}_o + \overline{B}_o\right) \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(\overline{A}_k, \overline{A}_o\right) + d\left(\overline{B}_k, \overline{B}_o\right) \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(\overline{A}_k, \overline{A}_o\right) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(\overline{B}_k, \overline{B}_o\right) \geq \frac{\varepsilon}{2}\right\}\right|.$$

Thus, $\overline{s}_\lambda^\alpha - \lim\left(\overline{A}_k + \overline{B}_k\right) = \overline{A}_o + \overline{B}_o$.

Note that $\overline{s}_\lambda^\alpha$ is different from $\overline{s}_\lambda$ defined in Esi [5] in general. If we take $\lambda_n = n^\alpha$ for $0 < \alpha < 1$ then $\overline{s}^\alpha \subset \overline{s}_\lambda$. If $\lambda_n = n$ and $\alpha = 1$ then $\overline{s}^\alpha = \overline{s}_\lambda = \overline{s}$ (see Esi [5]), that is the statistical convergence of interval numbers of order α, statistical convergence and statistical λ-convergence of interval numbers coincide when $\lambda_n = n$ and $\alpha = 1$.

It easy to see that every convergent sequence of interval numbers is statistically convergent of interval numbers of order α, that is $c^i \subset \overline{s}^\alpha$ for each $0 < \alpha \leq 1$, where the space c^i was defined in Şengönül and Eryılmaz [17]. But it follows from the following example that the converse does not hold.

Example 2 *Let the interval number sequence $\overline{A} = \left(\overline{A}_k\right)$ defined by*

$$\overline{A}_k = \begin{cases} \overline{1} & , \quad k = n^3, n = 1,2,\ldots \\ \overline{0} & , \quad \text{otherwise} \end{cases}.$$

Then $\overline{A} = \left(\overline{A}_k\right)$ is statistically convergent of order $\alpha > \frac{1}{3}$, but it is not convergent.

Theorem 2 $0 < \alpha \le \beta \le 1$. *Then* $\overline{s}_\lambda^\alpha \subset \overline{s}_\lambda^\beta$ and the inclusion is strict for some α and β such that $\alpha \le \beta$.

Proof. If $0 < \alpha \le \beta \le 1$, then

$$\frac{1}{\lambda_n^\beta}\left|\left\{k \in I_n: d\left(\overline{A}_k, \overline{A}_o\right) \ge \varepsilon\right\}\right| \le \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n: d\left(\overline{A}_k, \overline{A}_o\right) \ge \varepsilon\right\}\right|$$

for every $\varepsilon > 0$, which gives that $\overline{s}_\lambda^\alpha \subset \overline{s}_\lambda^\beta$. It follows from the following example that the inclusion is strict.

Example 3 Consider the interval sequence $\overline{A} = \left(\overline{A}_k\right)$ defined by

$$\overline{A}_k = \begin{cases} \overline{k} & , \quad k = m^2, m = 1,2, \dots \\ \overline{0} & , \quad \text{otherwise} \end{cases}.$$

Let $\lambda_n = n$ and then for every ε $(0 < \varepsilon \le 1)$

$$\frac{1}{n^\beta}\left|\left\{k \in I_n: d\left(\overline{A}_k, \overline{0}\right) \ge \varepsilon\right\}\right| \le \frac{[\sqrt{n}]}{n^\beta} \to 0 \text{as} n \to \infty \text{for} \frac{1}{2} < \beta \le 1$$

i.e., $\overline{A} = \left(\overline{A}_k\right) \in \overline{s}_\lambda^\beta$ but $\overline{A} = \left(\overline{A}_k\right) \notin \overline{s}_\lambda^\alpha$ for $0 < \alpha \le \frac{1}{2}$.

If we take $\beta = 1$ in Theorem 2 then we obtain the following result.

Corollary 1 If an interval sequence $\overline{A} = \left(\overline{A}_k\right)$ is statistically λ-convergent to interval number $\overline{A}_o$ of order α, then it is statistically λ-convergent to $\overline{A}_o$, *that is* $\overline{s}_\lambda^\alpha \subset \overline{s}_\lambda$ *for each* α $(0 < \alpha \le 1)$ and the inclusion is strict.

The following result is a consequence of Theorem 2.

Corollary 2 ***(i)*** $\overline{s}_\lambda^\beta = \overline{s}_\lambda^\alpha$ *if and only if* $\alpha = \beta$,

(ii) $\overline{s}_\lambda^\beta = \overline{s}_\lambda \Leftrightarrow \alpha = 1$.

Theorem 3 $\overline{s}_\lambda^\alpha \subset \overline{s}$ *for all* λ *and each* α $(0 < \alpha \le 1)$.

Proof. It is easy to see that $\overline{s}_\lambda \subset \overline{s}$ for all λ, since $\frac{\lambda_n}{n}$ is bounded 1 (Mursaleen [15]). From Corollary 1, we have $\overline{s}_\lambda \subset \overline{s}$.

Theorem 4 $\overline{s} \subset \overline{s}_\lambda^\alpha$ *if and only if*

$$\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{n} > 0. \tag{2.1}$$

Proof. For a given $\varepsilon > 0$, we have

$$\{k \leq n: d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\} \supset \{k \in I_n: d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\}.$$

Therefore

$$\frac{1}{n}\left|\{k \leq n: d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\}\right| \geq \frac{1}{n}\left|\{k \in I_n: d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\}\right|$$

$$= \frac{\lambda_n^\alpha}{n}\frac{1}{\lambda_n^\alpha}\left|\{k \in I_n: d(\overline{A}_k, \overline{A}_o) \geq \varepsilon\}\right|.$$

Taking limit as $n \to \infty$ and using (2.1), we get $\overline{s} - \lim \overline{A}_k = \overline{A}_o \Rightarrow \overline{s}_\lambda^\alpha - \lim \overline{A}_k = \overline{A}_o$.

Conversely, suppose that $\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{n} > 0$. We can choose a subsequence $(n(j))_{j=1}^\infty$ such that

$$\frac{\lambda_{n(j)}^\alpha}{n(j)} < \frac{1}{j}.$$

Define an interval number sequence $\overline{A} = (\overline{A}_k)$ by

$$\overline{A}_i = \begin{cases} \overline{1} & , \quad i \in I_{n(j)};\ \ j = 1,2,3, \dots \\ \overline{0} & , \quad \text{otherwise} \end{cases}.$$

Then $\overline{A} = (\overline{A}_k) \in \overline{s}$ but on the other hand $\overline{A} = (\overline{A}_k) \notin \overline{s}_\lambda$. From Corollary 1, since $\overline{s}_\lambda^\alpha \subset \overline{s}_\lambda$, we have $\overline{A} = (\overline{A}_k) \notin \overline{s}_\lambda^\alpha$. Hence (2.1) is necessary.

Theorem 5 *Let* $0 < \alpha \leq \beta \leq 1$. *Then* $[\overline{w}^\alpha]_\lambda \subset [\overline{w}^\beta]_\lambda$ *and the inclusion is strict for some* α *and* β *such that* $\alpha < \beta$.

Proof. Let $\overline{A} = (\overline{A}_k) \in [\overline{w}^{\alpha}]_{\lambda}$. Then given α and β such that $0 < \alpha \leq \beta \leq 1$, we may write

$$\frac{1}{\lambda_n^{\beta}} \sum_{k \in I_n} d(\overline{A}_k, \overline{A}_o) \leq \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} d(\overline{A}_k, \overline{A}_o)$$

which gives $[\overline{w}^{\alpha}]_{\lambda} \subset [\overline{w}^{\beta}]_{\lambda}$. To show that the inclusion is strict consider the following example.

Example 4 Consider the interval sequence $\overline{A} = (\overline{A}_k)$ defined by

$$\overline{A}_k = \begin{cases} \overline{1} & , \quad k = m^2, m = 1,2, \dots \\ \overline{0} & , \quad \text{otherwise} \end{cases}.$$

Let $\lambda_n = n$ for all $n \in \mathbb{N}$. Then it is easy to see that

$$\frac{1}{n^{\beta}} \sum_{k \in I_n} d(\overline{A}_k, \overline{0}) \leq \frac{\sqrt{n}}{n^{\beta}} = \frac{1}{n^{\beta - \frac{1}{2}}}.$$

Since $\frac{1}{n^{\beta-\frac{1}{2}}} \to 0$ as $n \to \infty$, then $[\overline{w}^{\beta}] - \lim \overline{A}_k = \overline{A}_o = \overline{0}$, i.e., $\overline{A} = (\overline{A}_k) \in [\overline{w}^{\beta}]$ for $\frac{1}{2} < \beta \leq 1$. But since

$$\frac{\sqrt{n}-1}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \sum_{k \in I_n} d(\overline{A}_k, \overline{0})$$

and $\frac{\sqrt{r}-1}{n^{\alpha}} \to \infty$ as $n \to \infty$, then $\overline{A} = (\overline{A}_k) \notin [\overline{w}^{\alpha}]$ for $0 < \alpha < \frac{1}{2}$. This completes the proof.

The following result is a consequence of Theorem 5.

Corollary 3 *Let* $0 < \alpha \leq \beta \leq 1$. *Then*

(i) $[\overline{w}^{\alpha}]_{\lambda} = [\overline{w}^{\beta}]_{\lambda}$ if and only if $\alpha = \beta$,

(ii) $[\overline{w}^{\alpha}]_{\lambda} \subset [\overline{w}]_{\lambda}$ for each $\alpha \in (0,1]$.

Theorem 6 *Let* $0 < \alpha \leq \beta \leq 1$. *Then* $[\overline{w}^{\alpha}]_{\lambda} \subset \overline{s}_{\lambda}^{\beta}$.

Proof. Let $\overline{A} = (\overline{A}_k) \in [\overline{w}^{\alpha}]_{\lambda}$ and $\varepsilon > 0$, we have

$$\frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} d(\overline{A}_k, \overline{A}_o) = \frac{1}{\lambda_n^{\alpha}} \sum_{\substack{k \in I_n \\ d(\overline{A}_k, \overline{A}_o) \geq \varepsilon}} d(\overline{A}_k, \overline{A}_o) + \frac{1}{\lambda_n^{\alpha}} \sum_{\substack{k \in I_n \\ d(\overline{A}_k, \overline{A}_o) < \varepsilon}} d(\overline{A}_k, \overline{A}_o)$$

$$\geq \frac{1}{\lambda_n^{\alpha}} \sum_{\substack{k \in I_n \\ d(\overline{A}_k, \overline{A}_o) \geq \varepsilon}} d(\overline{A}_k, \overline{A}_o)$$

$$= \frac{1}{\lambda_n^{\alpha}} \left| \{ k \in I_n : d(\overline{A}_k, \overline{A}_o) \geq \varepsilon \} \right| \varepsilon$$

$$\geq \frac{1}{\lambda_n^{\beta}} \left| \{ k \in I_n : d(\overline{A}_k, \overline{A}_o) \geq \varepsilon \} \right| \varepsilon.$$

So, it follows that $\overline{A} = (\overline{A}_k) \in \overline{s}_{\lambda}^{\beta}$. This completes the proof.

If we take $\alpha = \beta$ in Theorem 6, we obtain the following result.

Corollary 4 *Let* $\alpha \in (0,1]$. *Then* $[\overline{w}^{\alpha}]_{\lambda} \subset \overline{s}_{\lambda}^{\alpha}$.

Corollary 5 *Let* $\alpha \in (0,1]$. *Then* $[\overline{w}^{\alpha}]_{\lambda} \subset \overline{s}_{\lambda}$ *and the inclusion is strict for* $\alpha \in (0,1)$.

Proof. It is consequence of Corollary 1 and Corollary 4.

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Three dimensional unsteady visco-elastic flow with heat and mass transfer and suction along a vertical porous plate

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Abstract: A three-dimensional unsteady flow of visco-elastic fluid along a vertical porous plate with variable suction has been investigated. The effects of heat and mass transfer are taken into account. The resulting equations have been solved using multi-parameter perturbation scheme. The expressions of velocity components, temperature and concentration are derived. The influence of shearing stress, rate of heat transfer in terms of Nusselt number and the rate of mass transfer in terms of Sherwood number at the plate are discussed in detail and shown graphically. The variation of flow behaviour with the exertion of visco-elastic parameters in presence of other flow parameters are stimulating in this case.

Key-words: Visco-elastic, porous, suction, heat transfer, mass transfer.

1. Introduction

Three-dimensional fluid flows of non-Newtonian fluid have considerable significance in view of its immense practical applications in the fields of petroleum engineering, geophysics, chemical process industries and also in biological sciences. This kind of fluid flows are mainly used in plastic manufacturing, food processing, performance of lubricants, polymer processing, ice flows, biological fluid flows, dough floor, shampoos, paints etc. Furthermore, all the non-Newtonian fluids in nature cannot be predicted by single constitutive equation. The fluid flows of non-Newtonian fluids in presence of heat and mass transfer have special importance because of practical engineering applications like food processing and crude oil recovery. Some of the most recent works in this direction includes, Ashraf *et al*. [2017], they have studied the three-dimensional mixed convection flow of visco-elastic fluid in presence of chemical reaction and heat source. Hayat *et al.* [2010, 2014] have investigated the MHD three dimensional fluid flow of visco-elastic fluid with thermal radiation and variable conductivity. Three-dimensional flow with heat transfer of a visco-elastic fluid over a stretching surface in presence of magnetic field has been studied by Seshadri *et al.* [2016]. Choudhury *et al.* [2013] has investigated the visco-elastic effects on the three-dimensional hydrodynamic flow past a vertical porous plate. In this field, the works of Lie *et.al*. [2013], Ribeiro *et.al*. [2014], Singh [1993], Rajotia and Jat [2015] etc are noteworthy .The objective of the present work is to analyse the effects of heat and mass transfer past a vertical porous plate of a three dimensional visco-elastic fluid in presence of variable suction

2. Mathematical Formulation.

An unsteady visco-elastic fluid model past a semi-infinite porous vertical plate inpresence of porous medium is considered with heat and mass transfer. The x′-axis is considered along the vertical plate which is the direction of the flow, y′-axis is perpendicular to the plate and z′-axis is normal to x′y′-plane. The plate is subjected to periodic suction velocity distribution of the form

$$v' = V_0\left[1 + \in \cos\left(\frac{\pi u_\infty z'}{\eta_0} - ct'\right)\right] \tag{2.1}$$

where $\in (<<1)$ is the amplitude of suction velocity. Considering u', v', w' as velocity components in the directions x', y' and z' axes respectively, the basic equations of the flow are as follows:–

Equation of continuity: $\frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0$ (2.2)

Equation of motion:

$$\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'} = \frac{\eta_0}{\rho}\left(\frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right) - \frac{K_0}{\rho}\left(\frac{\partial^3 u'}{\partial t'\partial y'^2} + \frac{\partial^3 u'}{\partial t'\partial z'^2} + v'\frac{\partial^3 u'}{\partial y'^3} + v'\frac{\partial^3 u'}{\partial y'\partial z'^2}\right.$$
$$+ w'\frac{\partial^3 u'}{\partial z'\partial y'^2} + w'\frac{\partial^3 u'}{\partial z'^3} - 3\frac{\partial u'}{\partial y^1}\frac{\partial^2 v'}{\partial y'^2} - 3\frac{\partial u'}{\partial z'}\frac{\partial^2 w'}{\partial z'^2} - \frac{\partial u'}{\partial y'}\frac{\partial^2 w'}{\partial y'z'} - \frac{\partial u'}{\partial z'}\frac{\partial^2 v'}{\partial y'\partial z'}$$
$$- \frac{\partial v'}{\partial z'}\frac{\partial^2 u'}{\partial y'\partial z'} - 2\frac{\partial v'}{\partial y'}\frac{\partial^2 u'}{\partial y'^2} - 2\frac{\partial^2 w}{\partial z'^2}\frac{\partial u'}{\partial z'} - 2\frac{\partial w'}{\partial z'}\frac{\partial^2 w'}{\partial z'^2} - 2\frac{\partial w'}{\partial y'}\frac{\partial^2 u'}{\partial z'\partial y'} - \frac{\partial^2 v'}{\partial z'^2}\frac{\partial u'}{\partial y'}$$
$$\left. - \frac{\partial u'}{\partial z'}\frac{\partial^2 w'}{\partial y'\partial z'} - \frac{\partial^2 w'}{\partial y'^2}\frac{\partial u'}{\partial z'} - \frac{\partial^2 v'}{\partial z'^2}\frac{\partial u'}{\partial z'}\right) + g\beta_r\left(T' - T'_\infty\right) + g\beta_m\left(C' - C'_\infty\right) \tag{2.3}$$

$$\frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'} = -\frac{1}{\rho}\frac{\partial p}{\partial y'} + \frac{\eta_0}{\rho}\left(2\frac{\partial^2 v'}{\partial y'^2} + \frac{\partial^2 v'}{\partial z'^2} + \frac{\partial^2 w'}{\partial y'\partial z'}\right) - \frac{K_0}{\rho}\left(2\frac{\partial^3 v'}{\partial t'\partial y'^2} + 2\frac{\partial^3 v'}{\partial t'\partial z'^2}\right.$$

$$+2v'\frac{\partial^3 v'}{\partial y'^3}+w'\frac{\partial^3 v'}{\partial z'^3}+\frac{\partial^3 v'}{\partial z'^3}+2w'\frac{\partial^3 v'}{\partial z'\partial y'^2}+w'\frac{\partial^3 \omega'}{\partial y'\partial z'^2}-6\frac{\partial v'}{\partial y'}\frac{\partial^2 v'}{\partial y'^2}-5\frac{\partial^2 v'}{\partial z'\partial y'}\frac{\partial v'}{\partial z'}$$

$$-\frac{\partial v'}{\partial z'}\frac{\partial^2 w'}{\partial y'^2}-\frac{\partial w'}{\partial y'}\frac{\partial^2 v'}{\partial y'\partial z'}+\frac{\partial v'}{\partial z'}\frac{\partial^2 v'}{\partial z'^2}+2v\frac{\partial^3 v}{\partial z'^2\partial y'}+2\frac{\partial^3 v'}{\partial y'\partial z'\partial t'}+2v\frac{\partial^3 w'}{\partial z'\partial y'^2}-\frac{\partial v'}{\partial y'}\frac{\partial^2 w'}{\partial y'\partial z'}$$

$$\left.-3\frac{\partial v'}{\partial y'}\frac{\partial^2 w'}{\partial y'\partial z'}-3\frac{\partial^2 v'}{\partial z'^2}\frac{\partial w'}{\partial z'}-\frac{\partial w'}{\partial z'}\frac{\partial^2 w'}{\partial y'\partial z'}-3\frac{\partial v'}{\partial z'}\frac{\partial^2 w'}{\partial z'^2}-\frac{\partial^2 w'}{\partial z'^2}-\frac{\partial^2 v'}{\partial z'^2}\frac{\partial v'}{\partial y'}\right) \tag{2.4}$$

$$\frac{\partial w'}{\partial t'}+v'\frac{\partial w'}{\partial y'}+w'\frac{\partial w'}{\partial z'}=-\frac{1}{\rho}\frac{\partial p'}{\partial z'}+\frac{\eta_0}{\rho}\left(\frac{\partial^2 v'}{\partial z'\partial y'}+\frac{\partial^2 w'}{\partial y'^2}+2\frac{\partial^2 w'}{\partial z'^2}\right)-\frac{K_0}{\rho}$$

$$\left\{\left(\frac{\partial^3 w'}{\partial t'\partial y'\partial z'}+\frac{\partial^3 v'}{\partial t'\partial y'\partial z'}-4\frac{\partial w'}{\partial y'}\frac{\partial^2 w'}{\partial y'\partial z'}-2\frac{\partial w'}{\partial y'}\frac{\partial^2 v'}{\partial z'^2}\right)\right.$$

$$\left(+4\frac{\partial^3 w'}{\partial t'\partial z'^2}+2\frac{\partial^3 w'}{\partial t'\partial y'^2}+\frac{\partial v'}{\partial y'}\frac{\partial^3 w'}{\partial y'\partial z'}+v'\frac{\partial^3 w'}{\partial y'^2\partial z'}+2v'\frac{\partial^3 w'}{\partial y'^3}+4v'\frac{\partial^3 w'}{\partial y'\partial z'^2}+5w'\frac{\partial^3 w'}{\partial z'^3}\right.$$

$$+w'\frac{\partial^3 \omega'}{\partial z'^2\partial y'}+v'\frac{\partial^3 v'}{\partial y'^2\partial z'}+w'\frac{\partial^3 v'}{\partial z'^2\partial y'}+w'\frac{\partial^3 w'}{\partial y'^2\partial z'}-3\frac{\partial^2 v'}{\partial y'^2}\frac{\partial w'}{\partial y'}-3\frac{\partial w'}{\partial z'}\frac{\partial^2 v'}{\partial y'\partial z'}$$

$$\left.\left.+\frac{\partial w'}{\partial y'}\frac{\partial^2 w'}{\partial z'^2}-\frac{\partial^2 w'}{\partial y'^2}\frac{\partial w'}{\partial z'}-3\frac{\partial v'}{\partial z'}\frac{\partial^2 w'}{\partial y'\partial z'}-\frac{\partial^2 v'}{\partial y'\partial z'}\frac{\partial v'}{\partial y'}-\frac{\partial v'}{\partial z'}\frac{\partial^2 v'}{\partial y'^2}-6\frac{\partial w'}{\partial z'}\frac{\partial^2 w'}{\partial z'^2}\right)\right\} \tag{2.5}$$

Equation of energy: $$\frac{\partial T'}{\partial t'}+v'\frac{\partial T'}{\partial y'}+w'\frac{\partial T'}{\partial z'}=\frac{K}{\rho\, C_p}\left(\frac{\partial^2 T'}{\partial y'^2}+\frac{\partial^2 T'}{\partial z'^2}\right) \tag{2.6}$$

Equation of concentration: $$\frac{\partial C'}{\partial t'}+v'\frac{\partial C'}{\partial y'}+w'\frac{\partial C'}{\partial z'}=D\left(\frac{\partial^2 C'}{\partial y'^2}+\frac{\partial^2 C'}{\partial z'^2}\right) \tag{2.7}$$

where ρ is the density, p' is the fluid pressure, g is the acceleration due to gravity, β_r is the co-efficient of thermal expansion, β_m is the co-efficient of mass expansion, K is the co-efficient of heat conduction, C_p is the specific heat at constant pressure, T' is the fluid temperature, C' is the fluid concentration, η_0 is limiting viscosity.

The boundary conditions are:

$$\left.\begin{aligned} &y' = 0; u' = 0, v' = -V_0\left[1 + \in \cos\left(\frac{\pi\, u_\infty z'}{\eta_0} - ct'\right)\right], w' = 0, T' = T_\omega, C' = C_\omega \\ &y' \to \infty; u' = u_\infty, v' = V_0, w' = 0, p' = p_\infty, T' = T_\infty, C' = C_\infty \end{aligned}\right\} \tag{2.8}$$

We introduce following dimensionless quantities

$$\left.\begin{aligned} &y = \frac{u_\infty y'}{\eta_0}, z = \frac{u_\infty z'}{\eta_0}, t = ct', p = \frac{p'}{\rho u_\infty^2}, u = \frac{u'}{u_\infty}, v = \frac{v'}{u_\infty}, w = \frac{\omega'}{u_\infty}, \theta = \frac{T' - T'_\infty}{T_\omega - T_\infty}, \\ &C = \frac{C' - C'_\infty}{C_\omega - C_\infty}, G_m = \frac{g\beta_m (C_w - C_\infty)}{U_\infty^3}, G_r = \frac{g\beta_r (T_w - T_\infty)}{U_\infty^3}, \omega = \frac{C\eta_0}{U_\infty^2}, \\ &P_r = \frac{\rho\eta_0 C_p}{K}, S_c = \frac{\eta_0}{D} \end{aligned}\right\} \tag{2.9}$$

where G_m is the Grashof number for mass transfer, G_r is the Grashof number for heat transfer, ω is the frequency parameter, P_r is the Prandtl number, S_c is the Schmidt number, T_w is the temperature at the plate, T_∞ is the temperature far away from the plate, C_w is the concentration at the plate, C_∞ is the concentration far away from the plate.

Substituting (2.9) in the equations (2.2) to (2.8), we get the following dimensionless equations:

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.10}$$

$$\begin{aligned} \omega\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - K_1\Bigg(&\omega\frac{\partial^3 u}{\partial t\partial y^2} + \omega\frac{\partial^3 u}{\partial t\partial z^2} + v\frac{\partial^3 u}{\partial y^3} + v\frac{\partial^3 u}{\partial z\partial y^2} + w\frac{\partial^3 u}{\partial z^3} \\ &+ v\frac{\partial^3 u}{\partial y^3} + v\frac{\partial^3 u}{\partial y\partial z^2} + w\frac{\partial^3 u}{\partial z\partial y^2} + w\frac{\partial^3 u}{\partial z^3} - 3\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} - 3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} - \frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial y\partial z} \\ &- \frac{\partial u}{\partial z}\frac{\partial^2 v}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z} - 2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} - 2\frac{\partial w}{\partial y}\frac{\partial^2 u}{\partial z\partial y} \\ &- \frac{\partial^2 v}{\partial z^2}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial y\partial z} - \frac{\partial^2 w}{\partial y^2}\frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2}\frac{\partial u}{\partial z}\Bigg) + G_r\theta + G_m C \end{aligned} \tag{2.11}$$

$$\omega\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\left(2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}\right)-k_1\left(2\omega\frac{\partial^3 v}{\partial t\partial y^2}+2\omega\frac{\partial^3 v}{\partial t\partial z^2}+2w\frac{\partial^3 v}{\partial y\partial z\partial t}\right.$$

$$+2v\frac{\partial^3 v}{\partial y^3}+w\frac{\partial^3 v}{\partial z^2}+2w\frac{\partial^3 v}{\partial z\partial y^2}+w\frac{\partial^3 w}{\partial y\partial z^2}+2v\frac{\partial^3 v}{\partial z^2\partial y}+2v\frac{\partial^3 w}{\partial z\partial y^2}-6\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}$$

$$-5\frac{\partial^2 v}{\partial z\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}-3\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y\partial z}-3\frac{\partial^2 v}{\partial z^2}\frac{\partial w}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y\partial z}$$

$$\left.-3\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2}-\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial z^2}-\frac{\partial^2 v}{\partial z^2}\frac{\partial v}{\partial y}\right) \tag{2.12}$$

$$\omega\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\left(\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial z^2}\right)-k_1\left(\omega\frac{\partial^3 w}{\partial t\partial y\partial z}+\omega\frac{\partial^3 v}{\partial t\partial y\partial z}+4w\frac{\partial^3 w}{\partial t\partial z^2}\right.$$

$$+2w\frac{\partial^3 w}{\partial t\partial y^2}+2\frac{\partial v}{\partial y}\frac{\partial^3 w}{\partial y\partial z}+v\frac{\partial^3 w}{\partial y^2\partial z}+2v\frac{\partial^3 w}{\partial y^3}+4v\frac{\partial^3 w}{\partial y\partial z^2}+5w\frac{\partial^3 w}{\partial z^3}+w\frac{\partial^3 w}{\partial z^2\partial y}+v\frac{\partial^3 v}{\partial y^2\partial z}$$

$$+w\frac{\partial^3 w}{\partial y^2\partial z}-3\frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y}-3\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial z^2}-\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial z}-3\frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y\partial z}-\frac{\partial^2 v}{\partial y\partial z}\frac{\partial v}{\partial y}$$

$$\left.-\frac{\partial^2 v}{\partial y\partial z}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y^2}-6\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}-4\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z}-2\frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z^2}\right) \tag{2.13}$$

$$\omega\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial y}+w\frac{\partial\theta}{\partial z}=\frac{1}{P_r}\left(\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right) \tag{2.14}$$

$$\omega\frac{\partial c}{\partial t}+v\frac{\partial c}{\partial y}+w\frac{\partial c}{\partial z}=\frac{1}{S_c}\left(\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 c}{\partial z^2}\right) \tag{2.15}$$

The dimensionless boundary conditions are:

$$\left.\begin{aligned} &y=0;\ u=0,\ w=0,\ \theta=1,\ c=1,\ v=-S[1+\in\cos(\pi z-t)] \\ &y\to\infty;\ u=1,\ w=0,\ \theta=0,\ c=0,\ v=-S \end{aligned}\right\} \tag{2.16}$$

where $S=\frac{V_0}{U_\infty}$ is the suction parameter and $k_1=\frac{K_0u_\infty^2}{\rho\eta_0^2}$ is the visco-elastic parameter.

3. Method of Solution:

To solve the system of differential equations (2.11) to (2.15) under the modified boundary conditions (2.16), the physical quantities u, v, w, p, θ and c may be expanded in powers of $\in$ as

$$u(x,y)=u_0(y)+\in u_1(y,z,t)+0\left(\in^2\right) \tag{3.1}$$

$$v(x,y)=v_0(y)=\in v_1(y,z,t)+0\left(\in^2\right) \tag{3.2}$$

$$w(x,y)=w_0(y)+\in w_1(y,z,t)+0\left(\in^2\right) \tag{3.3}$$

$$p(x,y)=p_0(y)+\in p_1(y,z,t)+0\left(\in^2\right) \tag{3.4}$$

$$\theta=(x,y)=\theta_0(y)+\in \theta_1(y,z,t)+0\left(\in^2\right) \tag{3.5}$$

$$c(x,y)=c_0(y)+\in c_1(y,z,t)+0\left(\in^2\right) \tag{3.6}$$

Substituting (3.1) to (3.6) in (2.10) to (2.15) respectively, equating the coefficients of like powers of $\in$ and neglecting higher powers of $\in$, we get

Zeroth-order equations:

$$v_0' = 0 \tag{3.7}$$

$$u'' - v_0 u_0' - k_1\left(v_0 u_0''' - 3u_0' v_0'' - 2v_0' u_0''\right) + G_r\theta_0 + G_m C_0 = 0 \tag{3.8}$$

$$v_0 v_0' = -p_0 + 2v_0' - k_1\left(2v_0 v_0''' - 6v_0 v_0''\right) \tag{3.9}$$

$$v_0 w_0' = w_0'' - k_1\left(2v_0 w_0''' - 3v_0'' w_0'\right) \tag{3.10}$$

$$\theta_0'' - v_0\theta_0' p_r = 0 \tag{3.11}$$

$$c_0'' - c_0' s_c v_0 = 0 \tag{3.12}$$

The corresponding boundary conditions are:

$$\left.\begin{array}{l} y=0;\, u_0=0,\, v_0=-s,\, \theta_0=0,\, c_0=0 \\ y\to\infty;\, u_0=1,\, v_0=-s,\, \theta_0=0,\, c_0=0 \end{array}\right\} \tag{3.13}$$

The solutions of (3.7) and (3.12) under the boundary conditions (3.13) are given by

$$v_0(y)=-s,\, \theta_0(y)=e^{-sp_r y},\, c_0(y)=e^{-ss_c y} \tag{3.14}$$

To solve (3.8), we use multi-parameter perturbation scheme following Nowinski and Ismail considering k_1 as perturbation parameter $\left(k_1 << 1 \text{ for small shear rate}\right)$, and write

$$u_0(y)=u_{00}(y)+k_1u_{01}(y)+0(k_1^2) \tag{3.15}$$

Substituting (3.15) in (3.8) and comparing the like powers of k_1 with the neglect of higher order terms we get,

$$u''_{00}+su'_{00}=-G_r e^{-sp_r y}-G_m e^{-ss_c y} \tag{3.16}$$

$$u''_{01}+su'_{01}-su''_{00}=0 \tag{3.17}$$

The corresponding boundary conditions are:

$$\left.\begin{array}{l} y=0;\, u_{00}=0,\, v_{00}=-s,\, \theta_0=1,\, c_0=1,\, u_{01}=0,\, v_{01}=0 \\ y\to\infty;\, u_{00}=1,\, v_{00}=-s,\, \theta_{01}=0,\, c_0=0,\, u_{01}=0,\, v_{01}=0 \end{array}\right\} \tag{3.18}$$

Solving (8.3.16) under the boundary conditions (8.3.18) we get,

$$u_0(y)=1-e^{-sy}+\frac{G_r}{S^2P_r(P_r-1)}\left(e^{-sy}-e^{-sp_r y}\right)+\frac{G_m}{S^2S_c(S_c-1)}\left(e^{-sy}-e^{-ss_c y}\right)+k_1\left[\frac{G_rP_rc^{-sp_r y}}{(P_r-1)^2}\right.$$

$$\left.+\frac{G_mS_c}{(S_c-1)^2}e^{-ss_c y}-\left\{\frac{G_rP_r}{(P_r-1)^2}+\frac{G_mS_c}{(S_c-1)^2}\right\}e^{-sy}+\left\{\frac{G_mS}{S_c(S_c-1)}+\frac{G_rS}{P_r(P_r-1)}-S^3\right\}ye^{-sy}\right] \tag{3.19}$$

for $S_c\neq 1,\ P_r\neq 1$

First-order equations:

$$\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}=0 \tag{3.20}$$

$$\omega\frac{\partial u_1}{\partial t}+v_1\frac{\partial u_0}{\partial y}-s\frac{\partial u_1}{\partial y}=\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_1}{\partial z^2}+G_r\theta_1+G_cC_1-k_1\left(\omega\frac{\partial^3 u_1}{\partial y^2\partial t}+\omega\frac{\partial^3 u_1}{\partial t\partial z^2}-S\frac{\partial^3 u_1}{\partial y\partial z^2}\right.$$

$$\left.+v_1\frac{\partial^3 u_0}{\partial y^3}-s\frac{\partial^3 u_1}{\partial y^3}-3\frac{\partial u_0}{\partial y}\frac{\partial^2 v_1}{\partial y}-2\frac{\partial v_1}{\partial y}\frac{\partial^2 u_0}{\partial y^2}-\frac{\partial^2 v_1}{\partial z^2}\frac{\partial u_0}{\partial y}\right) \tag{3.21}$$

$$\omega\frac{\partial v_1}{\partial t}-s\frac{\partial v_1}{\partial y}=-\frac{\partial p_1}{\partial y}+\left(2\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 w_1}{\partial y\partial z}\right)-k_1\left(2\omega\frac{\partial^3 v_1}{\partial t\partial y^2}+2\omega\frac{\partial^3 v_1}{\partial t\partial z^2}+2\omega\frac{\partial^3 v_1}{\partial y\partial z\partial t}\right.$$
$$\left.-2s\frac{\partial^3 v_1}{\partial y^3}-2s\frac{\partial^3 v_1}{\partial z^2\partial y}\right) \tag{3.22}$$

$$\omega\frac{\partial w_1}{\partial t}-s\frac{\partial w_1}{\partial y}=-\frac{\partial p_1}{\partial z}+\left(\frac{\partial^2 v_1}{\partial z\partial y}+\frac{\partial^2 w_1}{\partial y^2}+2\frac{\partial^2 w_1}{\partial z^2}\right)-k_1\left(\omega\frac{\partial^3 w_1}{\partial t\partial y\partial z}+\omega\frac{\partial^3 v_1}{\partial t\partial y\partial z}+4\omega\frac{\partial^3 w_1}{\partial t\partial z^2}\right.$$
$$\left.+2\omega\frac{\partial^3 w_1}{\partial t\partial y^2}-s\frac{\partial^3 w_1}{\partial y^2\partial z}-2s\frac{\partial^3 w_1}{\partial y^3}-4s\frac{\partial^3 w_1}{\partial y\partial z^2}-s\frac{\partial^3 v_1}{\partial y^2\partial z}\right) \tag{3.23}$$

$$\omega\frac{\partial \theta_1}{\partial t}+v_1\frac{\partial \theta_0}{\partial y}-s\frac{\partial \theta_1}{\partial y}=\frac{1}{P_r}\left(\frac{\partial^2\theta_1}{\partial y^2}+\frac{\partial^2\theta_1}{\partial z^2}\right) \tag{3.24}$$

$$\omega\frac{\partial c_1}{\partial t}+v_1\frac{\partial c_0}{\partial y}-s\frac{\partial c_1}{\partial y}=\frac{1}{s_c}\left(\frac{\partial c_1}{\partial y^2}+\frac{\partial^2 c_1}{\partial z^2}\right) \tag{3.25}$$

The relevant boundary conditions are:

$$\left.\begin{array}{l} y=0;\, u_1=0,\, v_1=-s\cos(\pi z-t),\, w_1=0,\, \theta_1=0,\, c_1=0 \\ y\to\infty;\, u_1=0,\, v_1=0,\, w_1=0,\, \theta_1=0,\, c_1=0 \end{array}\right\} \tag{3.26}$$

We presume the velocity components as

$$\left.\begin{array}{l} u_1(y,z,t)=u_{11}(y)e^{i(\pi z-t)} \\ v_1(y,z,t)=v_{11}(y)e^{i(\pi z-t)} \\ w_1(y,z,t)=\dfrac{i}{\pi}v_{11}'(y)e^{i(\pi z-t)} \\ p_1(y,z,t)=p_{11}(y)e^{i(\pi z-t)} \\ \theta_1(y,z,t)=\theta_{11}(y)e^{i(\pi z-t)} \\ c_1=(y,z,t)=c_{11}(y)e^{i(\pi z-t)} \end{array}\right\} \tag{3.27}$$

Substituting (3.27) in (3.21) to (3.25) we get

$$u''_{11} + su'_1\left(\pi^2 - i\omega\right)u_{11} = v_{11}u'_0 - G_r\theta_1 - G_m c_{11} - k_1\left(iwu''_{11} - i\omega\pi^2 u_{11} - s\pi^2 u'_{11} - v_{11}u'''_0 + su'''_{11}\right.$$
$$\left. + 3u'_0 v''_{11} - \pi^2 v_{11}u''_0 + 2v'_{11}u''_0\right) \quad (3.28)$$

$$v''_{11} + sv'_{11} - \left(\pi^2 - i\omega\right)v_{11} = p'_{11} - k_1\left(2iwv''_{11} - 2i\omega\pi^2 v_1 - 2\omega\pi v'_{11} + 2sv'''_{11} - 2s\pi^2 v'_{11}\right) \quad (3.29)$$

$$\theta''_{11} + SP_r\theta'_{11} - \left(\pi^2 - iP_r\omega\right)\theta_{11} = P_r v_{11}\theta'_0 \quad (3.30)$$

$$c''_{11} + ss_c c'_{11} - \left(\pi^2 - is_c\omega\right)c_{11} = s_c v_{11}c'_0 \quad (3.31)$$

subject to boundary conditions:

$$\left.\begin{array}{l} y = 0;\, u_{11} = 0,\, v_{11} = -s,\, \theta_{11} = 0,\, c_{11} = 0 \\ y \to \infty;\, u_{11} = 0,\, v_{11} = 0,\, \theta_{11} = 0,\, c_{11} = 0 \end{array}\right\} \quad (3.32)$$

Eliminating p''_1 from (3.28) and (3.29) we get,

$$v'^{v}_{11} + sv'''_{11} - \left(\pi^2 - iw\right)v''_{11} - 2\pi^2 v''_{11} + \pi^2 v'_{11} - s^2\pi^2 v'_{11} + \pi^2\left(\pi^2 - iw\right)v_{11} = k_1\left\{6i\omega\pi^2 v''_{11} - 2i\omega\pi^4 v_{11}\right.$$
$$- 2\omega\pi^3 v'_{11} + 2s\pi^2 v'''_{11} - 2\pi^2 v'''_{11} - 2\pi^4 sv'_{11} + \omega\pi v'''_{11} - i\omega\pi^2 v'''_{11} - \left(2i\omega + 2is\pi\right)v'^{v}_{11}$$
$$\left. + 2sv^{v}_{11} - 4is\pi^2 v'''_{11}\right\} \quad (3.33)$$

Again, we consider

$$u_{11}(y) = u_{110}(y) + k_1 u_{111}(y) + 0\left(\alpha_1^2\right) \quad (3.34)$$

$$v_{11}(y) = v_{110}(y) + k_1 v_{111}(y) + 0\left(\alpha_1^2\right) \quad (3.35)$$

$$\theta_{11}(y) = \theta_{110}(y) + k_1\theta_{111}(y) + 0\left(\alpha_1^2\right) \quad (3.36)$$

$$c_{11}(y) = c_{110}(y) + k_1 c_{111}(y) + 0\left(\alpha_1^2\right) \quad (3.37)$$

Substituting the above expressions into the equations (3.29) and (3.33) and using (3.32), after comparing the like terms with the neglect of higher order terms we get

$$v'^{v}_{110} + sv'''_{110} - \left(\pi^2 - iw\right)v'''_{110} - \pi^2 v'''_{110} - \pi^2 sv'_{110} + \left(\pi^2 - iw\right)\pi^2 v_{110} = 0 \quad (3.38)$$

$$v'^{v}_{110} + sv'''_{111} - \left(\pi^2 - iw\right)v''_{11} - \pi^2 v''_{111} - \pi^2 sv'_{111} + \pi^2\left(\pi^2 - iw\right)v_{111} = 2i\omega\pi^2 v''_{110} - 2i\omega\pi^4 v_{110}$$
$$-2\omega\pi^3 v'_{110} + 2s\pi^2 v'''_{110} - 2s\pi^4 v'_{110} + \omega\pi v'''_{110} - i\omega\pi^2 v'''_{110} + 4i\omega\pi^2\pi^2 v''_{110} - 2i\omega v'^{v}_{110}$$

$$-2is\pi v_{110}'^{v}+2sv_{110}^{v}-4is\pi^{2}v_{110}''' \tag{3.39}$$

$$u_{110}'''+su_{110}'-\left(\pi^{2}-i\omega\right)u_{110}=v_{110}u_{0}'-G_{r}\theta_{11}-G_{m}C_{11} \tag{3.40}$$

$$u_{111}''+su_{111}'-\left(\pi^{2}-i\omega\right)u_{111}=-i\omega u_{110}''+i\omega\pi^{2}u_{110}-s\pi^{2}u_{110}'+v_{110}u_{0}'''-su_{110}'''-3u_{0}'v_{110}''$$
$$+\pi^{2}v_{110}u_{0}''-2v_{110}'u_{0}'' \tag{3.41}$$

The corresponding boundary conditions are:

$$\left.\begin{array}{l} y=0\,;u_{110}=u_{111}=0,v_{110}=-s,v_{111}=0,v_{110}'=v_{111}=0 \\ y\to\infty\,;u_{110}=u_{111}=0,v_{110}=v_{111}=0,v_{110}'=v_{111}'=0 \end{array}\right\} \tag{3.42}$$

Solving (3.25) to (3.29) and (3.38) to (3.41) with the relevant boundary conditions (3.32) and (3.42) we get,

$$v_{110}=A_{1}e^{-\pi y}-A_{2}e^{-\pi_{1}y} \tag{3.43}$$

$$v_{111}=A_{6}e^{-\pi y}+A_{5}e^{-r_{1}y}+A_{3}ye^{-\pi y}+A_{4}ye^{-\pi_{1}y} \tag{3.44}$$

$$\theta_{11}=A_{15}e^{-r_{2}y}+A_{11}e^{-(\pi+sp_{r})y}+A_{12}e^{-(r_{1}+sp_{r})y}+A_{13}ye^{-(\pi+sp_{r})y}+A_{14}ye^{-(r_{1}+sp_{r})y} \tag{3.45}$$

$$c_{11}=A_{20}e^{-r_{3}y}+A_{18}e^{-(\pi+ss_{c})y}+A_{19}e^{-(r_{1}+ss_{c})y}+A_{16}ye^{-(\pi+ss_{c})y}+A_{17}ye^{-(r_{1}+ss_{c})y} \tag{3.46}$$

$$u_{110}=A_{38}e^{-r_{1}y}+A_{32}e^{-(\pi+s)y}+A_{33}e^{-(\pi+sp_{r})y}+A_{34}e^{-(\pi+ss_{c})y}+A_{35}e^{-(r_{1}+s)y}+A_{36}e^{-(r_{1}+sp_{r})y}+A_{37}e^{-(r_{1}+ss_{c})y}$$
$$+A_{26}ye^{-(\pi+s)y}+A_{27}ye^{-(r_{1}+s)y}+A_{28}ye^{-(\pi+sp_{r})y}+A_{29}ye^{-(r_{1}+sp_{r})y}+A_{30}ye^{-(\pi+ss_{c})y}+A_{31}ye^{-(r_{1}+ss_{c})y}$$
$$+D_{1}e^{-r_{2}y}+D_{2}e^{-r_{3}y} \tag{3.47}$$

$$u_{111}=A_{54}e^{-r_{1}y}+A_{39}e^{-r_{1}y}.y+A_{40}e^{-(\pi+s)y}+A_{41}e^{-(\pi+sp_{r})y}+A_{42}e^{-(\pi+ss_{c})y}+A_{43}e^{-(r_{1}+s)y}+A_{44}e^{-(r_{1}+sp_{r})y}$$
$$+A_{45}ye^{-(r_{1}+s)y}+A_{46}ye^{-(\pi+sp_{r})y}+A_{47}ye^{-(\pi+s)y}+A_{48}ye^{-(r_{1}+sp_{r})y}+A_{49}e^{-(r_{1}+ss_{c})y}+A_{50}ye^{-(\pi+ss_{c})y}$$
$$+A_{51}ye^{-(r_{1}+ss_{c})y}+A_{52}e^{-r_{2}y}+A_{53}e^{-r_{3}y} \tag{3.48}$$

$$u_{11}=A_{55}e^{-r_{1}y}+A_{56}e^{-(\pi+s)y}+A_{57}e^{-r_{3}y}+A_{58}e^{-(\pi+ss_{c})y}+A_{59}e^{-(r_{1}+s)y}+A_{60}e^{-(r_{1}+sp_{r})y}+A_{61}e^{-(r_{1}+ss_{c})y}$$
$$+A_{62}ye^{-(\pi+s)y}+A_{63}ye^{-(r_{1}+s)y}+A_{64}ye^{-(\pi+sp_{r})y}+A_{65}ye^{-(r_{1}+sp_{r})y}+A_{66}ye^{-(\pi+ss_{c})y}+A_{67}ye^{-(r_{1}+ss_{c})y}$$
$$+A_{68}e^{-r_{2}y}+A_{69}e^{-r_{3}y}+A_{70}ye^{-r_{1}y} \tag{3.49}$$

$$v_{11} = E_1 e^{-\pi y} + E_2 e^{-r_1 y} + E_3 y e^{-\pi y} + E_4 y e^{-r_1 y} \tag{3.50}$$

Thus the respective velocity, temperature and concentration expressions are given as follows:

$$u = 1 - e^{-Sy} + \frac{G_r}{s^2 p_r (p_r - 1)}\left(e^{-Sy} - e^{-SP_r y}\right) + k_1\left[\frac{G_r P_r}{(P_r - 1)^2} e^{-SP_r y} + \frac{G_m S_c}{(S_c - 1)^2} e^{-SS_c y} - \left(\frac{G_r P_r}{(P_r - 1)^2}\right) e^{-Sy} + \left\{\frac{G_m S}{(S_c - 1) S_c} + \frac{G_r S}{P_r (P_r - 1)} - S^3\right\} y e^{-Sy}\right] + \left[A_{55} e^{-r_1 y} + A_{56} e^{-(\pi + S)y} + A_{57} e^{-(r_1 + SP_r)y} + A_{58} e^{-(\pi + SS_c)y}\right.$$

$$+ A_{59} e^{-(r_1 + S)y} + A_{60} e^{-(r_1 + SP_r)y} + A_{61} e^{-(r_1 + SS_c)y} + A_{62} y e^{-(\pi + S)y} + A_{63} y e^{-(r_1 + S)y} + A_{64} y e^{-(\pi + SP_r)y}$$

$$\left. A_{65} y e^{-(r_1 + SP_r)y} + A_{66} y e^{-(\pi + SS_c)y} + A_{67} e^{i(\pi z - t)} + A_{68} e^{-r_2 y} + A_{69} e^{-r_3 y} + A_{70} y e^{-r_1 y}\right]$$

$$v = -S + \in \left[E_1 e^{-\pi y} + E_2 e^{-r_1 y} + E_3 y e^{-\pi y} + E_4 y e^{-r_1 y}\right] e^{i(\pi z - t)}$$

$$w = \frac{\in_i}{\pi}\left[-\pi E_1 e^{-\pi y} - E_2 r_1 e^{-r_1 y} + E_3 e^{-\pi y} - e\pi y e^{-\pi y} + E_4 e^{-r_1 y} - E_4 r_1 y e^{-r_1 y}\right] e^{2i(\pi z - t)}$$

$$\theta = e^{-sp_r y} + \in \left[A_{15} e^{-r_2 y} + A_{11} e^{-(\pi + sp_r)y} + A_{12} e^{-(r_1 + sp_r)y} + A_{13} y e^{-(\pi + sp_r)y} + A_{14} y e^{-(r_1 + sp_r)y}\right] e^{i(\pi z - t)}$$

$$C = e^{-ss_c y} + \in \left[A_{20} e^{-r_3 y} + A_{18} e^{-(\pi + ss_c)y} + A_{19} e^{-(r_1 + ss_c)y} + A_{16} y e^{-(\pi + ss_c)y} + y A_{17} e^{-(r_1 + sp_r)y}\right]$$

The dimensionless forms of shearing stress, Nusselt number and Sherwood number at the plate $(y = 0)$ are given below:

$$\sigma = \left[\frac{\partial u}{\partial y}\right]_{y=0} - k_1\left[\omega \frac{\partial^2 u}{\partial t \partial y} + v \frac{\partial^2 u}{\partial y^2} + \omega \frac{\partial^2 u}{\partial y \partial z} - 3 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - 2 \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right]_{y=0}$$

$$= A_{71} - k_1 A_{81}$$

$$N_u = \left[\frac{\partial \theta}{\partial y}\right]_{y=0} \quad \text{and} \quad S_h = \left[\frac{\partial c}{\partial y}\right]_{y=0}$$

4. RESULTS AND DISCUSSION

In this chapter, an attempt is made to find out the effects of visco-elasticity of a three-dimensional unsteady fluid flow in presence of heat and mass transfer with variable suction. The successive results are discussed for Newtonian and non-Newtonian fluids. For Newtonian fluid flow, we set $k_1 = 0$ and characteristics of visco-elasticity are shown by setting $k_1 = 0.05$ and 0.1.The real part is inferred throughout the discussion. To guide the physical behaviour of the fluid flow, the flow velocity and shearing stress at the plate have been illustrated graphically and the effects of the visco-elastic parameter on the governing flow have been discussed in detail. For numerical calculation we consider $\omega = 8,\ z = 0.03,\ t = 0.1,\ s = 1,\ S_c = 0.6,\ P_r = 3,\ G_r = 5$ and $G_m = 4$ unless otherwise stated. The figures 1 to 5 depict the flow velocity u against y for different flow parameters. In all the cases, the fluid velocity accelerates near the plate and after attaining the peak, it shows uniformity when away from the plate in both Newtonian and non-Newtonian cases. The variation of the flow parameters Schmidt number S_c, Prandtl number P_r, Grashof number for heat transfer G_r, Grashof number for mass transfer G_m and the visco-elastic parameter k_1 does not alter the pattern of fluid velocity. The cross flow velocity w against y has been plotted in figure 6 and in this case the fluid velocity accelerates with the increase of visco-elastic parameter in comparison with Newtonian fluid flow phenomenon. Also, it is noticed that the variation of flow parameters do not alter the nature of flow curve significantly. Figures 7 to 10, exhibit the effects of shearing stress against Schmidt number S_c, Prandtl number P_r, Grashof number for heat transfer G_r and Grashof number for mass transfer G_m. The shearing stress exhibits decelerating trend in case of P_r, G_r and G_m but opposite pattern is noticed in case of S_c. Also in all the cases, the enhancement of the shearing stress is prominent with the growth of visco-elastic parameter in comparison with simple flow fluid. From the expressions of the temperature and mass concentration it is inferred that the visco-elastic parameter has no significant effect on the respected field.

5. CONCLUSION:

This study endeavors the effects of visco-elasticity on three-dimensional flow of a fluid past a vertical porous plate in presence of heat and mass transfer with variable suction. Some explicit conclusions are highlighted below:

- The velocity field is influenced noticeably by visco-elasticity in presence of pertinent flow parameters.

• With the magnification of visco-elastic parameter, the velocity component u reveals accelerating trend as compared to the simple Newtonian fluid whearas opposite result is observed for the velocity component w of the flow.

• The shearing stress exhibits enhancement with the increasing values of Prandtl number, Grashof number for heat transfer and Grashof number for mass transfer but diminishing trend for increase of Schmidt number.

Figures:

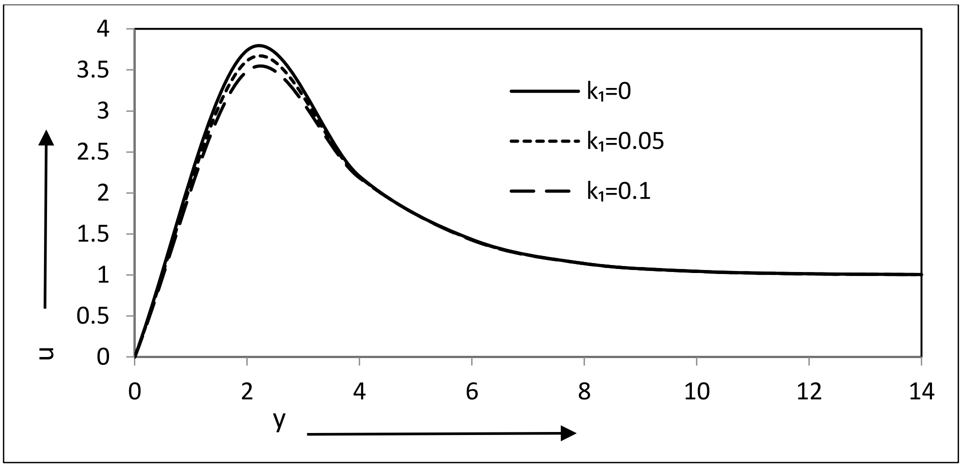

Figure 1: Fluid velocity u against y for S=1,Pr=3,Sc=0.6,Gr=5,Gm=4,t=0.2,z=0.03, $\omega = 8$,

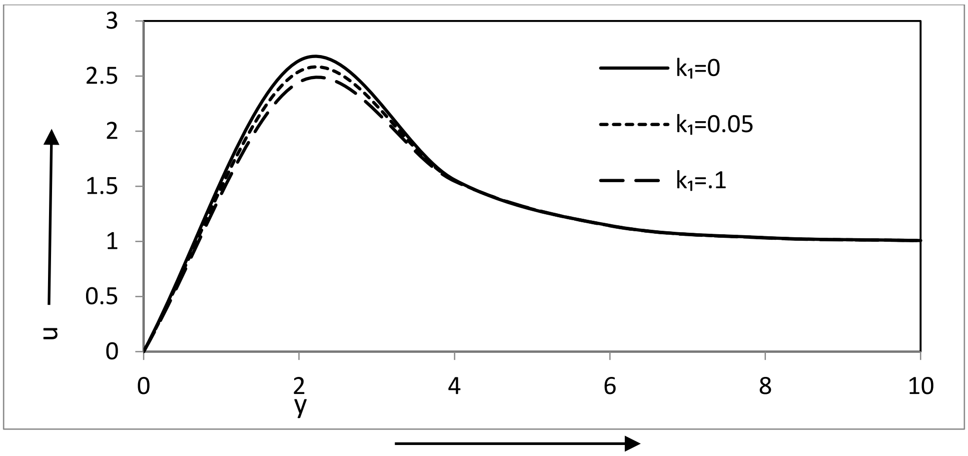

Figure 2: Fluid velocity u against y for S=1,Pr=3,Sc=0.8,Gr=5,Gm=4,t=0.2,z=0.03, $\omega = 8$,

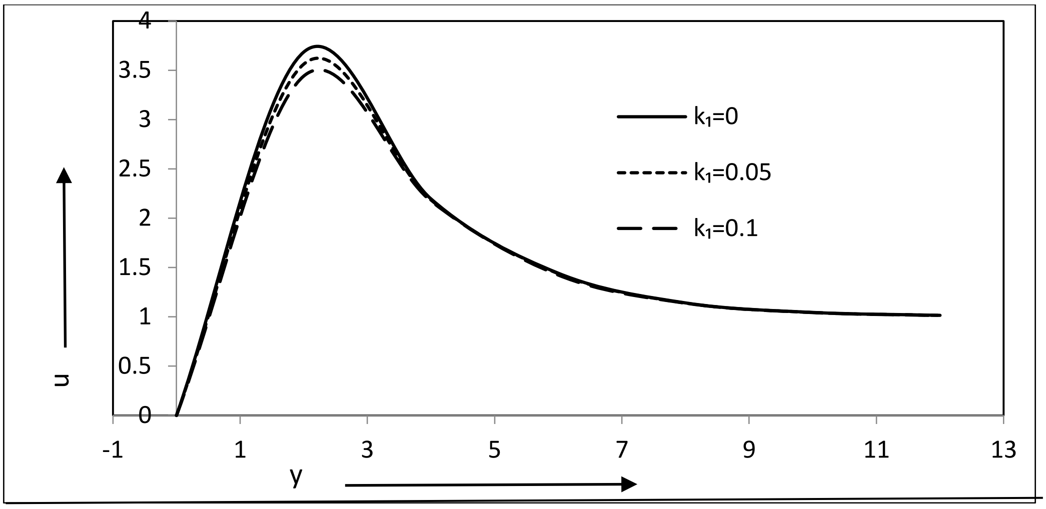

Figure 3: Fluid velocity u against y for S=1,Pr=4,Sc=0.6,Gr=5,Gm=4,t=0.2,z=0.03, $\omega = 8$,

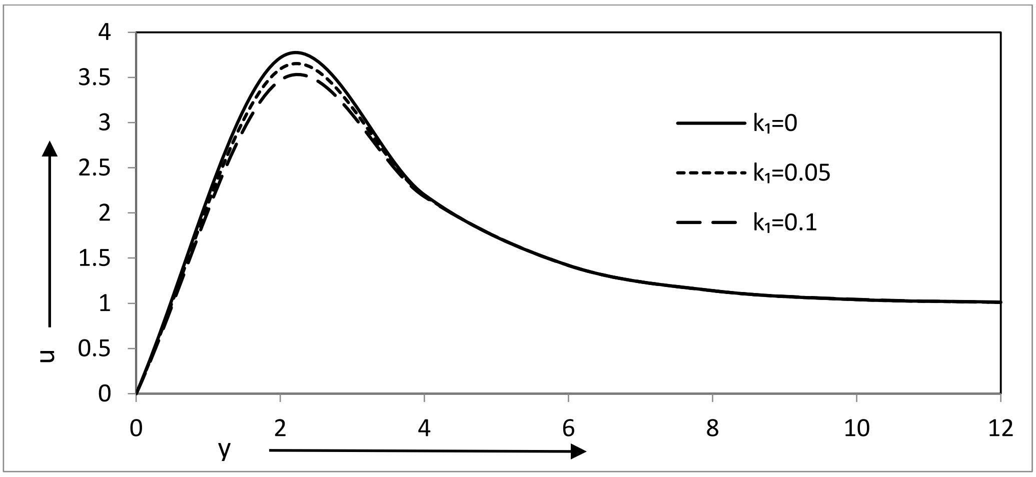

Figure 4: Fluid velocity u against y for S=1,Pr=3,Sc=0.6,Gr=4,Gm=4,t=0.2,z=0.03, $\omega = 8$,

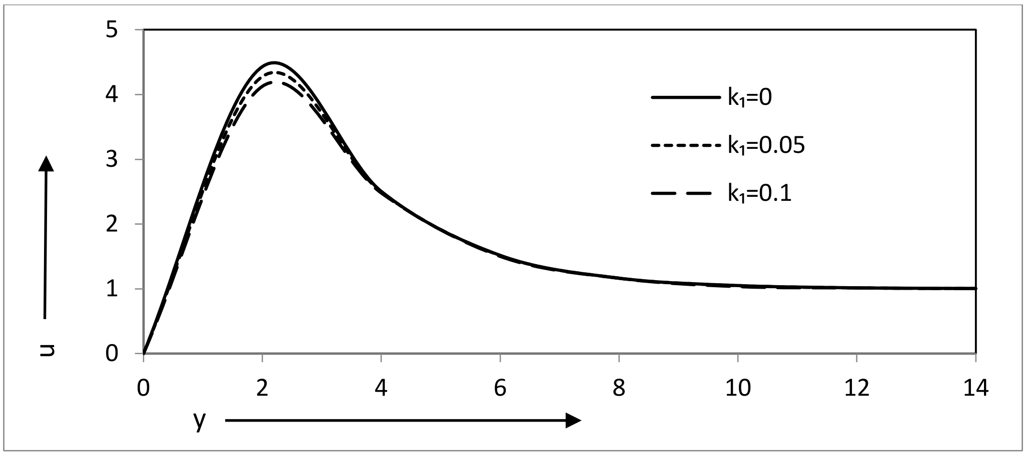

Figure 5: Fluid velocity u against y forS=1,Pr=3,Sc=0.6,Gr=5,Gm=5,t=0.2,z=0.03, $\omega = 8$,

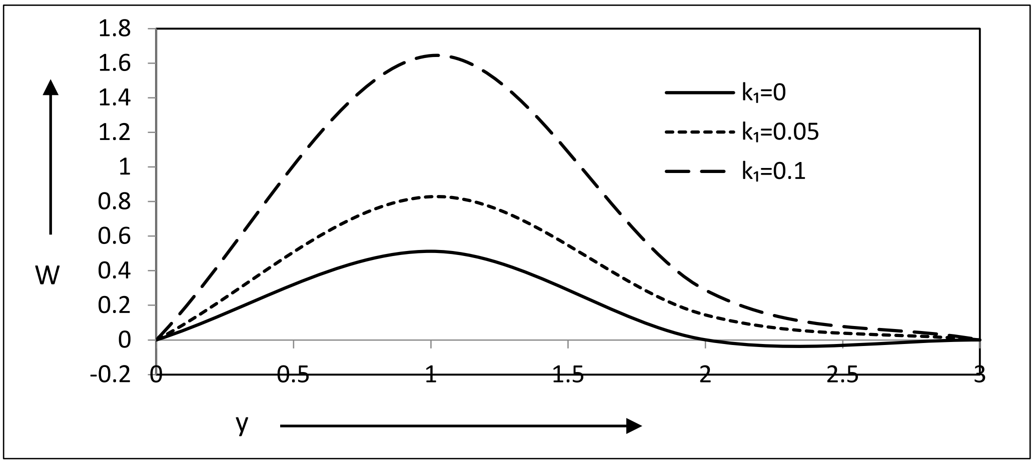

Figure 6: Fluid velocity w against y for S=1,Pr=3,Sc=0.6,Gr=5,Gm=4,t=0.2,z=0.03, $\omega = 8$,

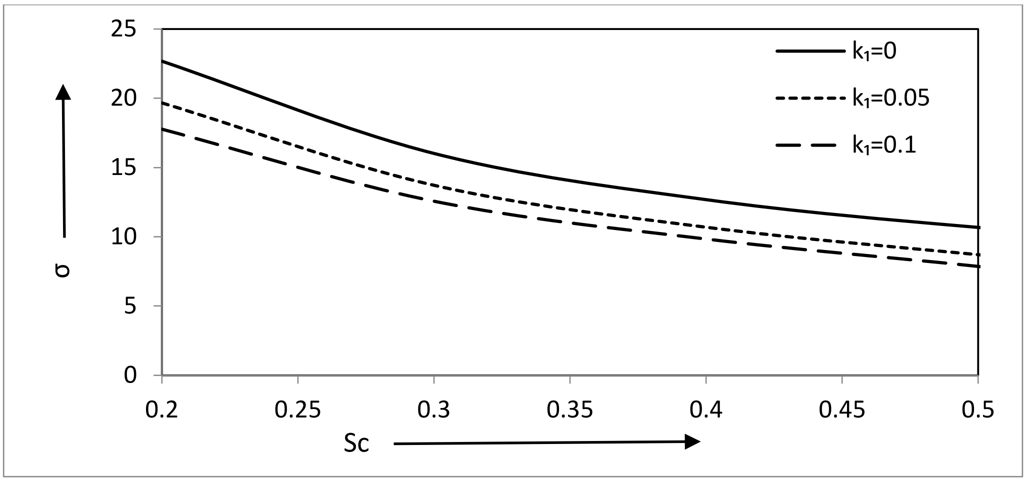

Figure 7: Effect of Sc on shearing stress for S=1,Pr=3,Gr=5,Gm=4,t=0.2,z=0.03, $\omega = 8$,

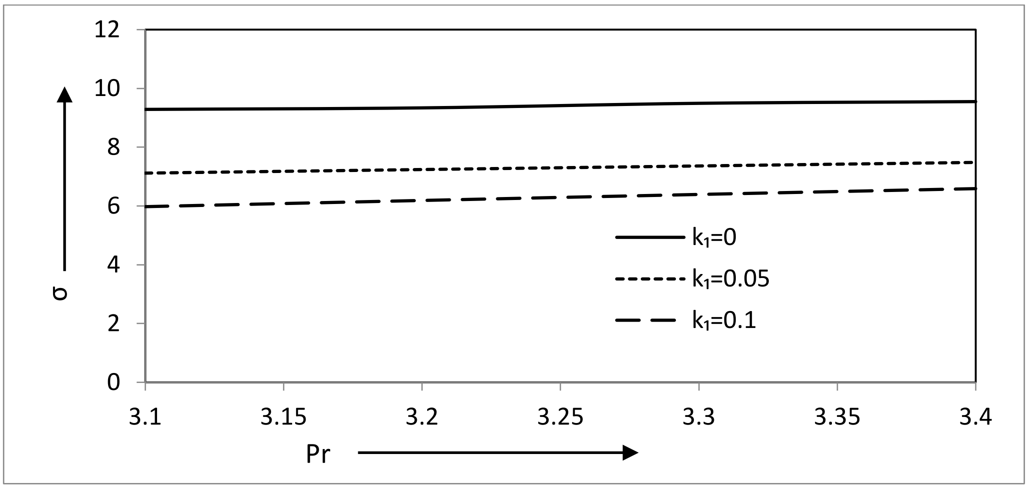

Figure 8: Effect of Pr on shearing stress for S=1,Sc=0.6,Gr=5,Gm=4,t=0.2,z=0.03, $\omega = 8$,

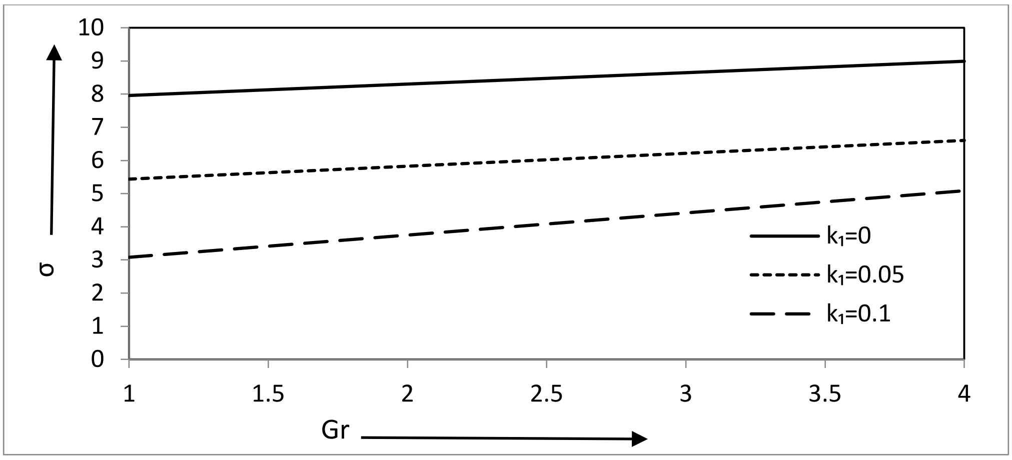

Figure 9: Effect of Gr on shearing stress for S=1,Pr=3,Sc=0.6,Gm=4,t=0.2,z=0.03, $\omega = 8$,

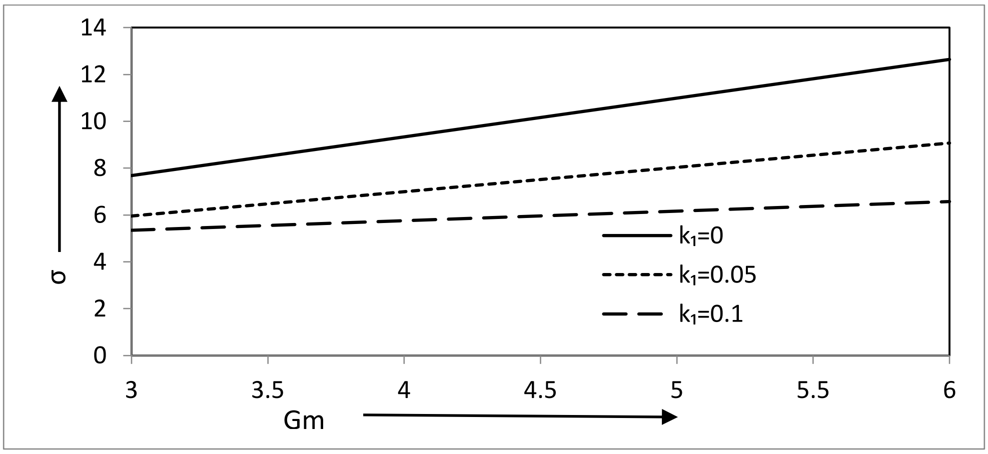

Figure 10: Effect of Gm on shearing stress for S=1,Pr=3,Sc=0.6,Gr=5,t=0.2,z=0.03 $\omega = 8$,

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ON FUZZY GLOBALLY DISCONNECTED SPACES

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ABSTRACT: In this paper, the concept of fuzzy globally disconnected spaces is introduced and studied. Several characterizations of fuzzy globally disconnected spaces are established. By means of fuzzy globally disconnectedness, conditions for fuzzy Baire spaces to become fuzzy σ-Baire spaces, fuzzy Volterra spaces and a condition for fuzzy strongly irresolvable spaces to become fuzzy submaximal spaces, are also obtained.

KEY WORDS : Fuzzy semi-open set, fuzzy G_δ -set, fuzzy F_σ-set, fuzzy nowhere dense set, fuzzy first category set, fuzzy Baire space, fuzzy Volterra space, fuzzy submaximal space.

AMS Classification No: 54A40, 03E72.

1. INTRODUCTION:

In 1965, Zadeh [17] introduced the concept of fuzzy sets as a new approach for modeling uncertainties. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. Topology provided the most natural framework for the concepts of fuzzy sets to flourish. In1968, Chang [3] introduced the concept of fuzzy topological spaces. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

In the recent years, there has been a growing trend among many fuzzy topologists to introduce and study different forms of fuzzy disconnected spaces and a considerable amount of research has been done on many types of fuzzy disconnectedness in fuzzy topology. In 1969, the concept of globally disconnected spaces was introduced and studied by El'kin [5] in classical topology. The purpose of this paper is to carry out globally disconnectedness to fuzzy setting. The concept of fuzzy globally disconnected spaces is introduced and studied in this paper. Several characterizations of fuzzy globally disconnected spaces are established. By means of fuzzy globally disconnectedness of fuzzy topological spaces, conditions for fuzzy Baire spaces to become fuzzy σ-Baire spaces, fuzzy Volterra spaces and a condition for fuzzy strongly irresolvable spaces to become fuzzy submaximal spaces, are obtained. Several examples are given to illustrate the concepts introduced in this paper.

2. PRELIMINARIES

In order to make the exposition self-contained, some basic notions and results used in the sequel are given. In this work (X,T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I, the unit interval [0,1]. A fuzzy set λ in X is a function from X into I. The null set 0 is the function from X into I which assumes only the value 0 and the whole fuzzy set 1 is the function from X into I which takes 1 only.

Definition 2.1[4] Let (X,T) be a fuzzy topological space and λ be any fuzzy set defined on X. The interior and the closure of λ are defined respectively as follows:

(i) $\mathrm{int}(\lambda) = \vee\{\mu / \mu \le \lambda, \mu \in T\}$
(ii) $\mathrm{cl}(\lambda) = \wedge\{\mu / \lambda \le \mu, 1 - \mu \in T\}$.

Lemma 2.1 [1] For a fuzzy set λ of a fuzzy topological space X,
(i) $1 - \mathrm{int}(\lambda) = \mathrm{cl}(1 - \lambda)$,
(ii) $1 - \mathrm{cl}(\lambda) = \mathrm{int}(1 - \lambda)$.

Definition 2.2 [9] A fuzzy set λ in a fuzzy topological space (X,T), is called fuzzy dense if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$. That is, $\mathrm{cl}(\lambda) = 1$, in (X,T).

Definition 2.3 [9] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < \mathrm{cl}(\lambda)$. That is, int cl(λ)=0 in (X,T).

Definition 2.4 [9] A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy first category set if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of fuzzy second category.

Definition 2.5 [2] A fuzzy set λ in a fuzzy topological space (X,T) is called a
(i). fuzzy G_δ - set in (X, T) if $\lambda = \bigwedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.
(ii). fuzzy F_σ - set in (X, T) if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$.
Definition 2.6 A fuzzy set λ in a fuzzy topological space (X,T) is called

(1) fuzzy pre-open if $\lambda \leq \text{int cl}(\lambda)$ and fuzzy pre-closed if $\text{cl int}(\lambda) \leq \lambda$ **[3].**

(2) fuzzy semi-open if $\lambda \leq \text{cl int}(\lambda)$ and fuzzy semi-closed if $\text{int cl}(\lambda) \leq \lambda$ **[1].**

(3) fuzzy α-open if $\lambda \leq \text{int cl int}(\lambda)$ and fuzzy α -closed if $\text{cl int cl}(\lambda) \leq \lambda$**[3].** .

Definition 2.7 [8] Let λ be a fuzzy set in a fuzzy topological space (X,T). The fuzzy boundary of λ is defined as $\text{Bd}(\lambda) = \text{cl}(\lambda) \wedge \text{cl}(1 - \lambda)$.

Definition 2.8 [15] A fuzzy set λ in a fuzzy topological space (X,T), is called a fuzzy simply open set if $\text{Bd}(\lambda)$ is a fuzzy nowhere dense set in (X,T).

Theorem 2.1 [12] If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X,T), then $1- \lambda$ is a fuzzy first category set in (X,T).

Theorem 2.2 [11] If λ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then $1-\lambda$ is a fuzzy dense set in (X,T).

Definition 2.9[13] A fuzzy set λ in a fuzzy topological space (X,T), is called a fuzzy σ-nowhere dense set if λ is a fuzzy F_σ-set in (X,T) with $\text{int}(\lambda) = 0$.

Definition 2.10 A fuzzy topological space (X,T) is called a

(i). fuzzy Baire space if $\text{int}(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T).

(ii). fuzzy σ-Baire space if $\text{int}(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T).

Definition 2.11 [14] A fuzzy topological space (X, T) is called a fuzzy Volterra space if $\text{cl}[\wedge_{k=1}^{N}(\lambda_k)] = 1$, where (λ_k)'s are fuzzy dense and fuzzy G_δ-sets in (X,T).

Definition 2.12 [10] A fuzzy topological space (X, T) is called a fuzzy strongly irresolvable space if for every fuzzy dense set λ in (X,T), $\text{cl int}(\lambda) = 1$ in (X, T).

Definition 2.13.[2] A fuzzy topological space (X, T) is called a fuzzy submaximal space if for each fuzzy set λ in (X, T) such that $\text{cl}(\lambda) = 1$, $\lambda \in T$, in (X,T).

Definition 2.14 A fuzzy topological space (X,T) is said to be fuzzy hyper-connected if every non-null fuzzy open subset of (X,T) is fuzzy dense in (X,T).

3.FUZZY GLOBALLY DISCONNECTED SPACES:

Definition 3.1 A fuzzy topological space (X,T) is called a fuzzy globally disconnected space if each fuzzy semi-open set in (X,T) is fuzzy open. That is, if $\lambda \leq$ cl int (λ) for a fuzzy set λ defined on X, then $\lambda \in T$.

Example 3.1 Let X={a, b, c}. Consider the fuzzy sets $\alpha, \beta, \gamma, \mu$ and δ defined on X as follows:

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.6$; $\alpha(b) = 0.4$; $\alpha(c) = 0.6$,

$\beta : X \to [0, 1]$ is defined as $\beta(a) = 0.5$; $\beta(b) = 0.5$; $\beta(c) = 0.5$,

$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.4$; $\gamma(b) = 0.6$; $\gamma(c) = 0.4$,

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.4$; $\mu(b) = 0.5$; $\mu(c) = 0.6$,

$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.5$; $\delta(b) = 0.6$; $\delta(c) = 0.4$.

Then T ={0, α, β, γ, $\alpha \vee \beta$, $\alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta$, $\alpha \wedge \gamma$, $\beta \wedge \gamma$,1} is a fuzzy topology on X. On computation, $\mathrm{cl}(\alpha) = 1 - \gamma = \alpha$; $\mathrm{cl}(\beta) = 1 - \beta = \beta$; $\mathrm{cl}(\gamma) = 1 - \alpha = \gamma$; $\mathrm{cl}(\alpha \vee \beta) = 1 - [\beta \wedge \gamma] = \alpha \vee \beta$; $\mathrm{cl}(\alpha \vee \gamma) = 1 - [\alpha \wedge \gamma] = \alpha \vee \gamma$; $\mathrm{cl}(\beta \vee \gamma) = 1 - [\alpha \wedge \beta] = \beta \vee \gamma$; $\mathrm{cl}(\alpha \wedge \beta) = 1 - [\beta \vee \gamma]\ \alpha \wedge \beta$; $\mathrm{cl}(\alpha \wedge \gamma) = 1 - [\alpha \vee \gamma] = \alpha \wedge \gamma$; $\mathrm{cl}(\beta \wedge \gamma) = 1 - [\alpha \vee \beta] = \beta \wedge \gamma$. Also $\mathrm{int}(\delta) = \gamma$; $\mathrm{int}(1 - \delta) = 1 - \gamma$; $\mathrm{int}(\mu) = \beta \wedge \gamma$; $\mathrm{int}(1 - \mu) = \beta \wedge \gamma$. Now $\delta \nleq$ cl $\mathrm{int}(\delta)$, $1 - \delta \nleq$ cl $\mathrm{int}(1 - \delta)$, $\mu \nleq$ cl $\mathrm{int}(\mu)$, $1 - \mu \nleq$ cl $\mathrm{int}(1 - \mu)$, implies that δ ,$1 - \delta, \mu$ and $1 - \mu$,are not fuzzy semi-open sets in (X,T). Hence the fuzzy semi-open sets α, β, γ, $\alpha \vee \beta$, $\alpha \vee \gamma$, $\beta \vee \gamma$, $\alpha \wedge \beta$, $\alpha \wedge \gamma$, $\beta \wedge \gamma$, α, β, γ, $\alpha \vee \beta$, $\alpha \vee \gamma$, $\beta \vee \gamma$, $\alpha \wedge \beta$, $\alpha \wedge \gamma$, $\beta \wedge \gamma$ are fuzzy open sets in (X,T), implies that (X,T) is a fuzzy globally disconnected space.

Proposition 3.1 If cl int (λ) = cl (λ) ,for a fuzzy set λ defined on X in a fuzzy globally disconnected space (X,T), then $\mathrm{int}(\lambda) = \lambda$ in (X,T).

Proof: Let λ be a fuzzy set defined on X such that cl $\mathrm{int}(\lambda)$=$\mathrm{cl}(\lambda)$. Now, $\lambda \leq \mathrm{cl}(\lambda)$ implies that $\lambda \leq$cl $\mathrm{int}(\lambda)$ and hence λ is a fuzzy semi-open set in (X,T). Since (X,T) is a fuzzy globally disconnected space, the fuzzy semi-open set λ is a fuzzy open set in (X,T) and hence $\mathrm{int}(\lambda) = \lambda$ in (X,T).

Remark 3.1 If cl $\mathrm{int}(\lambda) = \mathrm{cl}(\lambda)$ for a fuzzy set λ defined on X in a fuzzy topological space (X,T), then $\mathrm{int}(\lambda)$ need not be equal to λ in (X,T).
For, consider the following example:

Example 3.2 Let X ={a, b, c}. Consider the fuzzy sets α, β, γ and μ defined on X as follows:

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.5$; $\alpha(b) = 0.4$; $\alpha(c) = 0.6$,

$\beta : X \to [0, 1]$ is defined as $\beta(a) = 0.6$; $\beta(b) = 0.5$; $\beta(c) = 0.7$,

$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.6$; $\gamma(b) = 0.6$; $\gamma(c) = 0.7$,

$\mu : X \rightarrow [0, 1]$ is defined as $\mu(a) = 0.5$; $\mu(b) = 0.5$; $\mu(c) = 0.6$,

Then $T = \{0, \alpha, \beta, \gamma, 1\}$ is a fuzzy topology on X. On computation, $cl(\alpha) = 1$; $cl(\beta) = 1$; $cl(\gamma) = 1$; $cl(\mu) = 1$; $int(\mu) = \alpha$, $int(1 - \alpha) = 0$ $int(1 - \beta) = 0$; $int(1 - \gamma) = 0$ and $int(1 - \mu) = 0$. Now $cl\,int(\mu) = cl(\alpha) = 1$; $cl(\mu) = 1$, implies that $cl\,int(\mu) = cl(\mu)$ in (X,T). But $int(\mu) = \alpha \neq \mu$, in (X,T).

Proposition 3.2 If λ is a fuzzy nowhere dense set in a fuzzy globally disconnected space (X,T) then λ is a fuzzy closed set in (X,T).

Proof : Let λ be a fuzzy nowhere dense set in (X,T). Then $int\,cl(\lambda) = 0$ in (X,T). Now, $int\,cl(\lambda) \leq \lambda$, implies that λ is a fuzzy semi-closed set in (X,T). Then $(1 - \lambda)$ is a fuzzy semi-open set in (X,T) and since (X,T) is a fuzzy globally disconnected space, $(1 - \lambda)$ is a fuzzy open set in (X,T). Thus λ is a fuzzy closed set in (X,T).

Remark 3.2 In view of proposition 3.2, one will have the following result :**" The fuzzy nowhere dense sets are fuzzy closed sets in fuzzy globally disconnected spaces ".**

Proposition 3.3 If $cl\,int[cl(\lambda)] = cl(\lambda)$, for a fuzzy set λ defined on X in a fuzzy globally disconnected space (X,T), then λ is a fuzzy pre-open set in (X,T).

Proof : Let λ be a fuzzy set defined on X such that $cl\,int[cl(\lambda)] = cl(\lambda)$. Since $cl(\lambda)$ is a fuzzy closed set, $cl[cl(\lambda)] = cl(\lambda)$. Then, $cl\,int[cl(\lambda)] = cl[cl(\lambda)$, in (X,T). Since (X,T) is a fuzzy globally disconnected space, by proposition 3.1, $cl\,int[cl(\lambda)] = cl[cl(\lambda)]$ implies that $int[cl(\lambda)] = cl(\lambda)$, in (X,T). Now, $\lambda \leq cl(\lambda)$ implies that $\lambda \leq int\,cl(\lambda)$ and hence λ is a fuzzy pre-open set in (X, T).

Proposition 3.4 If λ is a fuzzy nowhere dense set in a fuzzy globally disconnected space (X,T), then $1 - \lambda$ is a fuzzy dense and fuzzy open set in (X,T).

Proof: Let λ be a fuzzy nowhere dense set in (X,T). Then by proposition 3.2, λ is a fuzzy closed set in (X,T) and hence $1 - \lambda$ is a fuzzy open set in (X,T). Since λ is a fuzzy nowhere dense set in (X,T), by theorem 2.2, $1 - \lambda$ is a fuzzy dense set in (X,T). Hence $1 - \lambda$ is a fuzzy dense and fuzzy open set in (X,T).

Proposition 3.5 If λ is a fuzzy first category set in a fuzzy globally disconnected space (X,T), then λ is a fuzzy F_{σ}-set in (X,T).

Proof: Let λ be a fuzzy first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$,where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy globally disconnected space, by proposition 3.2, the fuzzy nowhere dense sets (λ_i)'s are fuzzy closed sets in (X,T) and hence $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy closed sets in (X,T), implies that λ is a fuzzy F_σ-set in (X,T).

Proposition 3.6 If λ is a fuzzy residual set in a fuzzy globally disconnected space (X,T), then λ is a fuzzy G_δ-set in (X,T).

Proof: Let λ be a fuzzy residual set in (X,T). Then, $1- \lambda$ is a fuzzy first category set in (X,T). Since (X,T) is a fuzzy globally disconnected space, by proposition 3.5, the fuzzy first category set $1-\lambda$ is a fuzzy F_σ-set in (X,T) and hence λ is a fuzzy G_δ-set in (X,T).

Proposition 3.7 If λ is a fuzzy nowhere dense set in a fuzzy globally disconnected space (X,T), then λ is a fuzzy simply open set in (X,T).

Proof : Let λ be a fuzzy nowhere dense set in (X,T). Then, int cl(λ) $=0$, in (X, T). Since (X, T) is a fuzzy globally disconnected space, by proposition 3.4, $1 - \lambda$ is a fuzzy dense and fuzzy open set in (X,T). Then, cl $(1 - \lambda)= 1$ and int$(\lambda) = \lambda$ in (X, T). Now int cl[bd(λ)]=int cl[cl $(\lambda) \wedge$ cl $(1 - \lambda)$] = int cl [cl $(\lambda) \wedge 1$)] = int cl [cl (λ)] = int cl (λ) =0. Thus, int cl [bd(λ)] =0, implies that λ is a fuzzy simply open set in (X, T).

Proposition 3.8 If λ is a fuzzy semi-closed set in a fuzzy globally disconnected space (X, T), then λ is a fuzzy closed set in(X, T).

Proof: Let λ be a fuzzy semi-closed set in (X, T). Then $1-\lambda$ is a fuzzy semi-open set in (X, T). Since (X,T) is a fuzzy globally disconnected space, $1 - \lambda$ is a fuzzy open set in (X,T) and hence λ is a fuzzy closed set in (X,T).

Proposition 3.9 If λ is a fuzzy α-open set in a fuzzy globally disconnected space (X,T), then λ is a fuzzy open set in (X,T).

Proof: Let λ be a fuzzy α-open set in (X,T). Since each fuzzy α-open set is a fuzzy semi-open set in a fuzzy topological space, λ is a fuzzy semi-open set in (X, T). Since (X, T) is a fuzzy globally disconnected space, λ is a fuzzy open set in (X,T).

4. FUZZY GLOBALLY DISCONNECTED SPACES AND FUZZY BAIRE SPACES

Definition 4.1 [12] A fuzzy topological space (X,T) is called a fuzzy node space if every non-zero fuzzy nowhere dense set is fuzzy closed in (X,T).

Proposition 4.1 If (X, T) is a fuzzy globally disconnected space, then (X,T) is a fuzzy Nodec space.

Proof : Let λ be a fuzzy nowhere dense set in (X,T). Since (X,T) is a fuzzy globally disconnected space, by proposition 3.2, λ is a fuzzy closed set in (X,T) and hence fuzzy nowhere dense sets in (X,T) are fuzzy closed sets, implies that (X,T) is a fuzzy Nodec space.

Theorem 4.1 [11] Let (X,T) be a fuzzy topological space. Then the following are equivalent:

(1) (X,T) is a fuzzy Baire space.
(2) Int (λ)= 0 ,for every fuzzy first category set λ in (X,T).
(3) Cl (μ)= 1 ,for every fuzzy residual set μ in (X,T).

Proposition 4.2 If (X,T) is a fuzzy globally disconnected and fuzzy Baire space and λ is a fuzzy residual set in (X,T), then λ is a fuzzy dense and fuzzy G_δ-set in (X,T).

Proof: Let λ be a fuzzy residual set in (X,T). Since (X,T) is a fuzzy globally disconnected space, by proposition 4.1, λ is a fuzzy G_δ-set in (X,T). Also since (X,T) is a fuzzy Baire space, by theorem 2.3, $\mathrm{cl}\,(\lambda) = 1$, in (X,T). Hence, the fuzzy residual set λ is a fuzzy dense and fuzzy G_δ-set in (X,T).

Proposition 4.3 If (X,T) is a fuzzy globally disconnected and fuzzy Baire space and λ is a fuzzy first category set in (X,T), then λ is a fuzzy σ-nowhere dense set in (X,T).

Proof: Let λ be a fuzzy first category set in (X,T). Since (X,T) is a fuzzy Baire space, by theorem-4.1,int $(\lambda) = 0$,in (X,T). Also since (X,T) is a fuzzy globally disconnected space, by proposition- 3.5, the fuzzy first category set λ is a fuzzy F_σ-set in (X,T). Hence λ is a fuzzy F_σ-set in (X,T) with int $(\lambda) = 0$. Therefore λ is a fuzzy σ-nowhere dense set in (X,T).

Proposition 4.4 If (X,T) is a fuzzy globally disconnected and fuzzy Baire space and int $[\bigvee_{k=1}^{\infty}(\lambda_k)] = 0$,where $(\lambda_k)'s$ are fuzzy first category sets in (X,T), then (X,T) is a fuzzy σ-Baire space.

Proof: Let $(\lambda_k)'s$ be fuzzy first category sets in (X,T). Since (X,T) is a fuzzy globally disconnected and fuzzy Baire space, by proposition 4.3, $(\lambda_k)'s$ are fuzzy σ-nowhere dense sets in (X,T). Hence int $[\bigvee_{k=1}^{\infty}(\lambda_k)] = 0$, where $(\lambda_k)'s$ are fuzzy σ-nowhere dense sets in (X,T), implies that (X,T) is a fuzzy σ-Baire space.

Proposition 4.5 If cl $[\bigwedge_{k=1}^{N}(\lambda_k)] = 1$, where (λ_k)'s are fuzzy residual sets in a fuzzy globally disconnected and fuzzy Baire space (X,T), then (X,T) is a fuzzy Volterra space.

Proof. Let (λ_k)'s (k=1 to N) be fuzzy residual sets in (*X*, *T*). Since (*X*, *T*) is a fuzzy globally disconnected and fuzzy Baire space, by proposition 4.2, the fuzzy residual sets (λ_k)'s are fuzzy dense and fuzzy G_δ-sets in (*X*,*T*). Hence cl $[\wedge_{k=1}^{N}(\lambda_k)] = 1$, where ($\lambda_k$)'s are fuzzy dense and fuzzy G_δ-sets in (*X*,*T*), implies that (*X*,*T*) is a fuzzy Volterra space.

5. FUZZY GLOBALLY DISCONNECTED SPACES, FUZZY STRONGLY IRRESOLVABLE SPACES AND FUZZY HYPER–CONNECTED SPACES:

Theorem 5.1 [16] If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy strongly irresolvable space (*X*,*T*), then $(1 - \lambda)$ is a fuzzy nowhere dense set in (X, T).

Theorem 5.2 [16] If $\lambda \leq \mu$,where μ is a fuzzy set and λ is a fuzzy dense set in a fuzzy strongly irresolvable space (*X*,*T*), then $1 - \mu$ is a fuzzy nowhere dense set in (*X*, *T*).

Theorem 5.3 [16] If λ is a fuzzy first category set in a fuzzy Baire and fuzzy strongly irresolvable space (*X*, *T*), then λ is a fuzzy nowhere dense set in (*X*, *T*).

Proposition 5.1 If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy strongly irresolvable and fuzzy globally disconnected space (*X*,*T*), then λ is a fuzzy open set in (X, T).

Proof : Let λ be a fuzzy dense and fuzzy G_δ- set in (*X*,*T*). Since (*X*,*T*) is a fuzzy strongly irresolvable space, by theorem 5.1, $(1 - \lambda)$ is a fuzzy nowhere dense set in (*X*, *T*). Also since (*X*,*T*) is a fuzzy globally disconnected space, by proposition 3.2, $(1 - \lambda)$ is a fuzzy closed set in (*X*,*T*). Then λ is a fuzzy open set in (*X*, *T*).

Proposition 5.2 If $\lambda \leq \mu$ where μ is a fuzzy set and λ is a fuzzy dense set in a fuzzy strongly irresolvable and fuzzy globally disconnected space (*X*,*T*), then μ is a fuzzy open and fuzzy dense set in (*X*,*T*).

Proof : Let λ be a fuzzy dense set in (*X*,*T*) such that $\lambda \leq \mu$,where μ is a fuzzy set defined on *X*. Since (*X*,*T*) is a fuzzy strongly irresolvable space, by theorem 5.2, $1 - \mu$ is a fuzzy nowhere dense set in (*X*, *T*). Also since (*X*,*T*) is a fuzzy globally disconnected space, by proposition 3.2, $1 - \mu$ is a fuzzy closed set in (X,T)and hence μ is a fuzzy open set in (*X*,*T*). Now $\lambda \leq \mu$, implies that cl $(\lambda) \leq$ cl(μ) and then $1 \leq$ cl(μ).That is, cl$(\mu) = 1$, in (*X*, *T*). Hence μ is a fuzzy open and fuzzy dense set in (*X*,*T*).

Proposition 5.3 If λ is a fuzzy first category set in a fuzzy Baire, fuzzy strongly irresolvable and fuzzy globally disconnected space (*X*, *T*), then λ is a fuzzy closed set in (*X*,*T*).

Proof : Let λ be a fuzzy first category set in (X,T). Since (X,T) is a fuzzy Baire and fuzzy strongly irresolvable space, by proposition 5.3, λ is a fuzzy nowhere dense set in (X,T). Also since (X,T) is a fuzzy globally disconnected space, by proposition 3.2, λ is a fuzzy closed set in (X,T).

Theorem 5.4 [16] Let (X,T) be a fuzzy topological space. Then the following are equivalent:
(i). (X, T) is a fuzzy strongly irresolvable space.
(ii). Each fuzzy dense set in (X,T) is fuzzy semi-open in (X, T).

Proposition 5.4 If a fuzzy topological space (X,T) is a fuzzy strongly irresolvable and fuzzy globally disconnected space, then (X,T) is a fuzzy submaximal space.

Proof : Let λ be a fuzzy dense set in (X, T). Since (X,T) is a fuzzy strongly irresolvable space,by theorem 5.4, λ is a fuzzy semi-open set in (X,T). Also since (X,T) is a fuzzy globally disconnected space, the fuzzy semi-open λ is a fuzzy open set in (X, T). Hence the fuzzy dense set λ is a fuzzy open set in (X, T), implies that (X, T) is a fuzzy submaximal space.

Theorem 5.5 [6] Let (X,T) be a fuzzy topological space. Then the following conditions are equivalent:
(i) (X,T) is fuzzy hyper-connected.
(ii) Every fuzzy pre-open set is fuzzy dense set.

Proposition 5.5 If $\text{cl int}[\text{cl}(\lambda)] = \text{cl}(\lambda)$, for a fuzzy set λ defined on X in a fuzzy globally disconnected and fuzzy hyper-connected space (X,T), then λ is a fuzzy dense set in (X,T).

Proof: Let λ be a fuzzy set defined on X such that $\text{cl int}[\text{cl}(\lambda)] = \text{cl}(\lambda)$. Since (X,T) is a fuzzy globally disconnected space , by proposition 3.3, λ is a fuzzy pre-open set in (X,T). Also since (X,T) is a fuzzy hyper-connected space, by theorem 4.6, the fuzzy pre-open set λ is a fuzzy dense set in (X,T).

Theorem 5.6 [6] In a fuzzy hyper-connected space (X,T), any subset λ of X is a fuzzy semi-open set if $\text{int}(\lambda) \neq 0$.
Proposition 5.6 If $\text{int}(\lambda) \neq 0$, for a fuzzy set λ defined on X in a fuzzy globally disconnected and fuzzy hyper-connected space (X,T), then λ is a fuzzy open set in (X,T).

Proof : Let λ be a fuzzy set defined on X such that $\text{int}(\lambda) \neq 0$. Since (X,T) is a fuzzy hyper-connected space, by theorem 5.6, λ is a fuzzy semi-open set in (X,T). Since (X,T) is a fuzzy globally disconnected space, the fuzzy semi-open set λ is a fuzzy open set in (X,T).

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Effects of Magnetic Field and Radiation on Peristaltic Flow of Non-Newtonian Fluid and Heat Transfer

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Abstract: The present paper investigates a two dimensional peristaltic flow of non-Newtonian fluid and heat transfer through a porous channel under the influence of magnetic field and radiation. The heat source term has been considered in the energy equation. The flow is investigated in a wave frame of reference moving with a velocity of the wave. The governing equations of motion and energy are simplified using long wave length and low Reynolds number approximation. The transformed equations have been solved numerically for velocity, temperature and heat transfer coefficient. The effects of various parameters are illustrated graphically and discussed from the physical point of view. The numerical results of heat transfer coefficient are also tabulated. The study reveals that the velocity decreases at the central region of the channel with the increase of magnetic field parameter and temperature decreases for increasing thermal radiation parameter.

Keywords: Viscosity, Numerical solution, Casson fluid, Radiation.

Introduction:

The study of peristaltic flow of non-Newtonian fluids through a porous medium has great attention to the researchers due to their vast applications in engineering and industry. This mechanism is naturally occurred with a progressive wave of area expansion and contraction along the length of a fluid filled channel/tube, moving the mixed fluid in the direction of the wave propagation. It has a vital role in physiology, for examples, urine transports from kidney to bladder, swallowing food material through esophagus, semen movement in the vas deferens of male reproductive tract, ovum movement in the fallopian tube and blood circulation in small blood vessels. It was reported by De Vries *et al*. [1] that the intra-uterine contraction due to the myometrial (myometrium is the middle layer of the uterine wall) contraction is peristaltic type motion and this myometrial contraction may occur in both symmetric and asymmetric directions. Latham [2] conducted the primary effort on peristaltic mechanism in a viscous fluid and after that we found many analytical and experimental studies [3-10].

We know that blood carries a large quantity of heat to different parts of our body. When we work hard and when our body is exposed to extreme heat environment, then the blood flow of our body increases. It is also experimentally known that when the surrounding temperature crosses 20℃, heat transfer takes places from our skin surface and the temperature is below 20℃, our body loses heat. So we can say that the study of heat transfer analysis is a significant area in connection with peristaltic motion. Such flows with heat transfer have many applications in biomedical sciences and industry such as analysis of tissues, blood oxygenation, dialysis, crude oil refinement and food processing [11].

On the other hand, blood in small vessels and fluids in the intestine, urine under certain conditions behave as non-Newtonian fluids. Human blood shows this property mainly due to the suspension of red blood cells in the plasma. But the mechanism of non-Newtonian fluids is complex. So it is not possible to show all non-Newtonian fluid properties in a single constitutive equation. Thus a number of non-Newtonian fluid models have been proposed [12]. Casson fluid is one of the non-Newtonian fluids which behaves like a elastic solid and a yield stress exists in its constitutive equation. This model was first introduced by Casson [13]. Blair *et al*. [14] was reported that human blood can be presented by Casson's model. To the best of our knowledge, no investigation is made to the peristaltic flow of non-Newtonian fluid and heat transfer in a porous asymmetric channel with magnetic field and radiation effects. The main objective is to study the peristaltic flow of non-Newtonian fluid and heat transfer through a porous asymmetric channel in presence of magnetic field and radiation. The governing equations are reduced under low Reynolds number and long wave length approximation. The transformed equations have been solved numerically for velocity, temperature and heat transfer coefficient. The effects of various important parameters are displayed graphically and discussed.

Mathematical Formulation:

We consider the peristaltic flow of non-Newtonian fluid through a porous medium in two dimensional asymmetric channel. Let (U, V) be the velocity components in the stationary frame (X, Y) such that X is taken along the axis of the channel and Y perpendicular to it. The flow is supposed to be induced by a progressive sinusoidal wave train propagating with a constant speed c along the channel walls. The geometry (in Fig.1) of the upper and lower wall surfaces are assumed to be

$$Y = H_1 = d_1 + a_1 cos\left\{\frac{2\pi}{\lambda}(X - ct)\right\} \tag{1}$$

$$Y = H_2 = -d_2 - a_2 cos\left\{\frac{2\pi}{\lambda}(X - ct) + \phi\right\} \tag{2}$$

where a_1, a_2 be the wave amplitudes, λ is the wave length, $d_1 + d_2$ is the channel width, t is the time and ϕ is the phase difference $(0 \leq \phi \leq \pi)$. A uniform magnetic field strength B_0 is applied in Y −direction of the flow and then the induced magnetic field is assumed to be negligible.

The constitutive equation [15] for non-Newtonian Casson fluid is defined as

$$\tau_{ij} = 2\left(\mu_b + \frac{P_y}{\sqrt{2\pi}}\right) e_{ij} \tag{3}$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial y_i}\right)$ is the (i, j)th component of deformation rate, τ_{ij} is the (i, j)th component of the stress tensor, π is the product of the component of deformation rate with itself, and μ_b is the plastic dynamic viscosity. The yield stress P_y is expressed as $P_y = \frac{\mu_b\sqrt{2\pi}}{\beta}$, where β Casson fluid parameter. For non-Newtonian Casson fluid flow $\mu = \mu_b + \frac{P_y}{\sqrt{2\pi}}$ which gives $\vartheta' = \vartheta\left(1 + \frac{1}{\beta}\right)$, where $\vartheta = \frac{\mu_b}{\rho}$ is the kinematic viscosity for Casson fluid. Again $P_y = 0$ for Newtonian case.

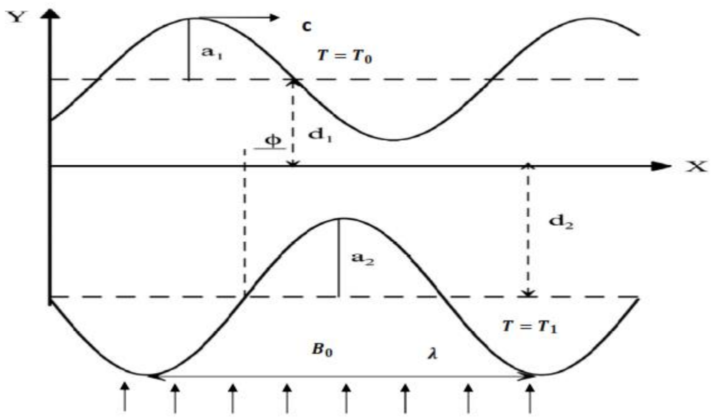

Fig.1: Geometry of the problem

With all the above mentioned considerations, the governing equations for of non-Newtonian Casson fluid are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{4}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}$$
$$= -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\left(1+\frac{1}{\beta}\right)\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) - \frac{\sigma B_0^2}{\rho}U - \nu\left(1+\frac{1}{\beta}\right)\frac{U}{K\prime} + g\beta_t(T-T_0) \quad (5)$$
$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(1+\frac{1}{\beta}\right)\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) - \nu\left(1+\frac{1}{\beta}\right)\frac{V}{K\prime} \quad (6)$$
$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2}\right) + \frac{Q\prime}{\rho C_p} - \frac{1}{\rho C_p}\left(\frac{\partial q_r}{\partial X} + \frac{\partial q_r}{\partial Y}\right) \quad (7)$$

The corresponding boundary conditions for velocity and temperature are

$$\left.\begin{array}{ll} U = 0, \quad T = T_0, & \text{at } Y = H_1 \\ U = 0, \quad T = T_1, & \text{at } Y = H_2 \end{array}\right\} \quad (8)$$

where, B_0 is the uniform magnetic field strength, σ is the electric conductivity, ρ is the fluid density, K' is the permeability of the porous space, g is the acceleration due to gravity, β_t is the coefficient of thermal expansion, T is the temperature, Q' is the heat absorption/addition, C_p is the specific heat at constant pressure, k is the thermal conductivity.

The radiative heat flux q_r in X direction is considered as negligible compared Y direction. Using Rosseland approximation for radiation we can get

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} \quad (9)$$

where σ^* is the Stefan- Boltzman constant and $3k^*$ is the absorption coefficient. Here we consider the temperature difference within the flow is very small such that T^4 may be expanded as a linear function of temperature. Using Taylor series and neglecting the higher order terms, we get, $T^4 \cong 4T_0^3 T - 3T_0^4$. Thus equation (9) implies

$$q_r = \frac{-16\sigma^* T_0^3}{3k^*}\ \frac{\partial T}{\partial Y} \quad (10)$$

The flow is unsteady in the stationary frame (X, Y) but it is assumed to be steady in the wave frame (x, y). The variables of these two frames are related to

$$x = X - ct, y = Y, u = U - c, v = V, p = P \quad (11)$$

To minimize the complexity of the governing equations, the following non-dimension variables and parameters are used.

$$\left.\begin{array}{c} x' = \frac{x}{\lambda}, y' = \frac{y}{d_1}, u' = \frac{u}{c}, v' = \frac{v}{c\delta}, \delta = \frac{d_1}{\lambda}, t' = \frac{ct}{\lambda}, p' = \frac{pd_1^2}{\lambda c\mu_b} \\ h_1 = \frac{H_1}{d_1}, h_2 = \frac{H_2}{d_1}, d = \frac{d_2}{d_1}, a = \frac{a_1}{d_1}, b = \frac{a_2}{d_1}, \theta = \frac{T-T_0}{T_1-T_0} \end{array}\right\} \quad (12)$$

The governing equations $(4)-(7)$ under the assumptions of long wave length and low Reynolds number in terms of stream function ψ (dropping the das symbols) become

$$\frac{\partial p}{\partial x} = \left(1+\frac{1}{\beta}\right)\left[\frac{\partial^3 \psi}{\partial y^3} - \alpha^2\left(\frac{\partial \psi}{\partial y} + 1\right)\right] + Gr\theta \quad (13)$$

$$\frac{\partial p}{\partial y} = 0 \tag{14}$$

$$\left(\frac{1}{Pr} + Rd\right)\frac{\partial^2 \theta}{\partial y^2} + Q_0 = 0 \tag{15}$$

The reduced boundary conditions became

$$\left.\begin{array}{llll} \psi = \frac{q}{2}, & \frac{\partial \psi}{\partial y} = -1, & \theta = 0, & \text{at } y = h_1 = 1 + a\cos 2\pi x \\ \psi = \frac{-q}{2}, & \frac{\partial \psi}{\partial y} = -1, & \theta = 1, & \text{at } y = h_2 = -d - b\cos(2\pi x + \phi) \end{array}\right\} \tag{16}$$

where $\alpha^2 = \frac{M^2}{1+1/\beta} + \frac{1}{K}$, $M = \sqrt{\frac{\sigma}{\mu_b}} B_0 d_1$ is the magnetic field parameter, $K = \frac{K\prime}{d_1^2}$ is the permeability parameter, q is the volume flow rate in the wave frame, $Re = \frac{\nu}{cd_1}$ is the Reynolds number, $Pr = \frac{\rho \nu C_p}{k}$ is the Prandtl number, $Gr = \frac{\rho g \beta_t (T_1 - T_0)}{\mu_b c}$ is the Grash of number, $Rd = \frac{16\sigma^* T_0^3}{3k^* \mu_b c}$ is the thermal radiation parameter and $Q_0 = \frac{Q\prime d_1^2}{k(T_1 - T_0)}$ is the heat generation parameter.

The dimensionless volume flow rate q is given by

$$q = \int_{h_2}^{h_1} u dy = \int_{h_2}^{h_1} \frac{\partial \psi}{\partial y} dy = \psi(h_1) - \psi(h_2) \tag{17}$$

The instantaneous flux at any axial situation is given by

$$Q = \int_{h_2}^{h_1} (u+1) dy = q + h_1 - h_2 \tag{18}$$

The average flux $\bar{Q}$ over one period ($T = \lambda/c$) is defined by

$$\bar{Q} = \frac{1}{T}\int_0^T Q dt = q + 1 + d \tag{19}$$

Again the heat transfer coefficient (Nusselt number) Z_1 at the upper wall is

$$Z_1 = h_{1x}\theta' \tag{20}$$

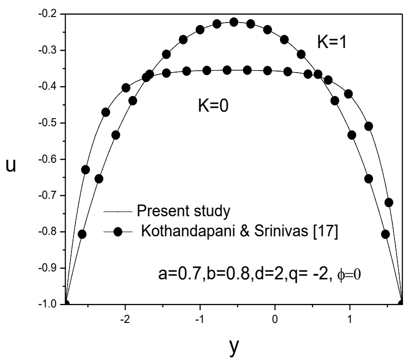

Fig. 2: Comparison of velocity profiles

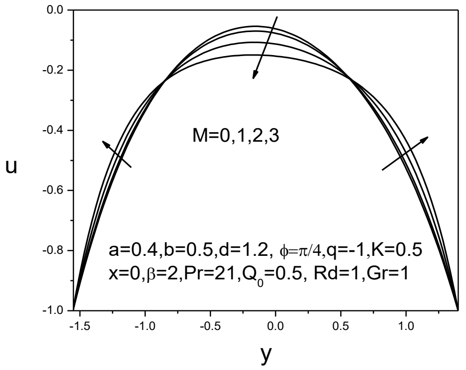

Fig. 3: Effect of M on velocity profile

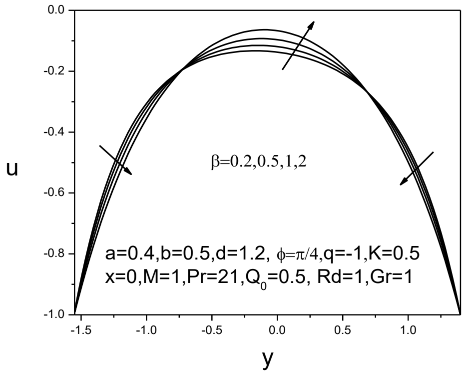

Fig. 4: Effect of β on velocity profile

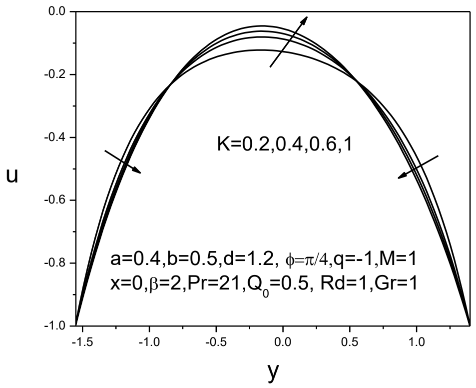

Fig. 5: Effect of K on velocity profile

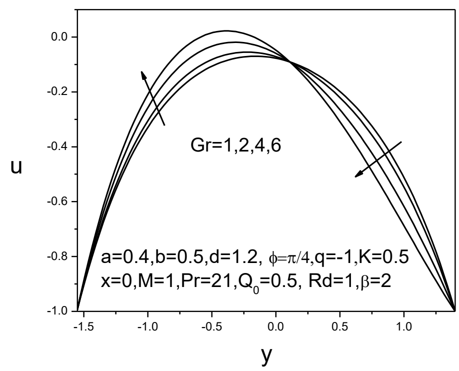

Fig. 6: Effect of Gr on velocity profile

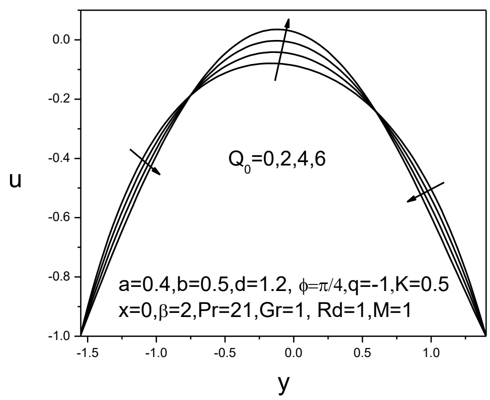

Fig. 7: Effect of Q_0 on velocity profile

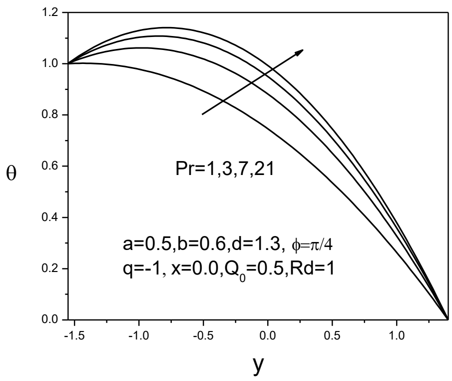

Fig. 8: Effect of Pr on temperature profile

Numerical Solution:

In order to get a numerical solution, first we eliminate the pressure terms from (13) and (14). A user defined code is written in MATLAB for numerical solution. We know that our body temperature is $T = 310\,K$ and the value of Prandtl number for human blood is $Pr = 21$ [16]. So Pr is kept 21 in this study. Also $a = 0.4$, $b = 0.5$, $d = 1.2$, $q = -1$, $x = 0.0$, $M = 1$, $K = 0.5$, $\beta = 2$, $Rd = 1$, $\phi = \pi/4$, $Q_0 = 0.3$ and $Gr = 1$ are taken fixed. Numerical computations have been carried out for various values of Magnetic field parameter (M), Casson fluid parameter (β), permeability parameter (K), Grashof number (Gr), heat generation parameter (Q_0), Prandtl number (Pr) and thermal radiation parameter (Rd). The software ORIGIN has been used to show the numerical results graphically. Fig.2 gives the comparison between the results obtained in the present study and the results of previous study [17]. To do so, both the studies have been brought to the same platform, by considering equal parameter values (Newtonian case).

Results and Discussion

Figs.3-7 represent the velocity profiles under the effect of Magnetic field parameter (M), Casson fluid parameter (β), permeability parameter (K), Grashof number (Gr) and heat generation parameter (Q_0). Hear we see that the characteristics of velocity due to these

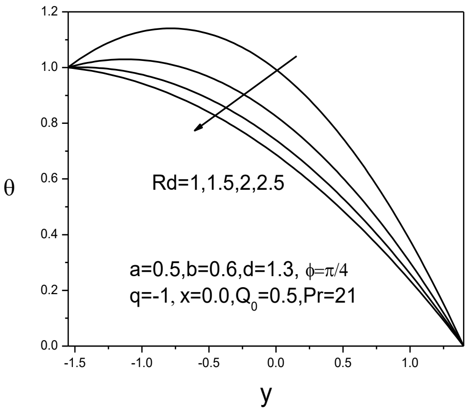

Fig. 9: Effect of Rd on temperature profile

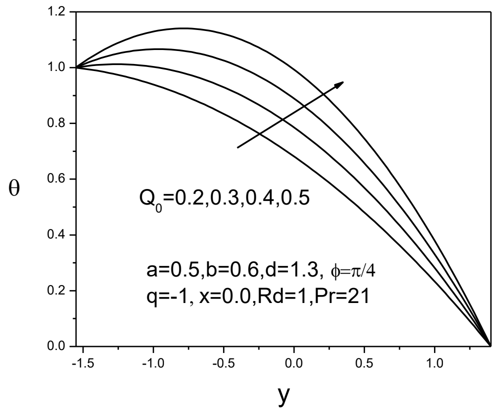

Fig. 10: Effect of Q_0 on temperature profile

parameters are quite different at the walls and near the centre of the channel. Fig. 3 is illustrated to see the effect of M on velocity profiles. It is noticed that the velocity profile declined at the central region with an increase in M. The reason is that the applied magnetic field produces a resistive force to the flow and this force diminishes the velocity of the fluid. Again an increase in velocity is noticed with increase in β near the centre of the channel while opposite behavior is observed towards the walls as seen in Fig. 4. The viscosity of Casson fluid is directly proportional to β and the fluid becomes thicker and becomes more viscous when we increase the values of β. Consequently the velocity at the centre of the channel increased. The variation of velocity for various values of K is explained in Fig. 5. It is evident from this figure that velocity is an increasing function of K. This is due to the fact that large K provides less resistance to the flow and accordingly there is a rise in the velocity of fluid near the middle of the channel. Fig. 6 shows the influence of Gr on axial velocity profiles. It is observed that the axial velocity decreases above the central line region of the channel but increases below it for large Gr. The Grashof number is proportional to buoyancy force and this buoyancy force accelerates the flow below the central line and decelerates it above the central line. Fig. 7 demonstrates that the velocity increases as the increasing values of heat generation parameter Q_0 near the middle of the channel.

Figs. 8-10 show the temperature profiles for different values of Prandtl number (Pr), thermal radiation parameter (Rd) and heat generation parameter (Q_0). Hear we see that temperature profiles sharply increases with an increase in Pr as seen in Fig. 8. It indicates that larger values of Pr correspond to the situation of less heat transfer from the boundary (wall) of the fluid. By definition, Pr is ratio of kinematic viscosity to thermal diffusivity. When $Pr > 1$, the momentum diffusivity surpasses the thermal diffusivity and this increases the heat transport in the channel. Fig. 9 presents the influence of thermal radiation parameter Rd on temperature profiles. This figure shows the decreasing nature of temperature as Rd increases. This is due to the loss of heat. Fig. 10 is plotted to represent the variation of temperature profiles for different values of heat generation parameter (Q_0). It is clear from this figure that temperature profile increases for large values of Q_0. That means when heat generates during the fluid flow there is a significant increase in the thickness of thermal boundary layer.

The numerical results for heat transfer coefficient Z_1 at upper wall have been prepared in Table 1.

Table 1: Heat transfer coefficient Z_1 at upper wall (at $y = h_1$)

$a = 0.4, b = 0.5, d = 1.2, \phi = \pi/3, q = -1, x = 0.1$			
Q_0	Rd	Pr	Z_1
0.1	1	21	0.75122
0.3	1	21	1.11811
0.7	1	21	1.85187
0.5	1.5	21	1.18867
0.5	2	21	1.03705
0.5	2.5	21	0.94496

Conclusion:

In this study, we have discussed peristaltic flow of non-Newtonian fluid and heat transfer through a porous asymmetric channel under the influence of magnetic field and radiation. Heat source term was considered. Numerical solutions for velocity, temperature and heat transfer coefficient have been developed. The features of flow characteristics are analyzed and discussed by sketching graphs. We have concluded the following important conclusions:

1. Velocity decreases at the central region of the channel and increases near the wall for increasing M. Also reverse behavior is seen for K and Q_0.
2. An increase in Rd result a decrease in temperature profiles.

3. Q_0 has increasing effect on temperature.
4. The heat transfer coefficient decreases at the upper wall when Rd increases.

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More on Intuitionistic Fuzzy Set

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Abstract: Behavior of two modal operators over the operations like complement, union, intersection, addition, multiplication, difference, symmetric difference, Cartesian product etc. in intuitionistic fuzzy set are rigorously studied and some new interesting relations are achieved. Some properties regarding normalization of intuitionistic fuzzy set are also discussed here.

Keywords: Intuitionistic fuzzy set, Modal operators, Normalization.

1. INTRODUCTION

In 1965, L.A.Zadeh [5] first introduced the concept of Fuzzy set. Atanassov [1] generalized this concept into intuitionistic fuzzy set in 1983. Since then so many authors [3,4] are concentrating as well as developing the concepts like algebraic laws of IFSs, basic operations on IFSs, modal operators and normalization of IFSs etc. In section two of this paper, we have established various interesting results using the modal logic operators over the well-known operations. Moreover the notion of normalization on intuitionistic fuzzy set are discussed in section three and various relations in this regard are also achieved.

2. Modal operators in IFS

We used some properties from the papers [2],[3], [4]. We can also prove the following.

Let A and B be two IFSs in a nonempty set X. Then

$$(A \ \Delta \ B) = A - B \text{ iff } B \subset A.$$

$$(A \ \Delta \ B) = B - A \text{ iff } A \subset B.$$

Here A, B ∈ IFSs means $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >: x \in X\}$.

Theorem 2.1 Let X be a nonempty set. A, B be two IFSs of X, then

(a) $(\square A)^C = \Diamond (A^C)$ (b) $(\Diamond A)^C = \square (A^C)$

Proof Obvious.

Theorem 2.2 Let X be a nonempty set. If A and B be two IFSs drawn from X, then,

(a) $[\square (A\cup B)]^C = \Diamond A^C \cap \Diamond B^C$ (b) $[\Diamond (A\cup B)]^C = \square A^C \cap \square B^C$

(c) $[\square (A\cap B)]^C = \Diamond A^C \cup \Diamond B$ (d) $[\Diamond (A\cap B)]^C = \square A^C \cup \square B^C$

Proof (a) $\square (A\cup B) = \square\{< x, \max(\mu_A(x), \mu_B(x)), \min (\nu_A(x), \nu_B (x)) >\}$

$= \{< x, \max(\mu_A(x), \mu_B(x)), 1- \max(\mu_A(x), \mu_B(x)) >\}$

Therefore $[\square (A\cup B)]^C = \{< x, 1- \max(\mu_A(x), \mu_B(x)), \max(\mu_A(x), \mu_B(x)) >\}$

Again , $\Diamond A^C \cap \Diamond B^C = < x, 1- \mu_A(x), \mu_A(x) > \cap < x, 1- \mu_B(x), \mu_B(x) >$

$= \{< x, \min [((1-\mu_A(x)), (1- \mu_B(x))], \max (\mu_A(x), \mu_B(x)) >\}$

$= \{< x, 1- \max(\mu_A(x), \mu_B(x)), \max(\mu_A(x), \mu_B(x)) >\}$

Hence the result.

The proof of (b), (c) and (d) are similar to (a).

Theorem 2.3 Let A and B be two IFSs in a nonempty set X. Then

(a) $[\square (A\oplus B)]^C = \Diamond A^C \otimes \Diamond B^C$ (b) $[\Diamond (A\oplus B)]^C = \square A^C \otimes \square B^C$

(c) $[\square (A\otimes B)]^C = \Diamond A^C \oplus \Diamond B^C$ (d) $[\Diamond (A\otimes B)]^C = \square A^C \oplus \square B^C$

Proof (a) $\square(A\oplus B) = \square\{< x, \mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x), \nu_A(x)\, \nu_B(x) >\}$

$= \{< x, \mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x), 1- (\mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x)) >\}$

Therefore $[\square (A\oplus B)]^C = \{< x, 1-(\mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x)) , \mu_A(x) + \mu_B(x)-\mu_A(x)\, \mu_B(x) >\}$

Again , $\Diamond A^C \otimes \Diamond B^C = \{< x, 1- \mu_A(x), \mu_A(x) > \otimes < x, 1- \mu_B(x), \mu_B(x) >\}$

$= \{< x, [(1- \mu_A(x))\, (1- \mu_B(x))], (\mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x)) >\}$

$= \{<x, 1- (\mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x)) , \mu_A(x) + \mu_B(x) - \mu_A(x)\, \mu_B(x)>\}$

Hence the result.

The proof of (b),(c) and (d) are similar to (a).

Theorem 2.4 Let X be a nonempty set. If A and B be two IFSs drawn from X, then

(a) $(\square A)\, \Delta\, (\square B) = \square B - \square A$ if $\square A \subset \square B$ (b) $(\Diamond A)\, \Delta\, (\Diamond B) = \Diamond B - \Diamond A$ if $\Diamond A \subset \Diamond B$

(c) $(\square A)\, \Delta\, (\square B) = \square A - \square B$ if $\square B \subset \square A$ (d) $(\Diamond A)\, \Delta\, (\Diamond B) = \Diamond A - \Diamond B$ if $\Diamond B \subset \Diamond A$

Proof : The proof is straightforward.

Theorem 2.5 Let X be a nonempty set. If A and B be two IFSs drawn from X, then

(a) $\square\,(A\;\Delta\;B) = \square\,(B - A)$ if $A \subset B$ (b) $\Diamond\,(A\;\Delta\;B) = \Diamond\,(B - A)$ if $A \subset B$

(c) $\square\,(A\;\Delta\;B) = \square\,(A - B)$ if $B \subset A$ (d) $\Diamond\,(A\;\Delta\;B) = \Diamond\,(A - B)$ if $B \subset A$

Proof : Obvious.

Example 2.6 Let A= $< .6, .3, .1 >$ and B= $< .7, .1, .2 >$ so that $A \subset B$.

(a) $\square\,(A\;\Delta\;B) = < .3, .7 >$, $\square\,(A - B) = < .1, .9 >$, i.e., $\square\,(A\;\Delta\;B) \neq \square\,(A - B)$

(b) $\Diamond\,(A\;\Delta\;B) = < .4, .6 >$, $\Diamond\,(A - B) = < .3, .7 >$, i.e., $\Diamond\,(A\;\Delta\;B) \neq \Diamond\,(A - B)$

Let A= $< .8, .1, .1 >$ and B= $< .6, .3, .1 >$ so that $B \subset A$.

(c) $\square\,(A\;\Delta\;B) = < .3, .7 >$, $\square\,(B - A) = < .1, .9 >$, i.e., $\square\,(A\;\Delta\;B) \neq \square\,(B - A)$

(d) $\Diamond\,(A\;\Delta\;B) = < .4, .6 >$, $\Diamond\,(B - A) = < .2, .8 >$, i.e., $\Diamond\,(A\;\Delta\;B) \neq \Diamond\,(B - A)$

Theorem 2.7 Let X be a nonempty set and A and B be two IFSs in X such that $A \subset B$, then

(a) $\square\,(A\;\Delta\;B) = \square(A^C) - \square(B^C)$ (b) $\Diamond\,(A\;\Delta\;B) = \Diamond(A^C) - \Diamond(B^C)$ (c) $(\square\,(A\;\Delta\;B) = (\Diamond A)^C - (\Diamond B)^C$

(d) $\Diamond\,(A\;\Delta\;B) = (\square A)^C - (\square B)^C$

Proof (a) $\square(A^C) - \square(B^C) = < x, \nu_A(x), 1- \nu_A(x) > - < x, \nu_B(x), 1- \nu_B(x) >$

$= < x, \min(\nu_A(x), 1- \nu_B(x)), \max(\nu_B(x), 1- \nu_A(x)) >$

$= < x, \min(\nu_A(x), \mu_B(x)), \max(1 - \mu_B(x), 1- \nu_A(x)) >$

Again from 2.5, $\square\,(A\;\Delta\;B) = \square\,(B - A) = \square < x, \min(\mu_B(x), \nu_A(x)), \max(\nu_B(x), \mu_A(x)) >$

$= < x, \min(\mu_B(x), \nu_A(x)), 1 - \min(\mu_B(x), \nu_A(x)) >$

$= < x, \min(\nu_A(x), \mu_B(x)), \max(1 - \mu_B(x), (1- \nu_A(x)) >$.

(b) Similar to (a).

(c) $(\Diamond A)^C - (\Diamond B)^C = < x, \nu_A(x), 1- \nu_A(x) > - < x, \nu_B(x), 1- \nu_B(x) >$

$= < x, \min(\nu_A(x), 1- \nu_B(x)), \max(\nu_B(x), 1- \nu_A(x)) >$

$= < x, \min(\nu_A(x), \mu_B(x)), \max(1 - \mu_B(x)), 1- \nu_A(x)) >$

$= < x, \min(\nu_A(x), \mu_B(x)), 1 - \min(\mu_B(x), \nu_A(x)) >$

Again $\square\,(B - A) = \square < x, \min(\mu_B(x), \nu_A(x)), \max(\nu_B(x), \mu_A(x)) >$

$= < x, \min(\mu_B(x), \nu_A(x)), 1 - \min(\mu_B(x), \nu_A(x)) >$

Thus $\square\,(B - A) = \square\,(A\;\Delta\;B) = (\Diamond A)^C - (\Diamond B)^C$ [using theorem 2.5]

(d) Similar to (c).

Remark 2.8 : If $B \subset A$, the above theorem is not true, for example

Let A = $< .7, .1, .2 >$ and B = $< .4, .5, .1 >$ such that $B \subset A$.

(a) $\square\,(A\;\Delta\;B) = < .5, .5 >$ and $\square(A^C) - \square(B^C) = < .1, .9 >$

(b) $\Diamond\,(A\;\Delta\;B) = < .6, .4 >$ and $\Diamond(A^C) - \Diamond(B^C) = < .3, .7 >$

(c) $\square\,(A\;\Delta\;B) = < .5, .5 >$ and $(\Diamond A)^C - (\Diamond B)^C = < .1, .9 >$

(d) $\lozenge$ (A Δ B) = < .6, .4 > and $(\square A)^C - (\square B)^C$ = < .3, .7 >

Theorem 2.9 Let X be a nonempty set. If A and B be two IFSs in X such that $B \subset A$, then

(a) $\square (A \Delta B) = \square(B^C) - \square(A^C)$ (b) $\lozenge (A \Delta B) = \lozenge(B^C) - \lozenge(A^C)$

(c) $\square (A \Delta B) = (\lozenge B)^C - (\lozenge A)^C$ (d) $\lozenge (A \Delta B) = (\square B)^C - (\square A)^C$

Proof Obvious.

Theorem 2.10 Let X be a nonempty set. If A and B be two IFSs drawn from X, then

(a) $(\square A) \Delta (\square B) = \lozenge (A - B) \cup \lozenge (B - A)$

(b) $(\lozenge A) \Delta (\lozenge B) = \square (A - B) \cup \square (B - A)$

Proof (a) L.H.S = $(\square A) \Delta (\square B) = < x, \mu_A(x), 1 - \mu_A(x) > \Delta < x, \mu_B(x), 1 - \mu_B(x) >$ $= < x, \max[\min(\mu_A(x), 1 - \mu_B(x)), \min(\mu_B(x), 1 - \mu_A(x))], \min[\max(\mu_B(x), 1 - \mu_A(x)), \max(\mu_A(x), 1 - \mu_B(x))] >$

R.H.S = $\lozenge(A - B) \cup \lozenge(B - A) = \{\lozenge < x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x)) >\} \cup \{\lozenge < x, \min(\mu_B(x), \nu_A(x)), \max(\nu_B(x), \mu_A(x)) >\} = < x, 1 - \max(\nu_A(x), \mu_B(x)), \max(\nu_A(x), \mu_B(x)) > \cup < x, 1 - \max(\nu_B(x), \mu_A(x)), \max(\nu_B(x), \mu_A(x)) > = < x, \min(1 - \nu_A(x), 1 - \mu_B(x)), \max(\nu_A(x), \mu_B(x)) > \cup < x, \min(1 - \nu_B(x), 1 - \mu_A(x)), \max(\nu_B(x), \mu_A(x)) > = < x, \min(\mu_A(x), 1 - \mu_B(x)), \max(\nu_A(x), \mu_B(x)) > \cup < x, \min(\mu_B(x), 1 - \mu_A(x)), \max(\nu_B(x), \mu_A(x)) > = < x, \max[\min(\mu_A(x), 1 - \mu_B(x)), \min(\mu_B(x), 1 - \mu_A(x))], \min[\max(\mu_B(x), 1 - \mu_A(x)), \max(\mu_A(x), 1 - \mu_B(x))] >$

Hence the proof.

(b) is similar to (a).

Theorem 2.11 Let X be a nonempty set. If A and B be two IFSs drawn from X, then

(a) $(\square A) \Delta (\square B) = \square (A \cup B) - \square (A \cap B)$

(b) $(\lozenge A) \Delta (\lozenge B) = \lozenge (A \cup B) - \lozenge (A \cap B)$

Proof Obvious.

Corollary 2.12 If X be nonempty and A and B be two proper IFSs of X, then

(a) $\square (A \cup B) - \square (A \cap B) = \lozenge (A - B) \cup \lozenge (B - A)$

(b) $\lozenge (A \cup B) - \lozenge (A \cap B) = \square (A - B) \cup \square (B - A)$

Proof Combining 2.10 and 2.11, (a) & (b) can be shown easily.

Remark 2.13 If X be nonempty and A and B be two IFSs of X, then,

(a) $(\square A) \Delta (\square B) \neq \square (A \Delta B)$ (b) $(\lozenge A) \Delta (\lozenge B) \neq \lozenge (A \Delta B)$

Example : Let A = < .7, .2, .1> and B= < .6, .3, .1>
(a) (□A) Δ (□ B) = < .4, .6 > and □ (A Δ B) = < .3, .7 > . So (□A) Δ (□ B) ≠ □ (A Δ B)
(b) (◊A) Δ (◊ B) = < .3, .7 > and ◊ (A Δ B) = < .4, .6 > . So (◊A) Δ (◊ B) ≠ ◊ (A Δ B).

Theorem 2.14 Let X be a nonempty set. If A and B be two IFSs drawn from X, then
(a) $\Box (A \times B) = \Box A \otimes \Box B$ (b) $\Diamond (A \times B) = \Diamond A \oplus \Diamond B$

Proof (a) L.H.S = $\Box (A \times B) = \Box \{< x, \mu_A(x)\, \mu_B(x),\ \nu_A(x)\, \nu_B(x) >\}$
$= < x, \mu_A(x)\, \mu_B(x),\ 1 - \mu_A(x)\, \mu_B(x) >$.
Again R.H.S = $\Box A \otimes \Box B = < x, \mu_A(x) , 1- \mu_A(x) > \otimes < x, \mu_B(x) , 1- \mu_B(x) >$
$= < x, \mu_A(x)\, \mu_B(x),\ (1 - \mu_A(x)) + (1- \mu_B(x)) - (1 - \mu_A(x)) (1- \mu_B(x)) >$
$= < x, \mu_A(x)\, \mu_B(x),\ 1 - \mu_A(x)\, \mu_B(x) >$. Hence the proof.
Similarly (b) can be proved.

Theorem 2.15 Let X be a nonempty set. If A and B be two IFSs drawn from X, then
(a) $(\Box A \times \Box B)^C = (\Box A)^C \times (\Box B)^C$ (b) $(\Diamond A \times \Diamond B)^C = (\Diamond A)^C \times (\Diamond B)^C$

Proof Obvious.

Theorem 2.16 Let X be a nonempty set. If A and B be two IFSs drawn from X, then
(a) $\Box (A^C) \times \Box (B^C) = (\Diamond A)^C \times (\Diamond B)^C$ (b) $\Diamond (A^C) \times \Diamond (B^C) = (\Box A)^C \times (\Box B)^C$

Proof Obvious.

Theorem 2.17 Let X be a nonempty set. If A and B be two IFSs drawn from X, then
(a) $\Box (\Box A \times \Box B) = \Box (A \times B)$ (b) $\Diamond (\Diamond A \times \Diamond B) = \Diamond (A \times B)$

Proof (a) Here $\Box A \times \Box B = < x, \mu_A(x) , 1- \mu_A(x) > \times < x, \mu_B(x) , 1- \mu_B(x) >$
$= < x, \mu_A(x)\, \mu_B(x), (1- \mu_A(x)) (1- \mu_B(x)) >$.
Therefore $\Box (\Box A \times \Box B) = < x, \mu_A(x)\, \mu_B(x), (1- \mu_A(x)\, \mu_B(x)) >$. Again $\Box (A \times B) = < x, \mu_A(x)\, \mu_B(x),\ 1 - \mu_A(x)\, \mu_B(x) >$. Thus the proof.
Similarly (b) can be proved.

3. NORMALIZATION OF IFS

Definition 3.1[4] Let X be a nonempty universal set. The normalization of an IFS A denoted by NORM(A) is defined as,

$$NORM(A) = \{<x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x) , \pi_{NORM(A)}(x) > : x \in X \},$$

where $\mu_{\text{NORM(A)}}(x) = \frac{\mu_A(x)}{\sup(\mu_A(x))}$, $\nu_{\text{NORM(A)}}(x) = \frac{\nu_A(x) - \inf \nu_A(x)}{1 - \inf \nu_A(x)}$ for X = {x}.
and $\pi_{\text{NORM(A)}}(x) = 1 - \mu_{\text{NORM(A)}}(x) - \nu_{\text{NORM(A)}}(x)$.

Example 3.2 Let X= {x_1 , x_2 , x_3 } and let IFS A be such that

A = {<.6, .2, .2 >, <.8, .1, .1 >, <.7, .2, .1 >}

Then NORM(A) = {< 3/4, 1/9, 5/36 >, <1, 0, 0 >, <7/8, 1/9, 1/72 >}

□ NORM(A) = {< 3/4, 1/4 >, <1, 0 >, <7/8, 1/8 >}

◊ NORM(A) = {< 8/9, 1/9 >, <1, 0 >, < 8/9, 1/9 >}

Thus □ NORM(A) ≠ NORM A ≠ ◊ NORM(A)

Theorem 3.3 For an IFS A of the universe X

`(i) □□NORM(A) = □NORM(A)
(ii) ◊◊ NORM(A) = ◊ NORM(A)
(iii) ◊□ NORM(A) = □NORM(A)
(iv) □◊ NORM(A) = ◊ NORM(A)

Proof Obvious.

Theorem 3.4 For IFSs A,B of the universe X

(i) (NORM(A) ∪ NORM(B)) = (NORM(B)) ∪ (NORM(A))
(ii) (NORM(A) ∩ NORM(B)) = (NORM(B)) ∩ (NORM(A))
(iii) (NORM(A) ⊕ NORM(B)) = (NORM(B)) ⊕ (NORM(A))
(iv) (NORM(A) ⊗ NORM(B)) = (NORM(B)) ⊗ (NORM(A))

Proof Obvious.

Theorem 3.5 For IFSs A,B of the universe X

(i) □ (NORM(A) ∪ NORM(B)) = □ NORM(A) ∪ □ NORM(B)
(ii) □ (NORM(A) ∩ NORM(B)) = □ NORM(A) ∩ □ NORM(B)
(iii) ◊ (NORM(A) ∪ NORM(B)) = ◊ NORM(A) ∪ ◊ NORM(B)
(iv) ◊ (NORM(A) ∩ NORM(B)) = ◊ NORM(A) ∩ ◊ NORM(B)

Proof (i) L.H.S = □ (NORM(A) ∪ NORM(B))

$= < x, \max(\mu_{\text{NORM A}}(x), \mu_{\text{NORM B}}(x)), 1 - \max(\mu_{\text{NORM A}}(x), \mu_{\text{NORM B}}(x)) >$

$= < x, \max(\mu_{\text{NORM A}}(x), \mu_{\text{NORM B}}(x)), \min(1 - \mu_{\text{NORM A}}(x)), (1 - \mu_{\text{NORM B}}(x)) >$

R.H.S = □ NORM(A) ∪ □ NORM(B)

$= \{□< x, \mu_{\text{NORM A}}(x), \nu_{\text{NORM A}}(x) >\} \cup \{□< x, \mu_{\text{NORM B}}(x), \nu_{\text{NORM B}}(x) >\}$

$= < x, \mu_{\text{NORM A}}(x), 1 - \mu_{\text{NORM A}}(x) > \cup < x, \mu_{\text{NORM B}}(x), 1 - \mu_{\text{NORM B}}(x) >$

$= < x, \max(\mu_{\text{NORM A}}(x), \mu_{\text{NORM B}}(x)), \min(1 - \mu_{\text{NORM A}}(x)), (1 - \mu_{\text{NORM B}}(x)) >$

Similarly (ii), (iii) and (iv) can be proved.

Theorem 3.6 For IFSs A, B of the universe X

(i) $(NORM(A) \cup NORM(B))^C = (NORM(A))^C \cap (NORM(B))^C$

(ii) $(NORM(A) \cap NORM(B))^C = (NORM(A))^C \cup (NORM(B))^C$

(iii) $(NORM(A) \oplus NORM(B))^C = (NORM(A))^C \otimes (NORM(B))^C$

(iv) $(NORM(A) \otimes NORM(B))^C = (NORM(A))^C \oplus (NORM(B))^C$

Proof Obvious.

Theorem 3.7 For IFSs A, B of the universe X

(i) $NORM(A) \cap (NORM(A) \cup NORM(B)) = NORM(A)$

(ii) $NORM(A) \cup (NORM(A) \cap NORM(B)) = NORM(A)$

Proof Obvious.

Theorem 3.8 For IFSs A,B of the universe X

(i) $\square\, [NORM(A) - NORM(B)] = \Diamond\, NORM(A) - \Diamond\, NORM(B)$

(ii) $\Diamond\, [NORM(A) - NORM(B)] = \square\, NORM(A) - \square\, NORM(B)$

Proof (i) L.H.S = NORM(A) - NORM(B)

$= < x, \mu_{NORM\ A}(x), \nu_{NORM\ A}(x) > - < x, \mu_{NORM\ B}(x), \nu_{NORM\ B}(x) >$

$= < x, \min(\mu_{NORM\ A}(x), \nu_{NORM\ B}(x)), \max(\mu_{NORM\ B}(x), \nu_{NORM\ A}(x)) >$

Therefore, $\square\, (NORM(A) - NORM(B))$

$= < x, \min(\mu_{NORM\ A}(x), \nu_{NORM\ B}(x)), 1 - \min(\mu_{NORM\ A}(x), \nu_{NORM\ B}(x)) >$

$= < x, \min(\mu_{NORM\ A}(x), \nu_{NORM\ B}(x)), \max(1 - \mu_{NORM\ A}(x), 1 - \nu_{NORM\ B}(x)) >$

$= < x, \min(1 - \nu_{NORM\ A}(x), \nu_{NORM\ B}(x)), \max(\nu_{NORM\ A}(x), 1 - \nu_{NORM\ B}(x)) >$

Again, R.H.S = $\Diamond\, NORM(A) - \Diamond\, NORM(B)$

$= < x, 1 - \nu_{NORM\ A}(x), \nu_{NORM\ A}(x) > - < x, 1 - \nu_{NORM\ B}(x), \nu_{NORM\ B}(x) >$

$= < x, \min(1 - \nu_{NORM\ A}(x), \nu_{NORM\ B}(x)), \max(\nu_{NORM\ A}(x), 1 - \nu_{NORM\ B}(x)) >$

Hence the proof. (ii) Similar to (i).

Theorem 3.9 For IFSs A,B of the universe X,

(i) $NORM(A \Delta B) = NORM(B \Delta A)$ (ii) $NORM(A \times B) = NORM(B \times A)$

Proof Obvious.

Remark 3.10 For IFSs A,B of the universe X,

(i) $NORM(A - B) \neq NORM(A) - NORM(B)$

(ii) $NORM(A - B) \neq NORM(B - A)$

(iii) NORM (A Δ B) ≠ NORM (A) Δ NORM (B)

(iv) NORM (A × B) ≠ NORM (A) × NORM (B)

This can be shown by the following example where X = {x_1 , x_2 , x_3 } and A and B be such that

A = {<.6, .2, .2 >, <.8, .1, .1 >, <.7, .2, .1 >} and B = {<.5, .3, .2 >, <.6, .2, .2 >, <.4, .4, .2 >}. NORM (A) = {< 3/4, 1/9 >, < 1,0 >, < 7/8, 1/9 >}

NORM (B) = {< 5/6, 1/8 >, < 1,0 >, < 2/3, 1/4 >}.

Now

(i) NORM (A – B) = {< 1/5, 1/2 >, < 3/5,1/3 >, < 1, 0 >}
NORM (A) – NORM (B) = {< 1/8, 5/6 >, < 0, 1 >, < 1/4, 2/3 >}

(ii) NORM (A – B) = {< 1/5, 1/2 >, < 3/5,1/3 >, < 1, 0 >}
NORM (B – A) = {< 1, 0 >, < 1/2,1/2 >, < 1, 0 >}

(iii) NORM (A Δ B) = {< 3/4, 1/6 >, < 1/2,1/3 >, < 1, 0 >}
NORM (A) Δ NORM (B) = {< 1/8, 3/4 >, < 0, 1 >, < 1/4, 2/3 >}

(iv) NORM (A × B) = {< 1/16, 2/49 >, < 1, 0 >, < 7/12, 3/49 >}
NORM (A) × NORM (B) = {< 5/8, 1/72 >, < 1, 0 >, < 7/12, 1/36 >}

Conclusion : The results which are achieved in this paper will undoubtedly develop the literature. These will be very helpful for studying other new topics of IFSs further.

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Fuzzy Logic Approach to Digital Image Processing

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One picture is worth more than ten thousand word – Anonymous
(An Image means a picture)

Abstract

The Objective of this study is to show the application of fuzzy logic in Image processing with a brief introduction to topology, Digital topology, digital image processing, fuzzy sets and fuzzy logic. Fuzzy logic, one of the decision-making techniques of artificial intelligence, has been proven to be applicable into our everyday life and in almost all scientific fields.

A digital image is a function $f : Z{\times}Z \rightarrow [0,1,\ldots N\text{-}1]$ where N – 1 is a positive whole number belonging to the natural interval[1, 256]. Digital image processing is an expanding and dynamic area with applications into our daily life such as medicine, space exploration, surveillance, authentication, automated industry and many more application areas.

In image processing an object in the 2D or 3D-space is represented digitally by a set of ***pixels*** (picture elements) or ***voxels*** (volume elements). The properties of the set of pixels or voxels that correspond to topological properties of the original object are called **Digital Topology.**

Fuzzy image processing is the collection of all approaches that understand, represent and process the images, their segments and features as fuzzy sets. The representation and processing depend on the fuzzy technique and on the problem to be solved. Fuzzy image processing consists of image fuzzification, modification of membership values and image defuzzification. After the image data are transformed from grey level plane to the membership plane(fuzzification), appropriate fuzzy techniques modify the membership values.

Key words: Some concepts of Topology, Digital Topology, Image processing, Fuzzy image processing.

AMS Sub-classification: 03B52, 03E72, 54A, 68U10

1. Introduction

The aim of this talk is to give an introduction of the field of Fuzzy logic approach to the digital image processing. In order to make this talk self-contained, the following topics are discussed in a nutshell.

1.Topology 2. Digital Topology 3. Image Processing 4. Fuzzy Sets 5. Fuzzy Logic

1.1 Topology

There are many interesting and appealing applications of purely topological notions that are useful in theoretical computer science, such as digital image processing, picture processing, computer graphics, Computerized Axial Tomography (CAT) images, computer networks, etc.

(a) Topological Spaces based on open subsets:

In a non-empty set X, a topology consists of a collection τ of subsets of X, which satisfies the following conditions:

a) $\phi, X \in \tau$.
b) The finite intersection of τ members is a member of τ.
c) The union of arbitrary of τ members is a member of τ.

If τ is a topology for X, then the pair (X, τ) is called a topological Spaces. The members of τ are called τ -open or open with respect to the topology τ. A set F is τ -closed or closed, if its complement is open.

There are two types of Topology:

(i) **discrete topology**: $\tau = P(X)$, in which all subsets of X are open sets, and
(ii) **indiscrete topology**: $\tau' = \{\phi, X\}$, in which ϕ and X are the only open sets.

The following are the **basic concepts** in a topological space (X, τ):

Interior, Closure, Set boundary, isolated point:

(i) Interior point of A: A point $x \in A \subset X$ of a topological space is an *interior point* of A iff A is an open subset of X, and the set of all interior points of A is the interior of A, denoted by int(A) or A^0.

(ii) Closure of A: Let (X, τ) be a topological space and $A \subseteq X$. Then the closure of A is defined as the intersection of all closed sets containing A and is denoted by cl(A).

(iii) Boundary of A: Let (X, τ) be a topological space and $A \subseteq X$. Then the boundary of A is defined by the equation Bd(A) = A∩ (X −A).

The boundary of A $\subseteq$ X is Bd(A) = A ∩ Cl(X-A).

(iv) Isolated Point of A: A point x of A ⊂ X is isolated if there is a neighborhood U ∈U(x), such that (U\x) ∩A = φ.

(v) A subset S of a topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.

(vi) Let a and b be two points in a topological space X. A continuous function $f : [0,1] \to X$ such that $f(0)$ = a and $f(1)$ = b, is called a path from a to b.

(vii) Let the points a = $f(0)$ and b = $f(1)$ be the ends of the path f. Then f is said to be a closed path if a = b.

(viii) A set S ⊂ X is regarded as path-connected when any two points of S can be linked by a path f in S.

(ix) A point x is called open if the set {x} is open and is called closed if {x} is closed.

(x) If a point x is either open or closed it is called pure, otherwise it is called mixed.

(xi) Two distinct points x and y in X are called adjacent if the subspace {x,y} is connected or iff y ϵ N(x) or x ϵ N(y).

(xii) The adjacency set in X of a point x, denoted by AX(x)[or A(x)],is the set of points adjacent to x.

If P ⊂ X then A(P) is the set of points not in P but adjacent to some point in P.

Thus A(P) ∩ Q ≠ φ or P ∩A(Q) ≠ φ.

(b) Topology based on distance function

There are many topological spaces in which topology is obtained from a distance function. Let X be a set. A distance function d : X × X → R such that for all x, y, z ϵ X, satisfying

a) $d(x,y) \geq 0$

b) $d(x,y) = 0$ iff $x = y$ (Separation)
c) $d(x,y) = d(y,x)$ (Symmetry)
d) $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality)

A metric space is a pair (X, d) , where d is a metric for X. If d is a metric, then $B(x, r) = \{y \in X: d(x, y) < r\}$ is an open ball of centre x and radius r.

A set $A \subset X$ is open iff for every $x \in A$, there is an open ball included in A. Then the collection of all open sets constitutes a topology.

1.2. Digital Topology

The concepts of Digital Topology, introduced by Rosenfeld in 1979[9], refers to the use of topological terms in computer graphics and image processing. Digital Topology provides mathematical basis for various image processing applications.

Digital topology deals with properties and features of two-dimensional or three-dimensional digital images, defined in digital grids, that correspond to topological properties (e.g, Connectedness, path-connectedness, etc.) or topological features (e.g., boundaries, frontiers, etc) of objects.

Let N, Z and R be the sets of natural numbers, integers and real numbers respectively. Z_n and R_n are the Cartesian product of *n*-tuples of Z and the *n*-dimensional real space, respectively.

A pixel (picture element) is $[i_1 - ½ , i_1 + ½] \times [i_2 - ½ , i_2 + ½]$ in R^2 and a voxel (volume element) is $[i_1 - ½, i_1 + ½] \times [i_2 - ½ , i_2 + ½] \times [i_3 - ½, i_3 + ½]$ in R^3, where i's are integers.

Two pixels are 4-adjacent if two are distinct but share at least an edge and
8-adjacent if two are distinct but share at least a vertex.

If $p, q \in Z^2$ and $k \in [4,8]$, then p is said to be k-adjacent to q if the pixels centered at these two points are k-adjacent.

Similarly, (6 , 18 , 26) - adjacency for Z_3 – space.

The digital topology requires the discrete topological structure of a space $X \subset Z_n$ with k-adjacency.

The digital n-space Z^n is the set of $x = (x_1, x_2, \ldots, x_n)$ of the real Euclidean n-space having integer coordinates. A point with integer coordinates is called a digital point.

A digital image is a function $f : Z \times Z \to [0,1,\ldots N\text{-}1]$ where $N - 1$ is a positive whole number belonging to the natural interval [1, 256]. The functional value of f at any point p(x, y) is called the intensity or grey level of the image at that point and is denoted by f(p).

Let us consider the topologies on $Z\times Z$ with metrics d_1 and d_2 restricted to $Z\times Z$ defined by:

$$d_1(x, y) = \sum_{i=1}^{2} |xi - yi|$$
$$d_2(x, y) = \max_{i\leq i\leq 2} |xi - yi|.$$

The (4 - and 8 -)neighborhoods of $x \in Z \times Z$ are defined as follows:

i. **4-neighbours of a point p(x,y),** denoted by $\mathbf{N_4(p)}$, are its four horizontal & vertical neighbours (x±1, y)) and (x, y ±1).

$\mathbf{N_4(p)} = \{ y \in Z^2 : | y_1 - x_1| + |y_2\text{-}x_2|\}$

$= \{y \in Z^2 : d_1(x ,y) \leq 1 \}.$

ii. **8-neighbours of a point,** denoted by $\mathbf{N_8\,(p)}$, consists of 4-neighbours together with its four diagonal neighours ((x+1, y±1) & (x-1, y ± 1).

$\mathbf{N_8(p)} = \{y \in Z^2 : \max\{| y_1 - x_1|, |y_2 - x_2|\leq 1\}$

$= \{y \in Z^2 : d_2(x, y) \leq 1\}.$

The above neighborhoods of $x \in Z\times Z$ are shown as follows (See figure 1):

	#	
#	*	#
	#	

Fig.1 (a) : $\mathbf{N_{4(p)}}$

#	#	#
#	*	#
#	#	#

Fig.1(b): $\mathbf{N_{8(p)}}$

Fig.1: The neighborhoods $N_{4(p)}$ & $N_{8(p)}$ of a point x depending on the metrics employed in 2-D

6, 18, 26-nbds. of a middle point in 3-D:

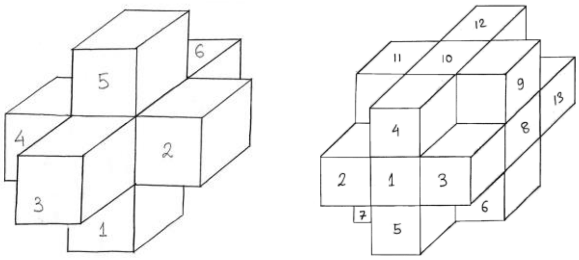


Fig.-2a: 6, 18 - nbd.

Fig.-2b: $\mathbf{N_{26}}$ **(x)** [Voxels]

Fig.-2: 6, 18 and 26-nbds. of a middle pt.

A digital image (in 2-D) contains cells of three kinds:

(i) The 2-dimensional cells are the **pixels**,

(ii) The 1-cells are called **edges** and

(iii) The 0-cells are the **points** or the **vertices**.

An edge may bound a pixel, but not vice versa and a point may bound an edge and a pixel.

The properties of the set of pixels or voxels that correspond to topological properties of the original object are called **Digital Topology.**

1.3 Digital Image Processing

(a) Historical Background:

One of the first application of digital images was in the newspaper industry, when pictures were first sent by submarine cable between London and New York. Introduction of the Bartlane cable picture transmission system in the early 1920s reducing the time required to transport a picture across the Atlantic from more than a week to less than three hours.

The history of digital image processing is tied to the development of the digital computer. In fact, digital images require so much storage and computational power that progress in the field of digital image processing has been dependent on the development of digital computers and of supporting technologies that include data storage, display, and transmission.

The computers appeared in the early 1960^s are powerful enough to carry out meaningful image processing tasks.

(b) Digital Image Processing

An image may be defined as a two-dimensional function, ***f***(x, y), where *x* and *y* are plane coordinates, and the amplitude of *f* at any pair of coordinates (*x*, *y*) is called the *intensity* or ***gray level* of the image** at that point. When *x*, *y* and the intensity values of *f* are all finite, discrete quantities, we call the image a *digital image*. The processing of digital images is called *digital image processing.*

In image processing an object (continuous image) in the 2D-space or 3D-space is represented digitally a set of pixels or voxels.

A digital image is composed of a finite number of elements, each of which has a particular location and value. These elements are called *picture elements*, *image elements*, *or pixels*.

There are three types of digital images: Grayscale image, colour image and binary image. A colour image is just three (Red, Green, Blue) functions pasted together:

$$f(x,y) = (f_r(x,y).f_g(x,y).f_b(x,y)).$$

1.4. Fuzzy sets

(a) Sets (or Crisp Sets): *A set is a collection of well-defined objects, called elements of the set.* If x is a <u>member</u> of *A*, we write $x \in A$. If x is not a member of A, we write $x \notin A$.

A subset A of a set X is a mapping from the elements of X to the elements of the set {0,1}. The value zero represents non-membership, while the value one represents membership. This mapping is called membership function (or characteristic function) on X (fig.-3).

$$A(x) = 1 \text{ if } x \in A$$
$$= 0 \text{ if } x \notin A.$$

Fig. - 3

Thus an element of the set X is either a member (value 1) or a non-member (0) of the subset A. There are no partial members in sets.

(b) Fuzzy sets

It is very *interesting phenomenon* that the ***impreciseness, Vagueness and uncertainty*** are the part & parcel of our thought processes. Thinking is a construction in mind which after being linguistic form, become liable of analysis and logical tests. It can neither be avoided from real life nor from the language.

Can the Science and Technology based on Cantor's Set Theory deal with our all real life Problems?

Obviously not.

There are many approaches to the problem of how to understand and manipulate imperfect knowledge.

There are many approaches to the problem of how to understand and manipulate imperfect knowledge. One of most successful approach is based on the fuzzy set notion proposed by Lotfi Zadeh in 1965[12].

We see that a subset A of a set X is a mapping from the elements of X to the elements of the set [0,1]. This is represented by the Venn-diagram:

The function $A : X \rightarrow [0,1]$ is denoted in Fig-4.

Consider a set X that contains all the real numbers between 0 and 1. A subset A is represented in fig.-5.

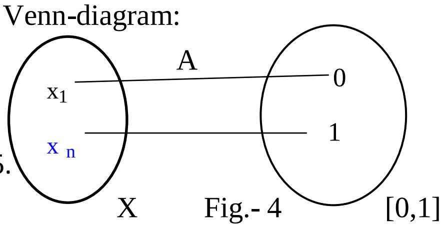


Fig.- 4

(It is different from Venn diagram).

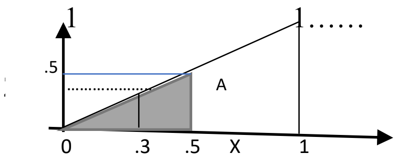


Fig.- 5

In the figure-5, the interval on the x-axis between 0 and 1 has y-value between 0 and 1. This indicates that any number in this interval is a member of the subset A. Any number that has a y value of zero is said to be a non-member of the subset A.

Fuzzy Subsets:

$A = \{(x, \mu_A(x)): x \in A, \mu_A(x) \in [0,1]\}$, where $\mu_A(x)$ is a grade or degree to which any element x in X belongs to the fuzzy set A. The value zero is used to represent non-membership; the value one is used to represent complete membership, and the values in between are used to represent degrees of membership.

Example. "Let X be the universal set of people and "tallness". Let us define a fuzzy subset TALL which will answer the question "to what degree is person x tall?"

To each person in the universe of discourse, we assign a degree of membership in the fuzzy subset TALL.

The easiest way to do this is with a membership function based on the **person's height.**

$$\text{tall}(x) = \begin{cases} 0 & \text{if } \text{height}(x) < 5 \text{ ft.}, \\ (\text{height}(x) - 5\text{ft.})/2\text{ft.} & \text{if } 5 \text{ ft.} \le \text{height} \le 6 \text{ ft.} \\ 1 & \text{if } \text{height}(x) > 6 \text{ ft.}\}. \end{cases}$$

Essentially this function calculates the membership value of a certain height.

For example:

1. If a person Is less 4'9", then this person has a membership value of 0 and thus is not a member of the set tall.

2. If a person is 5'6", then this person has a membership value of 0.50 and is a partial member of the set tall.

3. If a person is 7'6", then this person has a membership value of 1.0 and of 1.0 and thus is a member of the set TALL

Essentially this function calculates the membership value of a certain height.

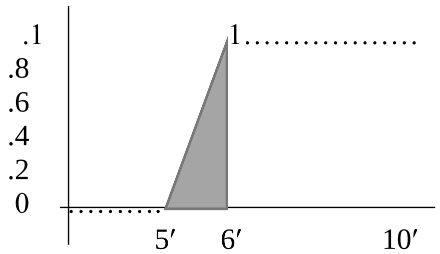


Fig.- 6 : The graphical representation of the fuzzy set for tallness.

Person	Height in ft.	Degrees in tallness
Asha	5′3″	0.25
Jiana	5′4″	0.33
Riana	5′ 6″	0.50
Ahana	5′9″	0.75
Bipasha	6′	1.00
Dipika	7′	1.00

Instead of just black and white, the color belonging to a set has degree of whiteness & blackness.

Definitions: Let $A = \{(x, \mu_A(x))\}$, $\mu_A(x) \in [0,1]$ and $B = \{(x, \mu_B(x))\}$, $\mu_B(x) \in [0,1]$.

1. Union of A and B: $A \cup B$, defined by $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

2. Intersection of A and B: $A \cap B$, defined by $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

3. Complement: A^c or 1 – A , defined by $\mu_{\text{not } A}(x) = 1 - \mu_A(x))$

.

4. A is contained in B , $A \subseteq B$, if $\mu_A(x) < \mu_B(x)$.

5. Set equality: **A = B** if and only if they have the same number of elements and their membership functions are also equal $[\mu_A(x) = \mu_B(x)]$.

Example: If $A=\{(x_1, .2), (x_2, .5), (x_3, .6), (x_4, 1)$, $B = \{(x_1, .3), (x_2, .4),(x_3, .7), (x_4, .9)\}$
$C = \{ (x_1, .1), (x_2, .3), (x_3, .5), (x_4, .7)\}$ be fuzzy subsets. Then

$A \cap B = \{(x_1, .2), (x_2, .4), (x_3, .6), (x_4, .9)\}$.

$A \cup B = \{(x_1, .3), (x_2, .5), (x_3, .7), (x_4, 1)\}$.

$1 – A = \{(x_1, .8), (x_2, .5), (x_3, .4), (x_4, 0)\}$.

$C \subseteq B$, since $\mu_C(x_1) < \mu_B(x_1)$; $\mu_C(x_2) < \mu_B(x)$; $\mu_C(x_3) < \mu_B(x_3)$; $\mu_C(x_4) < \mu_B(x_4)$.

The fuzzy set theory has attracted more and more attention in the area of image processing because of its inherent capability of handling uncertainty. The more important advantage of a fuzzy methodology lies in that the fuzzy membership function provides a natural means to model the uncertainty prevalent in an image scene.

1.5 FUZZY LOGIC

Logic is the discipline that deals with the study of language in **reasoning** and **argument**. Aristotle was the first systematizer of logic. The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments, i.e., in 2-valued logic, a statement is either true or false.

In 1973, Prof. L. Zadeh [12] proposed the concept of linguistic or "fuzzy" variables. Fuzzy logic is a superset of conventional (Boolean) logic that has been extended to handle the concept of partial truth - truth values between "completely true" and "completely false".

Fuzzy logic allows one to model uncertainty and subject concepts in a better form than crisp models. In fuzzy logic, a statement may be true or false or have an intermediate truth values {0, 0.5, 1}, i.e., false, may be true and true. In general, we may deal with multivalued logic, values in finer subdivision of the unit closed interval [0, 1].

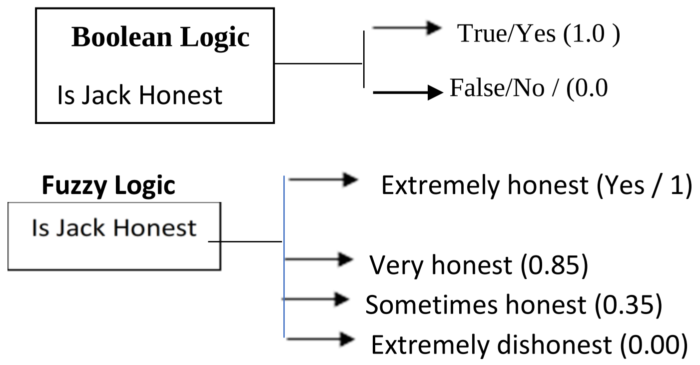
Linguistic Variables:

An algebraic variables take numbers as values, whereas linguistic variables take words or sentences as values.

The linguistic variables take words as values, e.g. "temperature", "velocity", "displacement", "flow", "pressure", hot, old etc.

In our daily life we use words, like old, tall, expansive, hot etc. , e.g.

1. He is *old* 2. The pen is *expensive*. 3. She is *tall*. 4. It is really *hot*.



Definition: *The variables whose values are words or sentences in natural or artificial languages are called linguistic variables.*

Age is a linguistic variable consisting of fuzzy sets like very young, young, middle age, old and very old. They are called terms of the linguistic variable 'age'.

Fuzzy Logic Operations:

The standard definitions of operations in fuzzy logic are:

1) Negate (negation criterion) : truth (not x) = 1.0 - truth (x)
2) Intersection (minimum criterion): truth (x and y) = minimum(truth(x), truth(y))
3) Union (maximum criterion) : truth (x or y) = maximum(truth(x), truth(y)).

If we interpret the image features as linguistic variables, then fuzzy if-then rule will be

***IF** the pixel is dark **AND** its neighbourhood is also dark **AND** homogeneous*
***THEN** it belongs to the background.*

2. Fuzzy Logic Approach to Digital Image Processing

The most important theoretical frameworks that can be used to construct the foundations of fuzzy image processing are: Fuzzy Topology, Fuzzy Digital Topology[10], fuzzy geometry, measures of fuzziness/ image information, rule-based approaches, fuzzy clustering algorithms, fuzzy measure theory, and fuzzy grammars.

Fuzzy logic allows one to model uncertainty in a better form than crisp models. Fuzzy logic is conceptually easy to understand and is flexible and tolerant of imprecise data.

Images as fuzzy sets

An image G of size $M \times N$ with L gray levels can be defined as an array of fuzzy singletons (fuzzy sets with only one supporting point) indicating the membership value μ_{mn} of each image point x_{mn} regarding a predefined image property (e.g., brightness, homogeneity, noisiness, edginess, etc.).

By **Fuzzy Image Processing** we understand the collection of all methodologies in digital image processing, with which the images, their segments or features, which represent these images or their segments, represented and processed as fuzzy sets.

Fuzzy Image Processing has five main stages:

i. Input Image - gray levels, features

ii. Image fuzzification - it means that the image with one or more membership values w. r. t. the brightness, edginess, etc.

iii. Modification of membership values- Expert Knowledge /Fuzzy Logic & Fuzzy Set theory.

iv. Image defuzzification (if necessary): the resulting output membership function (reversing fuzzification) may be assigned a numerical value by defuzzification. The step of defuzzification can be omitted if the result of the fuzzy inference system is given by crisp number.

v. Resulting Image – The resulting output membership function can be assigned a numerical value by defuzzification.

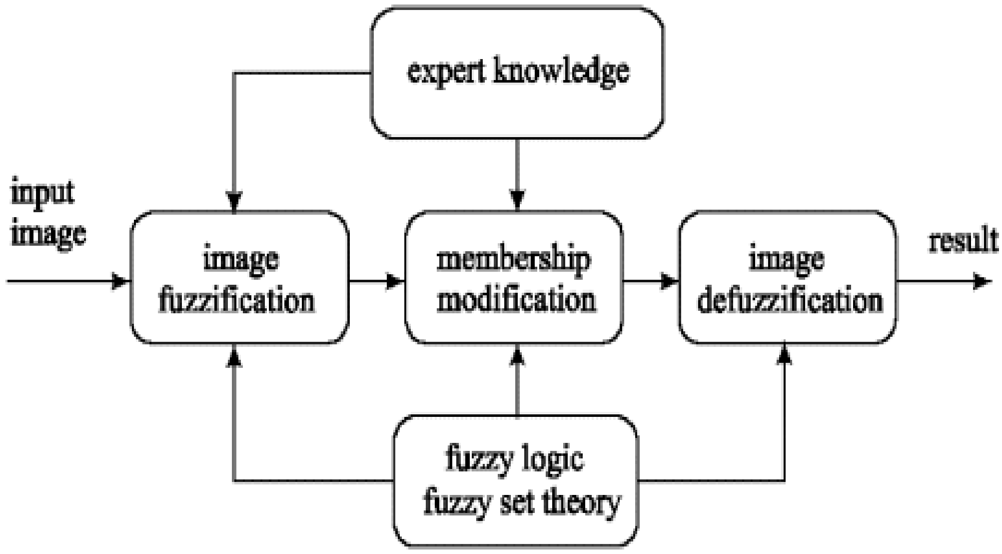


Figure 7: The General Structure of Fuzzy Image Processing.

The coding of *input image* **(fuzzification**) and decoding of the results (**defuzzification**) are steps that make possible to process images with fuzzy techniques.

The main power of fuzzy image processing is in the middle step (**modification of membership values)**. After the image data are transformed from **gray-level plane** to the **membership plane**(fuzzification), appropriate fuzzy techniques modify the membership values.

An example of the **General Structure of Fuzzy Image Processing** (Fig.-7) is given below (Fig.-8):

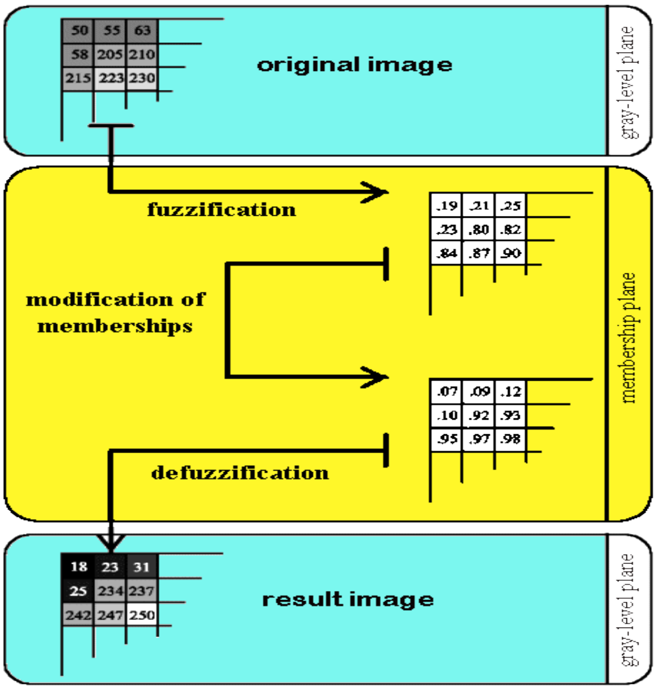


Fig- 8

3. Application

The use of fuzzy logic into the development of image processing and analysis has opened a new research area in the image-processing field. **Image processing** is a prominent area that supports applications in different fields, such as medical science, astronomy, national security, autonomous systems, product quality, industrial applications, agriculture, defense, moving object tracking, Forensic, Astro-photography, Fingerprint matching etc. are most remarkable.

1. Fuzzy Logic Technologies in Industrial Applications:

   Here the image processing is common to deal with subjective concepts like brightness, edges, uniformity, measurements, *etc*. For example, in cosmetic inspection of products, product quality is determined based on subjective observations of the inspector on a product. Therefore, a vision inspection system used in this type of application needs to manage subjectivity and uncertainty.

2. *X-rays:* X-ray are the oldest sources of EM radiation for imaging. It is used in medical diagnostics, industry, astronomy and other areas.

3. *Remote Sensing:* The sensors capture the pictures of earth's surface in remote sensing satellites.
4. *Intelligent Transportation:* Image processing technique can be used in Automatic number plate recognition and Traffic sign recognition.
5. *Defense surveillance:* Aerial surveillance methods are used to continuously keep an eye on the land and ocean.
6. *Biomedical Imaging techniques*: For medical diagnosis, different types of imaging tools such as X-ray, ultra-sound, CT scan, MR Images etc. are used.
7. *Automatic visual inspection system:* This application improves the quality and productivity of the product in industries.
8. *Vision Intelligence for Farming:* Using Fuzzy Logic Optimized Genetic Algorithm and Artificial Neural Network.
9. *Fuzzy Logic Approach to Numerical Water Level Gauge.*
10. *A Fuzzy Data Fusion Method for Improved Motion Estimation*
11. *Object Recognition in Robot Football by Using One Dimensional Image.*
12. *Finger print retrieval.*
13. *Remote sensing*: This application, sensors capture the pictures of the earth's surface in remote sensing satellites or multi – spectral scanner which is mounted on an aircraft.
14. *Moving object tracking:* This application enables to measure motion parameters and acquire visual record of the moving object.

Conclusion:

The concept of if-then rules is sophisticated bridge between human knowledge on side and the numerical framework of computers on other side. They are simple and easy to understand. It can be used to achieve a higher level of image quality considering the subjective perception and opinion of human observers. Fuzzy theory is used to overcome the drawbacks of spatial domain methods like thresholding and frequency domain methods to improve the contrast of an image.

Fuzzy clustering algorithms and rule-based approaches will certainly play an important role in developing new image processing algorithms. Here, the potentials of fuzzy if-then rule techniques seem to be greater than already estimated.

In which manner fuzzy image processing takes place, depends on the problem definition and on the respective fuzzy method. This definition refers naturally to other methodologies and not to image processing as a separate branch of science. Thus the misunderstanding has to be avoided that fuzzy image processing may replace the classical image processing

Neuro fuzzy techniques and rough image processing may be used to enhance the quality of images.

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A Survey On Fuzzy Topological Spaces

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Abstract: The concept of fuzzy sets was introduced by Professor LotfiA Zadeh and proved to be very useful. It is an essential tool for formulating the mathematical information of imprecise nature. A fuzzy subset of a given set X is by definition a function with domain X and range the unit closed interval I. Since 1965 it has been acknowledged with great importance by the community of Mathematician, Scientists, Engineers and Social Scientists. The novelty of the theory lies in the fact that it tries to take the inexact concepts that are shown everywhere in our daily life by an exact methodology.

The notion of a fuzzy topology was introduced by C.L Chang in 1968. It is an extension of the concept of ordinary topology
A family $F \subseteq I^X$ of fuzzy subsets is called a fuzzy topology (in the sense of Chang) for X if it satisfies the following

$0 = \mu_\phi$ and $1 = \mu_X \in F$
for all $\alpha, \beta \in F$ implies $\alpha \Lambda \beta \in F$
If $\alpha_j \in F$ for each j then Sup $\alpha_j \in F$; $j \in \Lambda$

Then F is called a fuzzy topology for X and the pair(X, F) is called a fuzzy topological space. Members of F are called fuzzy topological space. Members of F are called fuzzy open subsets and their pseudo complement $\alpha^C = 1-\alpha$ fuzzy closed subsets. In 1976 R Lowen introduced a new definition of fuzzy topology replacing the condition (i) by (i′) for all a(Constant function) $a \in F$

Both the definition are used by fuzzy topologists. One of the main advantage of Lowen's definition over Chang's definition is that all constant function are fuzzy continuous

In 1983, Atanassov introduced the concept of "Intuitionistic fuzzy set" . Using this type of generalized fuzzy set, Coker defined "Intuitionistic fuzzy topological spaces." In 1996, Coker and Demirci introduced the basic definitions and properties of intuitionistic fuzzy topological spaces in ˇSostak's sense, which is a generalized form of "fuzzy topological spaces" developed by ˇSostak.

It is seen that the family w(T) of lower semi continuous function from a topological spaces (X, T) to the unit closed interval I = [0,1] forms a fuzzy topology on X. The space (X, w(T)) is called induced fuzzy topological spaces. We will also discuss some old and recent works on generalized fuzzy topological spaces.

1. Introduction:

The concept of fuzzy sets was introduced by Professor Lotfi A Zadeh and proved to be very useful. It is an essential tool for formulating the mathematical information of imprecise nature. A fuzzy subset of a given set X is by definition a function with domain X and range the unit closed interval I. Since 1965 it has been acknowledged with great importance by the community of Mathematician, Scientists, Engineers and Social Scientists. The novelty of the theory lies in the fact that it tries to take the inexact concepts that are shown everywhere in our daily life by an exact methodology. Real situation are very often uncertain or vague in a number of ways. Due to lack of information or data the future state of the system might not be known.

1.1 In the classical or ordinary set theory a set is a "well defined" collection of objects. By "well defined" we mean that there is a given rule by means of which it is possible to know whether a particular object is contained in the collection or not. Let X is an ordinary set and A a subset of it. We write $x \in A$ if an element x of X is a member of A and $x \notin A$ if x of X is not a member of A .Membership in a subset A of X is based on two valued logic and can be restated in terms of characteristic function (or membership function) μ_A from X to {0, 1} i.e.

$$\mu_A(x) = 1 \text{ if } x\in A$$
$$= 0, \text{ if } x\notin A$$

Let us consider a finite set X = {a, b, c, d, e, f} and a subset A = {a, b, d, f}

A can be represented by the set of pairs i.e.

$$A = \{(x, \mu_A(x)): x \in X\}$$
$$= \{(a,1),(b,1),(c,0),(d,1),(e,0),(f,1)\}$$

where μ_A is the known characteristic function and μ_A: $X \rightarrow \{0,1\}$

Thus the subset A of a set X can be characterized by a characteristic function which associates with each x its grade of membership $\mu_A(x)$.

Let us consider the following two subfamilies of a set X of students

(i) B = {a collection of all students of Tripura University}

(ii) C = { a collection of all intelligence students of Tripura University}

In example (i) one can be sure whether a particular member belongs to the collection or not. Thus it is a collection of well defined and distinct objects, hence it is a set. The membership in the subset B of X is defined by $\mu_B(x) = 1$ if $x \in B$

$= 0$, if $x \notin B$

In example (ii) one cannot be sure whether a particular member belongs to the collection or not. The subfamily of the above kind is not precise, the simplest way to describe the above collection mathematically is to characterize the degree of belongness by a number from the closed interval [0,1]

Let X be an ordinary set. A fuzzy subset α in X is the collection of ordered pairs $(x, \mu_\alpha(x))$ with $x \in X$ and a membership function $\mu_\alpha : X \to [0,1]$. The value $\mu_\alpha(x)$ of x denotes the degree to which an element x may be a member of α . Thus a fuzzy subset α of X is denoted by $\alpha = \{(x, \mu_\alpha(x)) : x \in X\}$ where $\mu_\alpha(x) = 1$, indicates strictly the containment of the element x in α (full membership) and $\mu_\alpha(x) = 0$ denotes that x does not belong to α(non-membership). Thus an ordinary set is a special case of fuzzy set with a membership function which is reduced to a characteristic function. Because of these generalities the fuzzy set theory has a wider scope of applicability than the ordinary set theory in solving real problem

A fuzzy set α can also be represented in the following way $\alpha = \{ x / \mu_\alpha(x), \forall x \in X \}$ or $\alpha = \{ (x , \mu_\alpha(x)) : x \in X \}$The set of all fuzzy subset on X is denoted by I^X.

1.2. Basic Operations On Fuzzy Subsets

Let α and β be two fuzzy subsets of X with membership function μ_α and μ_β respectively. Then for all $x \in X$ we have

(i) α is equal to β i.e. $\alpha = \beta$ iff $\mu_\alpha(x) = \mu_\beta(x)$
(ii) α is a subset of β i.e. $\alpha \leq \beta$ iff $\mu_\alpha(x) \leq \mu_\beta(x)$
(iii) Union of α and β i.e. $\alpha\vee\beta$ iff $\mu_{\alpha\vee\beta}(x) = \max\{\mu_\alpha(x), \mu_\beta(x)\}$
(iv) Intersection of α and β i.e. $\alpha\wedge\beta$ iff $\mu_{\alpha\wedge\beta}(x) = \min\{\mu_\alpha(x), \mu_\beta(x)\}$
(v) Complement of β i.e. $\beta^C = 1 - \beta$ iff $\mu_\beta{}^C(x) = 1 - \mu_\beta(x)$

1.3 The notion of fuzzy topology was introduced by C.L Chang in 1968. It is the extension of the concepts of ordinary topological space. Let X be a non empty set and $I = [0,1]$ be the unit closed interval. For X, I^X denotes the collection of all mappings from X into I. A member $\lambda$of I^X is called a fuzzy set. The union λ_i, the intersection $(\cap\lambda_i)$ of a family $\{\lambda_i\}$ of fuzzy sets of X is defined to be the mapping $\sup\lambda_i$ $(\inf\lambda_i)$. For any two members λ and β of I^X,$\lambda\geq\beta$ if and only if $\lambda(x)\geq\beta(x)$ for each $x\in X$. 0 and 1 denotes the constant mappings family whole of X to 0 and 1 respectively. The complement λ^C of a fuzzy set λ of X is 1- λ defined as $(1-\lambda)(x) = 1-\lambda(x)$ for each $x\in X$. If λ is a fuzzy set of X and β is a fuzzy set of Y then $\lambda\times\beta$ is a fuzzy set of $X \times Y$ defined by $(\lambda\times\beta)(x,y) = \min\{\lambda(x), \beta(y)\}$ for each $(x, y) \in X \times Y$

1.4 A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one say $x \in X$. If its value at x is $p(0<p\leq1)$ we denote the fuzzy point by x_p where the point x is called its support and p its value.

Or equivalently a fuzzy point xp in X is a special fuzzy set with membership function denoted by $x_p(y) = p$,
$X = y$ $(0\leq p \leq1)$
$= 0$, $x\neq y$
Note: (i) Let α be a fuzzy set in X, then $x_p \subseteq \alpha$ implies $p \leq\alpha(x)$. In particular $x_p \subseteq y_q$ implies and implies by $x = y$, $p \leq q$

(ii) $x_p \in \alpha$ implies and implied by $p \leq\alpha(x)$

(b) (i) A fuzzy point x_p is said to be quasi coincident (q coincident) with a fuzzy subset α, denoted by $x_p q\alpha$ iff $p + \alpha(x) > 1$ or $p > 1-\alpha(x)$ or $p > \alpha^C(x)$

(ii) A fuzzy subset α is q coincident with another fuzzy subset β denoted by $\alpha q\beta$ iff there exist $x \in X$ such that $\alpha(x)+\beta(x) >1$, $\mu_\alpha(x) > 1- \beta(x)$ or $\alpha(x) >\beta^C(x)$. In this case we say that two fuzzy subsets α and β are q coincide (with each other) at x. It is clear that if α and β are q coincident at x then both $\alpha(x)$ and $\beta(x)$ are not zero, hence α and β intersect at x. If α does not q coincident with β then we write $\alpha q \beta$

Note: $\alpha \not\leq \beta$ iff α and β^C are not q coincident. In particular $x_p \in \alpha$ iff x_p is not q coincident with α^C.

It follows from the fact $\alpha(x) \not\leq \beta(x)$ implies and implied by $\alpha(x) + \beta^C(x) \leq \beta(x)+\beta^C(x) = 1$, i.e. $x_p\ q\ \alpha^C$

1.5 Let $f : X \to Y$ be a mapping . If λ be a fuzzy set of X we define $f(\lambda)$ as

$$f(\lambda)(y) = \sup_{x \in f^{-1}(y)} \lambda(x) \quad \text{iff}^{-1}(y) \neq \phi \ ,$$
$$= 0$$

for each $y \in Y$ and if β is a fuzzy set of Y we define $f^{-1}(\beta)$ as $f^{-1}(\beta)(x) = \beta f(x)$ for each $x \in X$.

1.6 Fuzzy Topology

The notion of a fuzzy topology was introduced by C.L Chang in 1968. It is an extension of the concept of ordinary topology

Definition: A family $F \subseteq I^X$ of fuzzy subsets is called a fuzzy topology (in the sense of Chang) for X if it satisfies the following

(i) $0 = \mu_\phi$ and $1 = \mu_X \in F$

(ii) for all $\alpha, \beta \in F$ implies $\alpha \Lambda \beta \in F$

(iii) If $\alpha_j \in F$ for each j, then $\text{Sup } \alpha_j \in F,\ j \in \Lambda$

Then F is called a fuzzy topology for X and the pair(X, F) is called a fuzzy topological space. Members of F are called fuzzy topological space. Members of F are called fuzzy open subsets and their pseudo complement $\alpha^C = 1-\alpha$ fuzzy closed subsets. In 1976 R Lowen introduced a new definition of fuzzy topology replacing the condition (i) by (i′) for all a(Constant function) $a \in F$

Both the definition are used by fuzzy topologists. One of the main advantage of Lowen's definition over Chang's definition is that all constant functions are fuzzy continuous.

1.7 Completely Induced Fuzzy Topological Spaces, (R.N. Bhaumik and A. Mukherjee, Fuzzy Sets and Systems,47(1992)387-390).

The concepts of induced fuzzy topological spaces was first introduced by Weiss(M. D. Wesis, Fixed points, separation and induced topologies for fuzzy sets,J.Math.Anal.Appl.50(1975)142-150). Lowen (R. Lowen, Fuzzy topological spaces and fuzzy compactness, J.Math.Anal.Appl.56(1976) 621-633)called these spaces as topological generated spaces. Martin (H.W. Martin, On weakly induced fuzzy topological spaces, J.Math.Anal.Appl.78(1980)634-639)introduced a generalized concept weakly induced fuzzy space, which was called semi induced space by Mashhour, Ghanim, Wakeil and Morsi (A.S. Mashhour, M.H. Ghanim, A. El-Wakeil and N.M. Morsi, Semi induced fuzzy topologies, Fuzzy Sets and Systems31(1989)1-18).

The notion of lower semi continuous functions plays an important tool in defining the above concepts. In the paper" Completely Induced Fuzzy Topological Spaces" we introduced two new classes of fuzzy topological spaces-Completely induced fuzzy topological spaces and Completely semi induced fuzzy topological spaces. These are defined with the generalized concept of completely continuous functions introduced by Arya and Gupta(S.P.Arya and R. Gupta, On strongly continuous mapping, Kyungpook Math. J.14(1974)131-143)).A mapping f: X →Y is completely continuous if the inverse image of every open subset of Y is a regular open subset of X.

Let (X, T) be a topological space. The family of all CLSC functions from the space (X, T) to the unit closed interval forms a fuzzy topology for X. The fuzzy topology obtained as above is called a completely induced fuzzy topological space and is denoted by (X, Ɵ(T)).

1.8 Some more results on completely induced fuzzy topological spaces (R.N. Bhaumik and A. Mukherjee, Fuzzy Sets and Systems 50(1992)113-117)

In this paper we have seen that every completely induced space is a fuzzy topological space. The converse is not true in generally. But under a certain condition the converse is also true. Next we studied the concept of c-initial topology associated with a completely induced fuzzy topological space.

Let (X,Q) be a completely induced fuzzy topological space. The family { $\sigma_r(\lambda)$: $\lambda \in Q$, $r \in I$ } of regular open subsets of X forms a subbase of some topology for X, called c-initial topology of Q and is denoted by i_c (Q).

(R,N,Bhaumik and A.Mukherjee, On c-initial spaces, Fuzzy sets and Systems 56(1993)209-213).

1.9: Recent Work: Possibility Interval Valued Intuitionistic Fuzzy Soft Expert Set theory and Its Application In Decision Making(A. Mukherjee, Bull. Cal. Math. Soc.109(6)(2017)501-524).

In many of the problems in engineering, medical science, economics and other real life problems have uncertainties. In 1999, Molodtsov (D.Molodtsov, Soft set theory – first result, Computers and Mathematics with Applications, 37(4-5) (1999) , 19-31). introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty. Among the significant development of the soft sets and its generalizations is the introduction of the possibility value which indicates the degree of possibility of belongingness of the elements in the universal set.

Let U be an initial universe of objects and E the set of parameters in relation to objects in U. Parameters are often attributes characteristic or properties of objects. Let P(U) denote the power set of U and $A \subseteq E$.

A pair (F,A) is called a soft set over U, where F is a mapping given by $F : A \rightarrow P(U)$. In this paper, the notion of possibility interval valued intuitionistic fuzzy soft expert sets(PIVIFSES) in complex phenomena is proposed and define some related matters pertaining to this notion as well as basic operations on this concept. A generalized algorithm is introduced and applied to the PIVIFSE sets in hypothetical decision making problem. In this study an interval of degree of possibility of each element in the universal set of elements is attached to the parameterization of interval valued intuitionistic fuzzy soft sets. Also the opinion of a set of experts is given. Lastly an explicit algorithm is proposed and is applied.

Let U be the universe, E be the set of parameters, X be the set of experts(agents), Q be a set of opinions, $Z = E \times X \times Q$ and $A \subseteq Z$.

Definition: Let $U = \{u_1, u_2, u_3, ..., u_n\}$ be the universal set of elements and E={e_1,e_2,e_3, ...,e_m}be the universal set of parameters, X={x_1,x_2,x_3,....,x_k} be the set of experts and Q={1=agree, 0=disagree}a set of opinions. Let $Z = \{E \times X \times Q\}$ and $A \subseteq Z$. Then the pair (U,Z) is called a soft universe. Let $F : Z \to IVIF(U)$ and P be an interval valued intuitionistic fuzzy sub set of Z defined by $P : Z \to IVIF(U)$, where IVIF(U) denote the collection of all interval valued intuitionistic fuzzy sub sets of U.

Suppose $F_P : Z \to IVIF(U) \times IVIF(U)$ be a function defined by $F_P(z) = \left(F(z)(u_i), P(z)(u_i)\right) \ \forall u_i \in U$. Then F_P is called a possibility interval valued intuitionistic fuzzy soft expert set(denoted by PIVIFSES) over the soft universe (U,Z).

For each $z_i \in Z$, $F_P(z_i) = \left(F(z_i)(u_i), P(z_i)(u_i)\right)$, where $F(z_i)$ represents the interval of the degree of belongingness and the interval of non-belongingness of the elements of U in $F_P(z_i)$ and P(z_i) represents the interval of degree of possibility of such belongingness.

Therefore $F_P(z_i) = \left\{ \dfrac{u_i}{F(z_i)(u_i)}, P(z_i)(u_i) \right\}$, $\forall i = 1, 2, 3, ..., n$, where

$F(z_i)(u_i) = \left\{ \left[\mu^-_{F(z_i)}(u_i), \mu^+_{F(z_i)}(u_i) \right], \left[\nu^-_{F(z_i)}(u_i), \nu^+_{F(z_i)}(u_i) \right] \right\}$, with $\mu^-_{F(z_i)}(u_i), \mu^+_{F(z_i)}(u_i)$ as left and right membership functions and $\nu^-_{F(z_i)}(u_i), \nu^+_{F(z_i)}(u_i)$ as left and right non membership functions of each element $u_i \in U$.

We write the PIVIFSES (F_P, Z) as F_P. If $A \subseteq Z$ it is also possible to have a PIVIFSES (F_P, A). For simplicity we take the set of opinion consists of only two values namely agree and disagree.

Example: Let U={u_1,u_2,u_3}be a set of elements and E={e_1,e_2}be a set of parameters, where e_1=beautiful and e_2=cheap and let X={x_1,x_2}be a set of be set of experts. Suppose that $F_P : Z \to IVIF(U) \times IVIF(U)$ is a function defined as follows;

$$F_P(e_1, x_1, 1) = \left\{ \left(\frac{u_1}{[0.6, 0.8][0.05, 0.1]}, [0.1, 0.3] \right), \left(\frac{u_2}{[0.2, 0.4][0.3, 0.5]}, [0.1, 0.2] \right), \left(\frac{u_3}{[0.3, 0.6][0.2, 0.4]}, [0.2, 0.3] \right) \right\}$$

$$F_P(e_2, x_1, 1) = \left\{ \left(\frac{u_1}{[0.4, 0.6][0.1, 0.3]}, [0.3, 0.6] \right), \left(\frac{u_2}{[0.1, 0.2][0.6, 0.8]}, [0.6, 0.8] \right), \left(\frac{u_3}{[0.6, 0.9][0.0, 0.05]}, [0.2, 0.5] \right) \right\}$$

$$F_P(e_1, x_2, 1) = \left\{ \left(\frac{u_1}{[0, 0][1, 1]}, [0.5, 0.9] \right), \left(\frac{u_2}{[0.1, 0.3][0.2, 0.4]}, [0.6, 0.8] \right), \left(\frac{u_3}{[0.1, 0.2][0.65, 0.75]}, [0.1, 0.15] \right) \right\}$$

$$F_P(e_2, x_2, 1) = \left\{ \left(\frac{u_1}{[0.2, 0.4][0.3, 0.6]}, [0.2, 0.4] \right), \left(\frac{u_2}{[0.2, 0.5][0.3, 0.5]}, [0.1, 0.3] \right), \left(\frac{u_3}{[0.6, 0.9][0, 0.1]}, [0.4, 0.7] \right) \right\}$$

$$F_P(e_1, x_1, 0) = \left\{ \left(\frac{u_1}{[0.1, 0.2][0.6, 0.8]}, [0.3, 0.5] \right), \left(\frac{u_2}{[0.1, 0.3][0.2, 0.7]}, [0, 0] \right), \left(\frac{u_3}{[0.6, 0.9][0, 0.1]}, [0, 0.1] \right) \right\}$$

$$F_P(e_2, x_1, 0) = \left\{ \left(\frac{u_1}{[0, 0][0.6, 0.8]}, [0.5, 0.7] \right), \left(\frac{u_2}{[0.1, 0.3][0.4, 0.7]}, [0.1, 0.2] \right), \left(\frac{u_3}{[0.6, 0.8][0.1, 0.2]}, [0.4, 0.6] \right) \right\}$$

$$F_P(e_1, x_2, 0) = \left\{ \left(\frac{u_1}{[0.5, 0.8][0, 0.1]}, [0.4, 0.6] \right), \left(\frac{u_2}{[1, 1][0, 0]}, [0.1, 0.25] \right), \left(\frac{u_3}{[0.1, 0.3][0.2, 0.55]}, [0.7, 0.9] \right) \right\}$$

$$F_P(e_2, x_2, 0) = \left\{ \left(\frac{u_1}{[0.2, 0.3][0.4, 0.5]}, [0.3, 0.55] \right), \left(\frac{u_2}{[0.4, 0.7][0.1, 0.3]}, [0, 1] \right), \left(\frac{u_3}{[1, 1][0, 0]}, [0.7, 0.9] \right) \right\}$$

The collection (F_P, Z) is a PIVIFSE set over the soft universe (U, Z).

Application of Possibility Interval Valued Intuitionistic Fuzzy Soft Expert Set in a Decision Making Problem:

A firm is looking to hire a person to fill in the vacancy for a position in their firm. Out of all the candidates who applied for the position, three candidates were short listed and these three candidates form the universe of elements $U = \{u_1, u_2, u_3\}$. The hiring committee consists of hiring manager, head of the department and HR director of the firm and this committee is represented by the set $X = \{p, q, r\}$ (a set of experts) , while the set Q={1= agree, 0 = disagree} represents the set of opinions of the hiring committee members. The hiring committee consider a set of parameters $E = \{e_1, e_2, e_3, e_4\}$, where the parameters $e_i (i = 1, 2, 3, 4)$ represents the characteristic or qualities that the candidates are assessed on namely “experience”, “academic qualifications”, “attitude towards the professionalism” and “technical knowledge” respectively. After finishing the interview of all the candidates and going through their certificates and other supporting papers the hiring committee constitutes the PIVIFSES (F_P, Z) as follows:

$$(F_P, Z) =$$

$$\left\{(e_1, p, 1) = \left\{\left(\frac{u_1}{[0.2,0.4][0.3,0.6]},[0.1,0.3]\right),\left(\frac{u_2}{[0.2,0.5][0.3,0.5]},[0.1,0.2]\right),\left(\frac{u_1}{[0.6,0.9][0.05,0.1]},[0.1,0.3]\right)\right\},\right.$$

$$(e_2, p, 1) = \left\{\left(\frac{u_1}{[0,0][0.7,0.9]},[0.6,0.9]\right),\left(\frac{u_2}{[0.1,0.3][0.4,0.65]},[0.5,0.8]\right),\left(\frac{u_3}{[0.1,0.2][0.5,0.7]},[0.1,0.15]\right)\right\},$$

$$(e_3, p, 1) = \left\{\left(\frac{u_1}{[0.2,0.4][0.4,0.6]},[0,0.1]\right),\left(\frac{u_2}{[0.3,0.5][0.2,0.3]},[0.6,0.8]\right),\left(\frac{u_3}{[0,0][0.2,0.4]},[0.1,0.3]\right)\right\},$$

$$(e_4, p, 1) = \left\{\left(\frac{u_1}{[0.1,0.15][0.6,0.8]},[0.7,0.9]\right),\left(\frac{u_2}{[1,1][0,0]},[0.3,0.5]\right),\left(\frac{u_3}{[0,0][1,1]},[0.5,1]\right)\right\},$$

$$(e_1, q, 1) = \left\{\left(\frac{u_1}{[0.6,0.8][0.1,0.2]},[0.2,0.3]\right),\left(\frac{u_2}{[0.2,0.4][0.3,0.5]},[0.1,0.2]\right),\left(\frac{u_3}{[0.4,0.6][0.1,0.3]},[0.1,0.3]\right)\right\},$$

$$(e_2, q, 1) = \left\{\left(\frac{u_1}{[0.4,0.6][0.2,0.4]},[0.4,0.6]\right),\left(\frac{u_2}{[1,1][0,0]},[0.6,0.8]\right),\left(\frac{u_3}{[0.6,0.9][0,0]},[0.3,0.5]\right)\right\},$$

$$(e_3, q, 1) = \left\{\left(\frac{u_1}{[0.7,0.9][0,0.1]},[0.8,0.9]\right),\left(\frac{u_2}{[0.3,0.5][0.1,0.2]},[0.3,0.4]\right),\left(\frac{u_3}{[0,0][0.7,0.9]},[0.4,0.5]\right)\right\},$$

$$(e_4, q, 1) = \left\{\left(\frac{u_1}{[0,0][0.6,0.8]},[0.5,0.7]\right),\left(\frac{u_2}{[0.1,0.2][0.4,0.6]},[0.1,0.25]\right),\left(\frac{u_3}{[1,1][0,0]},[0.2,0.4]\right)\right\},$$

$$(e_1, r, 1) = \left\{\left(\frac{u_1}{[0.1,0.3][0.3,0.55]},[0.5,0.7]\right),\left(\frac{u_2}{[0,0.1][0.5,0.7]},[0.2,0.25]\right),\left(\frac{u_3}{[0,0][1,1]},[0,0]\right)\right\},$$

$$(e_2, r, 1) = \left\{\left(\frac{u_1}{[0.1,0.25][0.2,0.45]},[0.4,0.6]\right),\left(\frac{u_2}{[0.4,0.6][0.1,0.25]},[0.7,0.95]\right),\left(\frac{u_3}{[0.2,0.4][0.4,0.6]},[0,0.1]\right)\right\},$$

$$(e_3, r, 1) = \left\{\left(\frac{u_1}{[0.6,0.8][0,0.05]},[0.7,0.9]\right),\left(\frac{u_2}{[0.1,0.2][0.5,0.75]},[0.6,0.8]\right),\left(\frac{u_3}{[0,0.05][0.8,0.9]},[0.2,0.4]\right)\right\},$$

$$(e_1, p, 0) = \left\{\left(\frac{u_1}{[0.1,0.2][0.5,0.6]},[0.3,0.5]\right),\left(\frac{u_2}{[0.1,0.3][0.2,0.4]},[0,0]\right),\left(\frac{u_3}{[0.7,0.9][0,0]},[0,0.1]\right)\right\},$$

$$(e_3, p, 0) = \left\{\left(\frac{u_1}{[0,0][0.8,0.9]},[0.6,0.75]\right),\left(\frac{u_2}{[0.1,0.3][0.5,0.7]},[0.1,0.2]\right),\left(\frac{u_3}{[0.6,0.8][0.1,0.15]},[0.4,0.6]\right)\right\},$$

$$(e_4, p, 0) = \left\{\left(\frac{u_1}{[0.6,0.8][0,0]},[0.4,0.6]\right),\left(\frac{u_2}{[1,1][0,0]},[0,0.05]\right),\left(\frac{u_3}{[0.1,0.3][0.2,0.55]},[0.7,0.9]\right)\right\},$$

$$(e_1, q, 0) = \left\{\left(\frac{u_1}{[0.1,0.3][0.3,0.5]},[0.3,0.55]\right),\left(\frac{u_2}{[0.5,0.7][0.1,0.3]},[0.8,1]\right),\left(\frac{u_3}{[1,1][0,0]},[0.7,0.9]\right)\right\},$$

$$(e_2, q, 0) = \left\{\left(\frac{u_1}{[0.1,0.2][0.5,0.7]},[0.1,0.2]\right),\left(\frac{u_2}{[0,0.1][0.5,0.75]},[0.2,0.3]\right),\left(\frac{u_3}{[0.3,0.5][0.2,0.4]},[0,0.1]\right)\right\},$$

$$(e_3, q, 0) = \left\{\left(\frac{u_1}{[0,0][1,1]},[0.4,0.6]\right),\left(\frac{u_2}{[0.2,0.3][0.4,0.65]},[0.6,0.8]\right),\left(\frac{u_3}{[0,0][0,0.1]},[0.7,1]\right)\right\},$$

$$(e_4,q,0)=\left\{\left(\frac{u_1}{[1,1][0,0]},[0.8,1]\right),\left(\frac{u_2}{[0.2,0.4][0.3,0.5]},[0.2,0.4]\right),\left(\frac{u_3}{[0,0][0.3,0.5]},[0.8,0.9]\right)\right\},$$
$$(e_1,r,0)=\left\{\left(\frac{u_1}{[0.1,0.3][0.3,0.5]},[0.2,0.4]\right),\left(\frac{u_2}{[0.3,0.55][0.1,0.3]},[0.5,0.7]\right),\left(\frac{u_3}{[0.2,0.4][0.3,0.6]},[0.1,0.3]\right)\right\},$$
$$(e_2,r,0)=\left\{\left(\frac{u_1}{[0,0][1,1]},[0.7,0.9]\right),\left(\frac{u_2}{[0.1,0.25][0.5,0.75]},[0.5,0.7]\right),\left(\frac{u_3}{[0,0.1][0.7,0.9]},[0.1,0.15]\right)\right\},$$
$$(e_4,r,0)=\left\{\left(\frac{u_1}{[0.3,0.5][0.2,0.45]},[0,0.1]\right),\left(\frac{u_2}{[0.1,0.3][0.5,0.7]},[0,0.1]\right),\left(\frac{u_3}{[0.2,0.4][0.2,0.4]},[0.2,0.4]\right)\right\},$$

Next the PIVIFSES (F_p,Z) is used together with an algorithm to solve the decision making problem. The algorithm given below is employed by the hiring committee to determine the or most suitable candidate to be hired for the position.

The steps of algorithm are as follows:

Step 1: Input the PIVIFSES (F_p,Z).

Step 2: Find the values of $\mu_{F_p(z_i)}(u_i)-\nu_{F_p(z_i)}(u_i)$ for each element $u_i \in U$. Where $\mu_{F_p(z_i)}(u_i)=\left[\underline{\mu}_{F_p(z_i)}(u_i),\bar{\mu}_{F_p(z_i)}(u_i)\right]$ and $\nu_{F_p(z_i)}(u_i)=\left[\underline{\nu}_{F_p(z_i)}(u_i),\bar{\nu}_{F_p(z_i)}(u_i)\right]$ are the lower and upper membership and lower and upper non-membership of each element $u_i \in U$ respectively. We use [a,b] – [c,d]= [a-d, b-c].

Step 3: Find the maximum numerical grade from the intervals for the agree-PIVIFSES and disagree-PIVIFSES.

Step 4: Compute the score of each element $u_i \in U$ by taking the sum of the products of the maximum numerical grade from the intervals of each element with the corresponding maximum degree of possibility μ_i , for the agree-PIVIFSES and disagree-PIVIFSES by A_i and D_i respectively.

Step 5: Find the values of the score $r_i = A_i - D_i$ for each element $u_i \in U$.

Step 6: Determine the values of the highest scores = $\max_{\mu_i \in U}\{r_i\}$. Then the decision is to choose element u_i as optional or best solution of the problem. If there are more than one element with the highest score r_i then we have to select the candidate having max degree of possibility of the element $u_i \in U$.

Let A_i and D_i represent the score of each numerical grade for the agree-PIVIFSES and disagree-PIVIFSES in table-4.

Table-1: Values of $\mu_{F_p(z_i)}(u_i) - \nu_{F_p(z_i)}(u_i)$ for all $u_i \in U$, with interval of degree of possibility of such belongingness.

	u_1	u_2	u_3
(e_1,p,1)	[-0.4,0.1],[0.1,0.3]	[-0.3,0.2],[0.1,0.2]	[0.5,0.85],[0.1,0.3]
(e_2,p,1)	[-0.9,-0.7],[0.6,0.9]	[-0.55,-0.1],[0.5,0.8]	[-0.6,-0.3],[0.1,0.15]
(e_3,p,1)	[-0.4,0],[0,0.1]	[0,0.3],[0.6,0.8]	[-0.4,-0.2],[0.1,0.3]
(e_4,p,1)	[-0.7,-0.45],[0.7,0.9]	[1,1],[0.3,0.5]	[-1,-1],[0.5,1]
(e_1,q,1)	[0.4,0.7],[0.2,0.3]	[-0.3,0.1],[0.1,0.2]	[0.1,0.5],[0.1,0.3]
(e_2,q,1)	[0,0.4],[0.4,0.6]	[1,1],[0.6,0.8]	[0.6,0.9],[0.3,0.5]
(e_3,q,1)	[0.6,0.9],[0.8,0.9]	[0.1,0.4],[0.3,0.4]	[-0.9,-0.7],[0.4,0.5]
(e_4,q,1)	[-0.8,-0.6],[0.5,0.7]	[-0.5,-0.2],[0.1,0.25]	[1,1],[0.2,0.4]
(e_1,r,1)	[-0.45,0],[0.5,0.7]	[-0.7,-0.5],[0.2,0.25]	[-1,-1],[0.0]
(e_2,r,1)	[-0.35,0.05],][0.4,0.6]	[0.15,0.5],[0.7,0.95]	[-0.4,0],[0,0.1]
(e_3,r,1)	[0.55,0.8],[0.7,0.9]	[-0.65,-0.3],[0.6,0.8]	[-0.9,-0.75],[0.2,0.4]
(e_1,p,0)	[-0.5,-0.3],[0.3,0.5]	[-0.3,0.1],[0,0]	[0.7,0.9],[0,0.1]
(e_3,p,0)	[-0.9,-0.8],[0.6,0.75]	[-0.6,-0.2],[0.1,0.2]	[0.45,0.7],[0.4,0.6]
(e_4,p,0)	[0.6,0.8],[0.4,0.6]	[1,1],[0.0.05]	[-0.45,0.1],[0.7,0.9]
(e_1,q,0)	[-0.4,0],[0.3,0.55]	[0.2,0.6],[0.8,1]	[1,1],[0.7,0.9]
(e_2,q,0)	[-0.6,-0.03],[0.1,0.2]	[-0.75,-0.4],[0.2,0.3]	[-0.1,0.03],[0,0.1]
(e_3,q,0)	[-1,-1],[0.4,0.6]	[-0.45,-0.1],[0.6,0.8]	[-0.1,0],[0.7,1]
(e_4,q,0)	[1,1],[0.8,1]	[-0.3,0.1],[0.2,0.4]	[-0.5,-0.3],[0.8,0.9]
(e_1,r,0)	[-0.4,0],[0.2,0.4]	[0,0.45],[0.5,0.7]	[-0.4,0.1],[0.1,0.3]
(e_2,r,0)	[-1,-1],[0.7,0.9]	[-0.65,-0.25],[0.5,0.7]	[-0.9,-0.6],[0.1,0.15]

(e_4,r,0)	[-0.15,0.3],[0,0.1]	[-0.6,-0.2],[0,0.1]	[-0.2,0.2],[0.2,0.4]

Table -2: Max numerical grade for agree-PIVIFSES

	u_i	Highest numerical grade of the interval(λ_i)	Corresponding max degree of possibility(μ_i)	$\lambda_i \times \mu_i$
(e_1,p)	u_3	0.85	0.3	0.255
(e_2,p)	u_2	-0.1	0.8	-0.08
(e_3,p)	u_2	0.3	0.8	0.24
(e_4,p)	u_2	1	0.5	0.50
(e_1,q)	u_1	0.7	0.3	0.21
(e_2,q)	u_2	1	0.8	0.80
(e_3,q)	u_1	0.9	0.9	0.81
(e_4,q)	u_3	1	0.4	0.40
(e_1,r)	u_1	0	0.7	0
(e_2,r)	u_2	0.5	0.95	0.475
(e_3,r)	u_1	0.8	0.9	0.72

Therefore score (u_1)=1.73, score(u_2)=1.935, score(u_3)=0.65.

Table-3: Max numerical grade for disagree-PIVIFSES

	u_i	Highest numerical grade of the interval(λ_i)	Corresponding max degree of possibility(μ_i)	$\lambda_i \times \mu_i$
(e_1,p)	u_3	0.9	0.1	0.92
(e_3,p)	u_3	0.7	0.6	0.42
(e_4,p)	u_2	1	0.05	0.05
(e_1,q)	u_3	1	0.9	0.90
(e_2,q)	u_3	0.3	0.1	0.03
(e_3,q)	u_3	0	1	0
(e_4,q)	u_1	1	1	1
(e_1,r)	u_2	0.45	0.7	0.315

(e_2,r)	u_2	-0.25	0.7	-0.175
(e_4,r)	u_1	0.3	0.1	0.3

Score (u_1)=1.03, score(u_2)=0.19 and score(u_3)=1.44.

Table-4: The score $r_i = A_i - D_i$

A_i	D_i	r_i
score(u_1)=1.730	score(u_1)=1.03	0.7
score(u_2)=1.935	score(u_2)=0.19	1.745
score(u_3)=0.655	score(u_3)=1.44	-0.785

Then $S = \max_{\mu_i \in U}(r_i) = r_2$, therefore the hiring committee should hire the candidate u_2 to fill the vacant position.

1.10: Interval Valued Neutrosophic Soft Topological Spaces **(**Anjan Mukherjee[1], Mithun Datta[2], Florentin Smarandache[3], Neutrosophic Sets and Systems, 6(2014)17-26)

In this work we formed a topological structure on interval valued neutrosophic soft sets and establish some properties of interval valued neutrosophic soft topological space with supporting proofs and examples.

A neutrosophicset A on the universe of discourse U is defined as (a) $A=\{\langle x,\mu_A(x),\gamma_A(x),\delta_A(x)\rangle : x\in U\}$, where $\mu_A,\gamma_A,\delta_A : U \to]^-0,1^+[$ are functions such that the condition: $\forall x\in U,\ \ ^-0 \le \mu_A(x)+\gamma_A(x)+\delta_A(x) \le 3^+$ is satisfied.

Here $\mu_A(x),\gamma_A(x),\delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership respectively of the element $x\in U$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0,1^+[$. Hence we have to consider the neutrosophic set which takes the value from the subset of $[0,1]$.

(b)An *interval valued neutrosophicset* A on the universe of discourse U is defined as $A=\{\langle x,\mu_A(x),\gamma_A(x),\delta_A(x)\rangle : x\in U\}$, where $\mu_A,\gamma_A,\delta_A : U\to Int]^-0,1^+[$ are functions such that the condition: $\forall x\in U,\ \ ^-0\le sup\mu_A(x)+sup\gamma_A(x)+sup\delta_A(x)\le 3^+$ is satisfied.

In real life applications it is difficult to use interval valued neutrosophic set with interval-value from real standard or non-standard subset of $Int(]^-0,1^+[)$. Hence we consider the interval valued neutrosophic set which takes the interval-value from the subset of $Int([0,1])$ (where $Int([0,1])$ denotes the set of all closed sub intervals of $[0,1]$). The set of all interval valued neutrosophic sets on U is denoted by *IVNS(U)*.

(c) Let U be an universe set, E be a set of parameters and $A\subseteq E$. Let *IVNs(U)* denotes the set of all interval valued neutrosophic sets of U . Then the pair (f,A) is called an *interval valued neutrosophicsoftset* (*IVNSs* in short) over U , where f is a mapping given by $f : A\to IVNs(U)$. The collection of all interval valued neutrosophic soft sets over U is denoted by *IVNSs(U).*

Let *U* be an universe set, *E* be the set of parameters, $\wp(U)$ be the set of all subsets of *U*, *IVNs(U)* be the set of all interval valued neutrosophic sets in *U* and *IVSNs(U;E)* be the family of all interval valued neutrosophic soft sets over *U* via parameters in *E*.

Definition: Let (ζ_A,E) be an element of *IVNSs(U;E)*, $\wp(\zeta_A,E)$ be the collection of all interval valued neutrosophic soft subsets of (ζ_A,E) . A sub family τ of $\wp(\zeta_A,E)$ is called an interval valued neutrosophic soft topology (in short *IVNS*-topology) on (ζ_A,E) if the following axioms are satisfied:

(i) $(\phi_{\zeta_A},E),(d_{\zeta_A},E)\in\tau$

(ii) $\{(f_A^k,E):k\in K\}\subseteq\tau\Rightarrow\bigcup_{k\in K}(f_A^k,E)\in\tau$

(iii) If $(g_A,E),(h_A,E)\in\tau$ then $(g_A,E)\cap(h_A,E)\in\tau$

The triplet (ζ_A,E,τ) is called interval valued neutrosophic soft topological space (in short *IVNS*-topological space) over (ζ_A,E) . The members of τ are called τ

–open *IVNS* sets (or simply open sets). Here $\phi_{\zeta_A}, d_{\zeta_A} : A \to IVNS(U)$ is defined as $\phi_{\zeta_A}(e) = \{(x,[0,0],[1,1],[1,1]) : x \in U\}$ $\forall e \in A$ an $d_{\zeta_A}(e) = \{(x,[1,1],[0,0],[0,0]) : x \in U\}$ $\forall e \in A$.

Definition: As every *IVNS*-topology on (ζ_A, E) must contains the sets (ϕ_{ζ_A}, E) and (ζ_A, E), so the family $\vartheta = \{(\phi_{\zeta_A}, E),(\zeta_A, E)\}$ forms a *IVNS*-topology on (ζ_A, E). The topology is called indiscrete *IVNS*-topology and the triplet (ζ_A, E, ϑ) is called an indiscrete interval valued neutrosophic soft topological space (or simply indiscrete *IVNS*-topological space).

Definition: Let ξ denotes the family of all *IVNS*-subsets of (ζ_A, E). Then we observe that ξ satisfies all the axioms of topology on (ζ_A, E). This topology is called discrete interval valued neutrosophic soft topology and the triplet (ζ_A, E, ξ) is called discrete interval valued neutrosophic soft topological space (or simply discrete *IVNS*-topological space).

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