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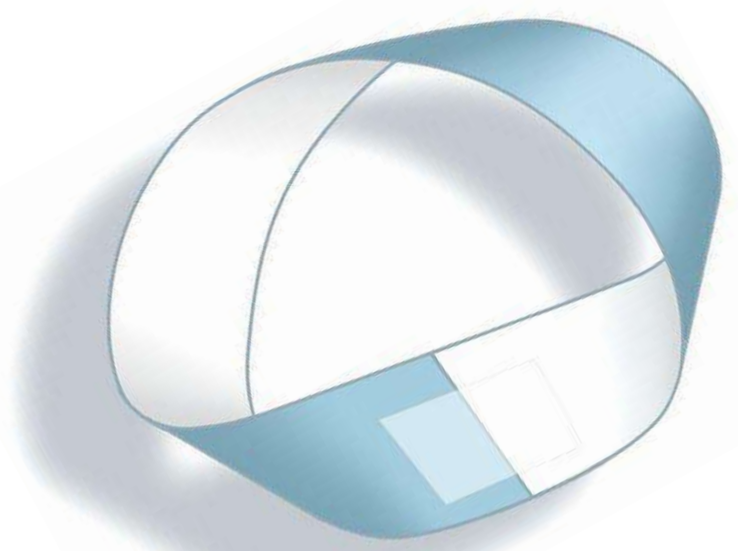

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Demystifying Fixed Point Theorems in Complete Rectangular b-Metric Spaces Using Chatterjea Contraction Mappings

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Abstract

The primary objective of this paper is to examine a fixed point through the use of Chatterjea contraction maps within a complete rectangular b metric space. This study is advancing the idea to apply this contraction to other pertinent rectangular metric spaces.

.

1 Introduction

Fixed point theory is now regarded as one of the most vital instruments across numerous scientific domains, such as applied science, computer science, engineering, and the advancement of nonlinear analysis. Within this discipline, a particularly powerful tool is the Banach contraction theorem, first put forth by Banach in 1922 [4]. Subsequent to that, this finding was elaborated upon by many researchers, employing different contractions and mappings within a range of metric spaces.

Additionally, in the year Chatterjea [8] established that " let (X, d) be a complete metric space and $S: X \to X$ be a mapping such that $\alpha \in \left[0, \frac{1}{2}\right)$ exists, then the inequality holds for all $x, y \in X$

$$d(Sx, Sy) \leq \alpha\big(d(Sx, y) + d(x, Sy)\big) \dots (1.1)$$

is satisfied, then there exists a unique fixed point of S on X ".

The counterpart of the Banach Contraction Principle in b-metric spaces was introduced by Bakhtin in 1989 [3], extending the classical concept of metric. Addressing convergence issues of measurable functions with respect to measure, Czerwik [10] was the first to generalize the Banach fixed point theorem in the context of b-metric spaces. Later, in 2009, Boriceanu et al. [6] explored fixed point theory for multivalued generalized contractions on sets equipped with two b-metrics. In a related development, Branciari [7] introduced the concept of rectangular metric spaces (RMS) by modifying the triangle inequality with a three-term expression, and provided an analogue of the Banach

Contraction Principle in this new framework.rectangular metric spaces were generalized as rectangular b-metric spaces by George et al. [12] at the year 2015.

Since then, numerous fixed point theorems involving various types of contractions have been established in rectangular metric spaces.

2 Mathematical Preliminary

In this section, we have collected a few fundamental preliminary findings that are pertinent to this study.

Definition 2.1. [3] Let $\bar{Y}$ be a non-empty set and $p \geq 1$ and $d: \bar{Y} \times \bar{Y} \rightarrow [0, \infty)$. If the mapping d satisfies the following conditions:

(b1) $d(x^*, y^*) \geq 0$;

(b2) $d(x^*, y^*) = 0$ if and only if $x^* = y^*$;

(b3) $d(x^*, y^*) = d(y^*, x^*)$;

(b4) $d(x^*, z^*) \leq p[d(x^*, y^*) + d(y^*, z^*)]$ for all $x^*, y^*, z^* \in X$

then $(\bar{Y}, d)$ is said to be a b-metric space and d is called a b-metric on $\bar{Y}$.

Example 2.2. [14] Let $K = L^p[0,1]$ be the collections of all real functions $z(t)$ such that $\int_0^1 |z(t)|^p dt < \infty$, where $t \in [0,1]$ and $0 < p < 1$. For the function $d: K \times K \rightarrow \mathbb{R}_0^+$defined by $b'(z, y) := \left(\int_0^1 (|z(t) - y(t)|^p dt\right)^{1/p}$, for each $z, y \in L^p[0,1]$, the ordered pair (K, b') forms a b-metric space with $s = 2^{1/p}$.

Definition 2.3. [7] Let $\bar{Y}$ be a non-empty set and $d: \bar{Y} \times \bar{Y} \rightarrow [0, \infty)$. If the mapping d satisfies the following conditions:

(b1) $d(x^*, y^*) \geq 0$;

(b2) $d(x^*, y^*) = 0$ if and only if $x^* = y^*$;

(b3) $d(x^*, y^*) = d(y^*, x^*)$;

(b4) $d(x^*, z^*) \leq d(x^*, p^*) + d(p^*, y^*) + d(y^*, z^*)$ for all $x^*, y^*, p^*, z^* \in X$

then $(\bar{Y}, d)$ is said to be a rectangular metric space and d is called a rectangular metric on $\bar{Y}$.

Definition 2.4. [12] Let $\bar{Y}$ be a non-empty set and $p \geq 1$ and $d: \bar{Y} \times \bar{Y} \rightarrow [0, \infty)$. If the mapping d satisfies the following conditions:

(b1) $d(x^*, y^*) \geq 0$;

(b2) $d(x^*, y^*) = 0$ if and only if $x^* = y^*$;

(b3) $d(x^*, y^*) = d(y^*, x^*)$;

(b4) $d(x^*, z^*) \leq p[d(x^*, p^*) + d(p^*, y^*) + d(y^*, z^*)]$ for all $x^*, y^*, p^*, z^* \in X$

then $(\bar{Y}, d)$ is said to be a rectangular b-metric space and d is called a rectangular b-metric on $\bar{Y}$.

Example 2.5. Let $X = P \cup Q$, where $P = \left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}\right\}$ and $Q = [1,2]$. Let us define a mapping $d: X \times X \to \mathbb{R}$ by $d(p,q) = |p - q|$ for all $p, q \in X$. Now from the definition (2.4), we easily verify that (X, d) is a rectangular b-metric space for any $S \geq 1$.

3 Main Results

Theorem 3.1. Let (X, d) be a complete rectangular b-metric space with the coefficient $S \geq 1$. Supposee that T be a self-mapping on X. If T satisfies the Chatterjea contraction mapping, i.e.,

$$d(Tx^*, Ty^*) \leq \lambda\big(d(Tx^*, y^*) + d(x^*, Ty^*)\big) \text{ ... } (3.1)$$

where,

$$0 \leq S\lambda < \frac{1}{2} \text{ ... } (3.2)$$

then T has a unique fixed point on X.

Proof. Let $x^*{}_0 \in X$ be an arbitrary point. Now we construct a sequence $\{x^*{}_n\}$ by $Tx^*{}_n = x^*{}_n$ for all $n \geq 0$. We shall show that $\{x^*{}_n\}$ be a Cauchy sequence. If $x^*{}_n = x^*{}_{n+1} = x$ for some $n \geq 0$ then trivially, x is fixed point of T.
Now we assume that $x^*{}_n \neq x^*{}_{n+1}$ for all $n \geq 0$. So by using (3.1), we have

$$\begin{aligned}
&d(x^*{}_n, x^*{}_{n+1})\\
&= d(Tx^*{}_n, Tx^*{}_{n+1})\\
&\leq \lambda d(Tx^*{}_{n-1}, x^*{}_n) + \lambda d(Tx^*{}_n, x^*{}_{n-1})\\
&= \lambda d(Tx^*{}_{n-1}, x^*{}_n) + \lambda d(x^*{}_{n+1}, x^*{}_{n-1})\\
&\leq \lambda d(x^*{}_{n-1}, x^*{}_n) + \lambda S[d(x^*{}_{n+1}, x^*{}_n) + d(x^*{}_n, Tx^*{}_{n-1}) + d(Tx^*{}_{n-1}, x^*{}_{n-1})]\\
&= \lambda S d(x^*{}_{n+1}, x^*{}_n) + \lambda S d(x^*, x^*{}_n) + \lambda S d(x^*{}_n, x^*{}_{n-1})\\
&(1 - \lambda S) d(x^*{}_n, x^*{}_{n+1}) \leq \lambda S d(x^*{}_n, x^*{}_{n-1})\\
&\Rightarrow d(x^*{}_n, x^*{}_{n+1}) \leq \frac{\lambda S}{(1 - \lambda S)} d(x^*{}_n, x^*{}_{n-1}) \leq K d(x^*{}_n, x^*{}_{n-1}).
\end{aligned}$$

Now by using induction method, we have

$$d(x^*{}_n, x^*{}_{n+1}) \leq K^n d(x^*{}_1, x^*{}_0) \qquad (3.3)$$

Here it is clear that $K < 1$ as $S\lambda < \frac{1}{2}$. Similarly, by using (3.1), (3.3) we have,

$$
\begin{aligned}
& d(x^*{}_{n}, x^*{}_{n+2}) \\
=& d(Tx^*{}_{n-1}, Tx^*{}_{n+1}) \\
\leq& \lambda d(Tx^*{}_{n-1}, Tx^*{}_{n+1}) + \lambda d(x^*{}_{n-1}, x^*{}_{n+1}) \\
\leq& \lambda d(x^*{}_{n}, x^*{}_{n+1}) + \lambda S d(x^*{}_{n-1}, x^*{}_{n}) + \lambda S d(x^*{}_{n}, x^*{}_{n+1}) + \lambda S d(x^*{}_{n+1}, x^*{}_{n+2}) \\
\leq& \lambda S d(x^*{}_{n}, x^*{}_{n+1}) + \lambda S d(x^*{}_{n+1}, x^*{}_{n+2}) + \lambda S d(x^*{}_{n+2}, x^*{}_{n+3}) + \lambda S d(x^*{}_{n+3}, x^*{}_{n+4}) \\
=& \lambda S K^{n} d(x^*{}_{1}, x^*{}_{0}) + \lambda S K^{n+1} d(x^*{}_{1}, x^*{}_{0}) + \lambda S K^{n+2} d(x^*{}_{1}, x^*{}_{0}) + \lambda S K^{n+3} d(x^*{}_{1}, x^*{}_{0}) \\
=& \lambda S K^{n} [1 + K + K^{2} + K^{3}] d(x^*{}_{1}, x^*{}_{0}) \\
\leq& \lambda S K^{n} \left[\sum_{n=0}^{\infty} K^{n}\right] d(x^*{}_{1}, x^*{}_{0}) \\
=& \frac{K^{n}}{1-K} \lambda S d(x^*{}_{1}, x^*{}_{0}) \\
=& \frac{K^{n}}{(1-K)} \lambda S d(x^*{}_{1}, x^*{}_{0})
\end{aligned}
$$

Now, for the sequence $\{x^*{}_{n}\}$, we consider $d\left(x^*{}_{n}, x^*{}_{n+j}\right)$ in two cases, as follows:
Case-1: If j is odd, say $j = 2m + 1 (n \in N)$, then we get

$$
\begin{aligned}
& d(x^*{}_{n}, x^*{}_{n+2m+1}) \\
\leq& S d(x^*{}_{n+2m+1}, x^*{}_{n+2m}) + S d(x^*{}_{n+2m}, x^*{}_{n+2m-1}) + S d(x^*{}_{n+2m-1}, x^*{}_{n}) \\
\leq& S[d(x^*{}_{n+2m+1}, x^*{}_{n+2m}) + \cdots\cdots + d(x^*{}_{n+1}, x^*{}_{n})] \\
=& S[K^{n+2m} + K^{n+2m-1} + \cdots\cdots + K^{n}] d(x^*{}_{1}, x^*{}_{0}) \\
\leq& S K^{n} [1 + K + K^{2} + \cdots\cdots + K^{2m-1} + K^{2m}] d(x^*{}_{1}, x^*{}_{0}) \\
\leq& S K^{n} \left(\sum_{p=0}^{\infty} K^{p}\right) d(x^*{}_{1}, x^*{}_{0}) \\
\leq& \left(\frac{S K^{n}}{1-K}\right) d(x^*{}_{1}, x^*{}_{0})
\end{aligned}
$$

Case-2: If j is even, say $j = 2m(n \in N)$, then we get

$$\begin{aligned}
& d(x^*{}_n, x^*{}_{n+2m}) \\
\leq & Sd(x^*{}_{n+2m}, x^*{}_{n+2m-1}) + Sd(x^*{}_{n+2m-1}, x^*{}_{n+2m-2}) + Sd(x^*{}_{n+2m-2}, x^*{}_n) \\
\leq & S[d(x^*{}_{n+2m}, x^*{}_{n+2m-1}) + \cdots\cdots + d(x^*{}_{n+2}, x^*{}_n)] \\
= & S[K^{n+2m-1} + K^{n+2m-2} + \cdots\cdots + K^{n+2}]d(x^*{}_1, x^*{}_0) \\
& +\lambda \frac{SK^n}{(1-K)} \lambda Sd(x^*{}_1, x^*{}_0) \\
\leq & SK^{n+2}\left[\sum_{n=0}^{\infty} K^n\right] d(x^*{}_1, x^*{}_0) + \lambda \frac{SK^n}{(1-K)} \lambda Sd(x^*{}_1, x^*{}_0) \\
\leq & \frac{SK^{n+2}}{1-K} d(x^*{}_1, x^*{}_0) + \lambda \frac{SK^n}{(1-K)} \lambda Sd(x^*{}_1, x^*{}_0) \\
\leq & \frac{SK^n}{(1-K)} (K^2 + \lambda) d(x^*{}_1, x^*{}_0)
\end{aligned}$$

Let, $n_0 = \min\{n_1, n_2\}$. Then from (3.4) and (3.5), we obtain

$$d\left(x_n^*, x^*{}_{n+j}\right) \leq n_0 K^n d(x^*{}_1, x_0^*) \tag{3.6}$$

Since, $K < 1$, so we get, $K^n \to 0$ as $n \to \infty$.

Therefore, we conclude that,

$$d\left(x^*{}_n, x^*{}_{n+j}\right) \to 0 \text{ as } n \to \infty.$$

$$\Longrightarrow \{x^*{}_n\} \text{ is a Cauchy Sequence on } X$$

Since X is complete. So, there exists a number $v \in X$ such that $x^*{}_n \to v$ as $n \to \infty$. Now we will demonstate that v is a fixed point of T. So, using (3.1), we have

$$\begin{aligned}
d(Tv, v) & \leq S\{d(Tv, x^*{}_{n+1}) + d(x^*{}_{n+1}, x^*{}_n) + d(x^*{}_n, v)\} \\
& = S\{d(Tv, Tx^*{}_n) + d(x^*{}_{n+1}, x^*{}_n) + d(x^*{}_n, v)\} \\
& \leq \lambda S\{d(Tv, x^*{}_n) + d(v, Tx^*{}_n)\} + Sd(x^*{}_{n+1}, x^*{}_n) + Sd(x^*{}_n, v) \\
& = \lambda S\{d(Tv, x^*{}_n) + d(v, x^*{}_{n+1})\} + Sd(x^*{}_{n+1}, x^*{}_n) + Sd(x^*{}_n, v) \\
& = \lambda S\{d(Tv, v) + d(v, v)\} + Sd(v, v) + Sd(v, v)
\end{aligned}$$

where $x^*{}_n \to v$ as $n \to \infty$

$$\Longrightarrow d(Tv, v) \leq \lambda Sd(Tv, v)$$

$$\Longrightarrow (1 - \lambda S) d(Tv, v) \leq 0$$

Since, $1-\lambda S>0$ as $\lambda S<\frac{1}{2}$. Therefore from (3.7) it follows that $d(Tv,v)\leq 0$, which leads to a contradiction. So, We conclude that $d(Tv,v)=0\Rightarrow Tv=v$.
Therefore v is a fixed point of T.
For uniqueness, let u be another fixed point of T. Then from (3.1) it follows that,

$$\begin{aligned} d(v,u) &= d(Tv,Tu) \\ &\leq \lambda\{d(Tv,u)+d(v,Tu)\} \\ &= \lambda\{d(v,u)+d(v,u)\} \\ &= 2\lambda d(v,u) < \frac{1}{S}d(v,u) \end{aligned}$$

$\Rightarrow Sd(v,u)<d(v,u)$, that leads to a contradiction.

Therefore, we must have

$$d(u,v)=0\Rightarrow u=v$$

Thus, fixed point v is unique. Hence, T has a unique fixed point on X.

Example 3.2. From the above example (2.5), we claim that (X,d) is a rectangular b-metric space with the coefficient $S=3$. Let we take $T:X\to X$ be a mapping defined by

$$T(x^*)=\begin{cases}\frac{1}{4} \text{ if } x^*\in P\\ \frac{1}{5} \text{ if } x^*\in Q\end{cases}$$

Then for the values of $\lambda=0.07$, the condition (3.1) and (3.2) is satisfied and $x^*=\frac{1}{4}$ is a unique fixed point for T.

4 Conclusion

In this paper, we have demonstrated a fixed-point theorem in the context of complete rectangular b-metric spaces using the Chatterjea contraction condition. The applicability of fixed-point results is improved by the generalization from conventional metric spaces to rectangular b-metric spaces, which permits consideration of a larger class of spaces. Our method shows that even in this more generalized context, the Chatterjea-type contraction still ensures the existence and uniqueness of fixed points under suitable conditions. These results aid in the continued development of fixed-point theory and could provide a basis for additional study of nonlinear analysis and its uses in a number of applied and mathematical domains.

Authors Contribution: All the authors have equal contributions to the preparation of this article.

Conflict of Interest: The authors declare that they have no conflict of interest.

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Multi-sequence Space of p –Absolutely Summable Sequences for $0 < p < 1$

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Abstract: In this article we introduce some multisequence spaces of real numbers related to p-absolutely Summable spaces associated with the multiplicities of elements. The main aim of this paper is to introduce the convergence of multisequences and study some basic topological and algebraic properties of multisequences.

Key words. Multisequence, ℓ_p-space, Solid space, Symmetric space, Banach space.

AMS(2010) Classification No. 40A05, 40C15, 40F05, 40H05.

1. Introduction. A multiset is a collection of objects (called elements) in which objects may occur more than once. The number of times of an element occurs in a multiset is called its multiplicity. The cardinality of a multiset is the sum of multiplicities of its elements. Multisets are of interest in a certain area of mathematics, computer science and physics. In classical set theory, a set is a well-defined collection of distinct elements. Therefore, a set is a multiset (shortly, mset) in which the multiplicity of each distinct element is one. The multiset theory which contains set theory as a special case was introduced by Cerf et al. [4] in 1971.The prime factorization of an integer $n> 0$, repeated roots of polynomials etc. are examples of multiset. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience, a multiset is written as $\{x_1/k_1, x_2/k_2, x_3/k_3, ... x_n/k_n, ...\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer.

Blizard [1] initiated the work on multiset in 1989. From 1989 to 1991, he made a thorough study of multiset theory, real valued multisets and further investigated on it [2, 3]. R. Roy et al. [6] studied multipoint, multi metric, multi open ball, multi closed ball, limit point in M-metric space, convergence of sequence of multipoints in *M*-metric space. Tripathy and Sen [7], Tripathy and Mahanta [8], Sargent [6] studied on sequence spaces in different directions.

A multisequence is a sequence whose terms may occur more than one. Here the multiplicity of an term in a multisequence is always under some restriction, it may be identical or less than a finite number. Let X be a set, then X^S denotes a multiset, created by the elements of X, where multiplicity of each element ≤ S, where w ∈ ℕ.

Throughout this paper w and ℓ_∞ denote the spaces of all sequences and bounded sequences respectively. The zero sequence is denoted by $\ddot{\Theta} = (\theta, \theta, \theta, \dots\dots\dots)$. Further by ℓ_p (for $0 < p < 1$) we denote the sequence space of all p-absolutely summable sequences, i.e.,

$$\ell_p = \left\{ x = (x_n) \in E: \sum_{n=1}^{\infty} | x_n |^p < \infty \right\}$$

and *p*-norm of this space is given by

$$\| x \| = \sum_{k=1}^{\infty} |x_k|^p < \infty.$$

2. Definitions and Background.

In this section we procure some definitions those will be used throughout the article.

Definition 2.1. A subset E of w is said to be *solid or normal* if $(x_n) \in E$ implies $(y_n) \in$ E for all sequences (y_n) such that $| y_n | \leq | x_n |$.

Definition 2.2. A sequence space E is said to be *symmetric* if $(x_n) \in E$ implies $\left(x_{\pi(n)}\right) \in E$, where π is a permutation of ℕ.

Note 2.1. If all the rearrangements of the terms of the sequence (x_n) belongs to *E*, then we say that the sequence space *E* is symmetric.

Definition 2.3. A subset E of w is said to be *convergence free*, if $(x_n) \in E$ and $x_n = 0 \Longrightarrow y_n = 0$ together implies that $(y_n) \in$ E.

Definition 2.4. Let E be a sequence space. Then E is said to be a *sequence algebra* if there is defined a product ⋆ on E such that $x, y \in E \Rightarrow x \star y \in$ E.

Definition 2.5. Let $K = \{k_1 < k_2 < k_3 \dots \dots \dots \dots < k_n\} \subset \mathbb{N}$. Let $(x_n) \in$ w. Then the *K-step space* of the sequence space E is defined by

$$\lambda_K^E = \{(x_{k_i}) \in \text{ w}: (x_n) \in E\}$$

Definition 2.6. A *canonical pre-image* (y_n) of a sequence $(x_n) \in E$, where K-step space λ_K^E is considered, is defined by

$$y_n = \begin{cases} x_n, & n \in K, \\ 0, & otherwise. \end{cases}$$

Definition 2.7. A sequence space E is said to be *monotone*, if it contains all its step spaces.

Definition 2.8. A collection of elements which are allowed to repeat is called a *multiset*. Formally, if X is a set of elements, a multiset A drawn from the set X is represented by a function $C_A: X \to \mathbb{N}_\theta$, where $\mathbb{N}_\theta$ represents the set of non-negative integers.

For each $x \in X$, $C_A(x)$ is the characteristic value or multiplicity of x in A. A multiset is a set if $C_A(x) = 0 \; or \; 1, \forall \, x \in X$.

Definition 2.9. Let $\mathbb{R}$ be the set of all real numbers. Then a set of real numbers where repetition of real numbers is allowed, is called *multiset of real numbers*, denoted by $m\mathbb{R}$, defined by
$m\mathbb{R} = \{x_i/c_i : x_i \in \mathbb{R}, c_i \in \mathbb{N}\}$.
Here, x_i/c_i represents real number x_i appears c_i times and $\mathbb{N}$ denotes the set of natural numbers.

Definition 2.10. A function whose domain is the set $\mathbb{N}$ of natural numbers and range set is the set m$\mathbb{R}$ (multiset of real numbers) is called a *Multi-sequence*.

Thus a multi-sequence is denoted symbolically as mx: $\mathbb{N} \to$ m$\mathbb{R}$, defined by

$(x_n/c_n) = (x_1/c_1, x_2/c_2, x_3/c_3, \dots \dots., x_n/c_n, \dots \dots \dots \dots)$, where $n \in \mathbb{N}$.

In this article we introduce the following definition of multisequence relating *p-absolutely summable for multiplicity c* and the space of this multisequences.

Definition 2.11. Let $0 < p < 1$. A multisequence $mx = (x_n/c_n)$ of mX^S is said to be *p-absolutely summable for multiplicity c* if

$$\sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} < \infty$$

where $mX^s = \{(x_n/c_n) \in m\mathbb{R}: c_n = card\ (x_n) \leq s,\ s \in \mathbb{N}\}$ denotes the set of all multi-sequences whose elements drawn from the sequence $X = (x_n)$ and no element in the multi-sequence occurs more than s times.

Example 1.1. Let $p = \frac{1}{2}$ and multisequence (x_n/c_n) be defined by

$$x_n/c_n = \begin{cases} \frac{1}{n^4}/5, & for\ n \leq 100 \\ \frac{1}{n^4}/4, & for\ n > 100 \end{cases}$$

This multi-sequence is $\frac{1}{2} - absolutely\ summable$ for multiplicity 4.

For,

$$\sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\}$$

$$= \sum_{n=1}^{\infty}\left\{|x_n|^{\frac{1}{2}} + |c_n - 4|^{\frac{1}{2}}\right\}$$

$$= \sum_{n=1}^{100}\left\{\left(\frac{1}{n^4}\right)^{\frac{1}{2}} + (5-4)^{\frac{1}{2}}\right\} + \sum_{n=101}^{\infty}\left\{\left(\frac{1}{n^4}\right)^{\frac{1}{2}} + (4-4)^{\frac{1}{2}}\right\}$$

$$= 2 + \left(\frac{1}{2^2} + 1\right) + \left(\frac{1}{3^2} + 1\right) + \left(\frac{1}{4^2} + 1\right) + \ \dots\dots\dots + \left(\frac{1}{100^2} + 1\right) + \sum_{n=101}^{\infty}\frac{1}{n^2}$$

= a finite number + a convergent series.

Hence, is finite.

Example 1.2. Let $p = \frac{1}{2}$ and multi-sequence (x_n/c_n) be defined by

$$x_n/c_n = \begin{cases} 1/4\,, & for\ \ n\,is\,odd \\ 2/6\,, & for\ \ n\,is\,even \end{cases}$$

Now we have, $\sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\}$

$= \sum_{n=1}^{\infty}\left\{|x_n|^{\frac{1}{2}} + |c_n - c|^{\frac{1}{2}}\right\}$

$= \sum_{n\ odd}\left(1 + |4 - c|^{\frac{1}{2}}\right) + \sum_{n\ even}\left(2^{\frac{1}{2}} + |6 - c|^{\frac{1}{2}}\right).$

So, this multi-sequence is not $p - absolutely\ summable$ for any multiplicity c.

Definition 2.12. Let 0 < p < 1. Then the *Class of* *p-absolutely summable multi-sequences* of real numbers with multiplicity c, denoted by $\ell_p^{M^c}$, is defined by

$\ell_p^{M^c} = \{(x_n/c_n) \in mX^s : \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} < \infty\}.$

For $0 < p < 1$, $the\ norm\ on\ the\ p - normed\ mspace$ is given by :

$\|\ (x_n/c_n)\ \| = \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\}.$

Definition 2.13. Let $(x_n/c_n) \in mX^S$ be a multi-sequence and $\alpha \in \mathbb{R}$. Then the scalar multiplication of this multi-sequence with, denoted by $\alpha(x_n/c_n)$, whose norm is given by

$$\|\ \alpha(x_n/c_n)\ \| = \alpha\sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\},\ \ for\ \ 0 < p < 1\,,$$

Definition 2.14. Let $(x_n/c)\ and\ (y_n/c) \in mX^S$ be two multi-sequences having same multiplicity c of each element both of the multi-sequences. Then the sum of the multi-sequences is defined by

$(x_n/c) + (y_n/c) = (x_n + y_n/c), \forall\ n \in \mathbb{N}.$

Definition 2.15. Let $(x_n/c)\ and\ (y_n/c) \in mX^S$ be two multi-sequences having same multiplicity c of each element both of the multi-sequences and let $\alpha, \beta \in \mathbb{R}$. Then the linear combination of these two multi-sequences is given by

$\alpha(x_n/c) + \beta(y_n/c) = (\alpha x_n + \beta y_n)/c.$

Definition 2.16. Let $(x_n/c) \in mX^S$ be a multi-sequence. Then the *modulus* of this multi-sequence, denoted by $|(x_n/c)|$ defined by,

$$|(x_n/c)| = \sqrt{x_n{}^2 + (c-1)^2}.$$

3. Main Results:

In this section we establish some results involving the sequence space $\ell_p^{M^c}$.

Theorem 3.1. The class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is a linear space for $0 < p < 1$.

Proof: Let the multi-sequences $(x_n/c_n), (y_n/c_n) \in \ell_p^{M^c}$, for multiplicity c, and let $\alpha, \beta \in \mathbb{R}$, where $0 < p < 1$

Then, $\sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} < \infty$

and $\sum_{n=1}^{\infty}\{|y_n|^p + |c_n - c|^p\} < \infty$.

Now we consider the multi-sequence

$$\alpha(x_n/c_n) + \beta(y_n/c_n) = (\alpha x_n + \beta y_n)/c_{n.}$$

Norm of this multi-sequence is given by

$$\|(\alpha x_n + \beta y_n)/c_n\| = \sum_{n=1}^{\infty}\{|\alpha x_n + \beta y_n|^p + |c_n - c|^p\}$$

$$\leq \sum_{n=1}^{\infty}\{|\alpha x_n|^p + |c_n - c|^p\} + \sum_{n=1}^{\infty}\{|\beta y_n|^p + |c_n - c|^p\}$$

$\leq \alpha\sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} + \beta\sum_{n=1}^{\infty}\{|y_n|^p + |c_n - c|^p\} < \infty.$

i.e., $\|(\alpha x_n + \beta y_n)/c_n\| = \sum_{n=1}^{\infty}\{|\alpha x_n + \beta y_n|^p + |c_n - c|^p\} < \infty$

Hence, the class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is a linear space for $0 < p < 1$.

Theorem 3.2. For $0 < p < 1$, the class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is a p-normed linear space with respect to the p-norm

$$\|x_n/c_n\| = \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\}.$$

Proof. Let $(x_n/c_n), (y_n/c_n) \in \ell_p^{M^c}$ and let $\lambda \in \mathbb{R}$. Then we have,

$$\| (x_n/c_n) \| = \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} = 0$$

$\Rightarrow \{|x_1|^p + |c_1 - c|^p\} + \{|x_2|^p + |c_2 - c|^p\} + \{|x_3|^p + |c_3 - c|^p\} + \ldots\ldots\ldots\ldots = 0$

$\Rightarrow \{|x_1|^p + |c_1 - c|^p\} = \{|x_2|^p + |c_2 - c|^p\} = \{|x_3|^p + |c_3 - c|^p\} = \ldots\ldots\ldots\ldots = 0$

$\Rightarrow |x_1|^p = |x_2|^p = |x_3|^p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$

and $|c_1 - c|^p = |c_2 - c|^p = |c_3 - c|^p = \ldots\ldots\ldots\ldots\ldots = 0$

$\Rightarrow x_1 = x_2 = x_3 = \ldots\ldots\ldots\ldots\ldots\ldots = 0$

and $c_1 = c_2 = c_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = c$.

Therefore, $\| (x_n/c_n) \| = 0 \Rightarrow (x_n/c_n) = (0/c) = m\widehat{\theta}$, the zero multi-sequence.

(ii)

$$\| \lambda(x_n/c_n) \|^p = \left\{\lambda \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\}\right\}^p$$

$$= \lambda^p \|x_n/c_n\|.$$

(iii) $\|(x_n/c_n) + (y_n/c_n)\| = \|(x_n + y_n/c_n)\|$

$$= \sum_{n=1}^{\infty}\{|x_n + y_n|^p + |c_n - c|^p\}$$

$$\leq \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\} + \sum_{n=1}^{\infty}\{|y_n|^p + |c_n - c|^p\}$$

$$= \| (x_n/c_n) \| + \| (y_n/c_n) \|$$

i.e., $\|(x_n/c_n) + (y_n/c_n)\| \leq \| (x_n/c_n) \| + \| (y_n/c_n) \|$.

Hence, $\ell_p^{M^c}$ is a p-normed linear space.

Theorem 3.3. The class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is solid, where $0 < p < 1$.

Proof. Let $(x_n/c_n) \in \ell_p^{M^c}$. Then $(x_n/c_n) \in mX^s$,

where, $mX^s = \{(x_n/c_n) \in m\mathbb{R}: c_n = card\ (x_n) \leq s,\ s \in \mathbb{N}\}$.

Let (y_n/c_n) be such that $|y_n/c_n| \leq |x_n/c_n|$

i.e., $\sqrt{{y_n}^2 + (c_n - 1)^2} \leq \sqrt{{x_n}^2 + (c_n - 1)^2}$

$\Rightarrow {y_n}^2 + (c_n - 1)^2 \leq {x_n}^2 + (c_n - 1)^2$

$\Rightarrow {y_n}^2 \leq {x_n}^2$

$\Rightarrow |y_n| \leq |x_n|$

$\Rightarrow |y_n|^p \leq |x_n|^p$

$\Rightarrow |y_n|^p + |c_n - c|^p \leq |x_n|^p + |c_n - c|^p$

$$\Rightarrow \sum_{n=1}^{\infty}\{|y_n|^p + |c_n - c|^p\} \leq \sum_{n=1}^{\infty}\{|x_n|^p + |c_n - c|^p\}.$$

$\Rightarrow \|y_n/c_n\| \leq \|x_n/c_n\| < \infty$.

Thus, $(y_n/c_n) \in \ell_p^{M^c}$. Hence, $\ell_p^{M^c}$ is solid, where $0 < p < 1$.

Theorem 3.4. For $0 < p < 1$, the class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is complete p-normed space with respect to the p-norm

$$\| (x_n/c_n) \| = \sum_{i=1}^{\infty} \{|x_n|^p + |c_n - c|^p\}$$

Proof: Let $(x_n/c_n) \in \ell_p^{M^c}$ $(0 < p < 1)$ be a Cauchy multi-sequence, $\forall\, n \in \mathbb{N}$. Then we have,

$$\|(x_r - x_s)/c_n\| \to 0, as\ \ r, s \to \infty.$$

i.e., $\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \{|x_r - x_s|^p + |c_n - c|^p\} \to 0, as\ \ r, s \to \infty$

$$\Rightarrow \sum_{r=1}^{\infty} \{|x_r - x_1|^p + |c_n - c|^p + |x_r - x_2|^p + |c_n - c|^p + |x_r - x_3|^p + |c_n - c|^p + \ \ldots\ldots\ldots upto\ \infty\}$$

$$\to 0, \qquad as\ \ r, s \to \infty$$

$\Rightarrow |x_r - x_s| \to 0\ \ and\ \ \ c_n \to c\ when\ r, s\ \to \infty.$

So, (x_n/c_n) is convergent and since it is arbitrary multi-sequence in $\ell_p^{M^c}$, therefore, $\ell_p^{M^c}$ is Banach space, where $0 < p < 1$.

Theorem 3.5. The class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is symmetric, where $0 < p < 1$.

Proof: Let $(x_n/c_n) \in \ell_p^{M^c}$, where $0 < p < 1$. Then we have,

$$\sum_{n=1}^{\infty} \{|x_n|^p + |c_n - c|^p\} < \infty.$$

We know that if a multi-sequence holds the above relation, then the multi-sequences formed by the rearrangements of the terms of the multi-sequence (x_n/c_n) also holds good the above relation. So, all the rearrangements of the terms of the multi-sequence (x_n/c_n) belongs to $\ell_p^{M^c}$. Hence, $\ell_p^{M^c}$ is symmetric, where $0 < p < 1$.

Theorem: 3.6. The class of p-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e**.** $\ell_p^{M^c}$ is not convergence free, where $0 < p < 1$.

Proof: The above result follows from the following example:

Let $p = \frac{1}{2}$. $(x_n/c_n) \in \ell_p^{M^c}$ be defined by

$$x_n/c_n = \begin{cases} \frac{1}{n^2}/5\,,\ when\ n \leq 200, \\ 0/4\ \ when\ \ n > 200. \end{cases}$$

Then this multisequence is $\frac{1}{2}$-absolutely summable for multiplicity 5.

Consider (y_n/c_n) be defined by

$$y_n/c_n = \begin{cases} \frac{1}{n}/2\,,\ when\ n\ is\ odd' \\ 0/4\,,\ \ when\ \ n\ is\ even. \end{cases}$$

Then $y_n/c_n \notin \ell_p^{M^c}$ for $p = 1\backslash 2$.

Theorem 3.7. The class of *p*-absolutely summable multi-sequences of multiset real numbers with multiplicity c, i.e. $\ell_p^{M^c}$ is a sequence algebra, where $0 < p < 1$.

Proof: Considering two multi-sequences $(x_n/c_n), (y_n/c_n) \in mX^w$ in $\ell_p^{M^c}$. Then using the definition of $\ell_p^{M^c}$ (where $0 < p < 1$) and term wise product of the multi-sequences $(x_n/c_n), (y_n/c_n) = (x_n y_n/c)$, it can be easily seen that $\ell_p^{M^c}$ is a sequence algebra.

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New results related to intuitionistic fuzzy operators and operations

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Abstract:

In intuitionistic fuzzy sets, several operators and operations were introduced and discussed by many researchers time to time. The characteristics of modal operators have been examined and their applications in different fields have been studied. Atanassov introduced modal operators in intuitionistic fuzzy sets and he examined some properties of these modal operators. The main objective of this paper is to investigate further some new results related to these operators over intuitionistic fuzzy sets.

Keywords:

Fuzzy sets, Intuitionistic fuzzy sets, Modal operators, Average operation.

1. Introduction

In 1983, Atanassov [1] introduced the concept of intuitionistic fuzzy set as an extension of fuzzy set earlier invented by Zadeh [12] in 1965. Since then many authors and researchers are giving much attention as well as concentration for developing intuitionistic fuzzy sets. In recent past, some results on algebraic laws in intuitionistic fuzzy sets [3,7,8,10,11] and some basic relation among modal operators [9] are discussed. It is also well known to us that every fuzzy set is intuitionistic fuzzy set but the reverse is not true. But more importantly there exist some operators by which we can transform intuitionistic fuzzy sets into fuzzy sets easily. As discussing the past, present and future of intuitionistic fuzzy sets, Atanassov[4] has remarkably mentioned about the importance of modal operators which are analogous of the modal logic operators 'necessity' and 'possibility'. Here we establish some new properties of intuitionistic fuzzy sets.

2. Preliminaries

Throughout this paper, intuitionistic fuzzy set and fuzzy set are denoted by IFS and FS respectively.

Definition 2.1 [12]. Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{<x, \mu_A(x)>: x \in X\}$, where μ_A: $x \rightarrow [0,1]$ is the membership function of the fuzzy set A. Fuzzy set is a collection of objects with graded membership i.e. having degrees of membership.

Definition 2.2 [2]. Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A= \{<x, \mu_A(x), \nu_A(x)>: x \in X\}$, where the functions μ_A, ν_A: $X \rightarrow [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Furthermore, we have $\pi_A(x)= 1- \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ i.e, π_A: $X \rightarrow [0,1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$.

$\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 2.3 [2]. LetA,B be two IFSs in X. The basic operations are defined as follows:

1. [inclution] $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall x \in X$.
2. [equality] $A= B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$.
3. [complement] $A^c = \{<x, \nu_A(x), \mu_A(x) >: x \in X\}$,
4. [union] $A \cup B = \{<x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))>: x \in X\}$.
5. [intersction] $A \cap B = \{<x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))>: x \in X\}$.
6. [addition] $A \oplus B = \{<x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)>: x \in X\}$.
7. [multiplication] $A \otimes B = \{<x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)>: x \in X\}$.
8. [difference] $A- B = \{<x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))>: x \in X\}$.
9. [symmetric difference] $A \Delta B = \{<x, \max[\min(\mu_A(x), \nu_B(x)), \min(\mu_B(x), \nu_A(x))], \min[\max(\nu_A(x)\mu_B(x), \max(\nu_B(x)\mu_A(x)]>: x \in X\}$.

.

Definition 2.4 [5] Let X be a nonempty set. If A is an IFS drawn from X, then,

(i) $\square A = \{< x, \mu_A(x), 1- \mu_A(x) > : x \in X \}$
(ii) $\Diamond A = \{< x, 1- \nu_A(x), \nu_A(x) > : x \in X\}$

For a proper IFS, $\square A \subset A \subset \Diamond A$ and $\square A \neq A \neq \Diamond A$

Definition 2.5 [5]. Let $\alpha, \beta \in [0,1]$ and $A \in$ IFS X. Then the operator $J_{\alpha,\beta}(A)$ and $J^*_{\alpha,\beta}(A)$ can be defined as

(i) $J_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha\, \pi_A(x), \beta\, \nu_A(x) > : x \in X\}$, where $\alpha + \beta \leq 1$.

(ii) $J^*_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha\,(1- \mu_A(x) - \beta\, \nu_A(x)), \beta\, \nu_A(x)> : x \in X\}$, where $\alpha + \beta \leq 1$.

Definition 2.6[6] Let A and B be two IFSs in a nonempty set X. We define the average operation denoted by$A \ominus B$ as $A \ominus B = <x, ½[\mu_A(x) + \mu_B(x)], ½[\nu_A(x) + \nu_B(x) >]$.

Theorem 2.7 [6] Let A and B be two IFSs in a nonempty set X. Then

(i) $(A\ \Delta\ B) = A - B$ iff $B \subset A$.

(ii) $(A\ \Delta\ B) = B - A$ iff $A \subset B$.

3. Main results

Here A, B ∈ IFSs means $A = \{<x, \mu_A(x), \nu_A(x)>: x \in X\}$ and $B = \{<x, \mu_B(x), \nu_B(x)>: x \in X\}$.

Theorem 3.1 For every IFS A, and for any real number $\alpha,\beta \in [0,1]$, we have

(i) $J_{\alpha,\beta}(A^c) \in$ IFS and $J^*_{\alpha,\beta}(A^c) \in$ IFS

(ii) $J_{\alpha,\beta}(A \cup B) = J_{\alpha,\beta}(A) \cup J_{\alpha,\beta}(B)$

(iii) $J^*_{\alpha,\beta}(A \cup B) = J^*_{\alpha,\beta}(A) \cup J_{\alpha,\beta}(B)$

(iv) $J_{\alpha,\beta}(A \cap B) = J_{\alpha,\beta}(A) \cap J_{\alpha,\beta}(B)$

(v) $J^*_{\alpha,\beta}(A \cap B) = J^*_{\alpha,\beta}(A) \cap J^*_{\alpha,\beta}(B)$

(vi) $J_{\alpha,\beta}(A \ominus B) = J_{\alpha,\beta}(A) \ominus J_{\alpha,\beta}(B)$

(vii) $J^*_{\alpha,\beta}(A \ominus B) = J^*_{\alpha,\beta}(A) \ominus J^*_{\alpha,\beta}(B)$

Proof (i)We have, $J_{\alpha,\beta}(A^c) = \{< x, \beta\, \nu_A(x), \mu_A(x) + \alpha\, \pi_A(x) > : x \in X\}$

Here $0 \leq \beta\, \nu_A(x) + \mu_A(x) + \alpha\, \pi_A(x) \leq 1$

Hence $J_{\alpha,\beta}(A^c) \in$ IFS.

Similarly, it can be shown that $J^*_{\alpha,\beta}(A^c) \in$ IFS.

(ii), (iii), (iv) and (v) are straightforward.

(vi) L.H.S = $J_{\alpha,\beta}(A \ominus B) = J_{\alpha,\beta}<½[\mu_A(x) + \mu_B(x)], ½[\nu_A(x) + \nu_B(x)] >$

$= <x, ½[(\mu_A(x) + \mu_B(x)) + \alpha(\pi_A(x) + \pi_B(x))], ½[\beta\,(\nu_A(x) + \nu_B(x))] >$

R.H.S = $J_{\alpha,\beta}(A) \ominus J_{\alpha,\beta}(B)$

$= < x, \mu_A(x) + \alpha\, \pi_A(x), \beta\, \nu_A(x)> \ominus < x, \mu_B(x) + \alpha\, \pi_B(x), \beta\, \nu_B(x)>$

$= \langle x, \tfrac{1}{2}[(\mu_A(x) + \mu_B(x)) + \alpha(\pi_A(x) + \pi_B(x))], \tfrac{1}{2}[\beta (\nu_A(x) + \nu_B(x))] \rangle$

Hence the proof.

(vii) Same as (vi).

Theorem 3.2 Let A and B be two IFSs in a nonempty set X and $B \subset A$. Then

(i) $\square J_{\alpha,\beta}(A \Delta B) = \square J_{\alpha,\beta}(A - B)$
(ii) $\Diamond J_{\alpha,\beta}(A \Delta B) = \Diamond J_{\alpha,\beta}(A - B)$
(iii) $\square J^*_{\alpha,\beta}(A \Delta B) = \square J_{\alpha,\beta}(A - B)$
(iv) $\Diamond J^*_{\alpha,\beta}(A \Delta B) = \Diamond J_{\alpha,\beta}(A - B)$

Proof. Obvious.

Remark 3.3 Let A and B be two IFSs in a nonempty set X and $B \not\subset A$. Then

(i) $\square J_{\alpha,\beta}(A \Delta B) \neq \square J_{\alpha,\beta}(A - B)$
(ii) $\Diamond J_{\alpha,\beta}(A \Delta B) \neq \Diamond J_{\alpha,\beta}(A - B)$
(iii) $\square J^*_{\alpha,\beta}(A \Delta B) \neq \square J_{\alpha,\beta}(A - B)$
(iv) $\Diamond J^*_{\alpha,\beta}(A \Delta B) \neq \Diamond J_{\alpha,\beta}(A - B)$

Let us consider an example. Suppose, $A = \langle .6, .1, .3 \rangle$ and $B = \langle .7, .2, .1 \rangle$ and $\alpha = .2$, $\beta = .4$.

Then we have, $A - B = \langle .2, .7, .1 \rangle$ and $A \Delta B = \langle .2, .6, .2 \rangle$.

Now, $\square J_{\alpha,\beta}(A \Delta B) = \langle .24, .76 \rangle$, $\square J_{\alpha,\beta}(A - B) = \langle .22, .78 \rangle$

$\Diamond J_{\alpha,\beta}(A \Delta B) = \langle .76, .24 \rangle$, $\Diamond J_{\alpha,\beta}(A - B) = \langle .72, .28 \rangle$

$\square J^*_{\alpha,\beta}(A \Delta B) = \langle .312, .688 \rangle$, $\square J^*_{\alpha,\beta}(A - B) = \langle .304, .696 \rangle$

$\Diamond J^*_{\alpha,\beta}(A \Delta B) = \langle .76, .24 \rangle$, $\Diamond J^*_{\alpha,\beta}(A - B) = \langle .72, .28 \rangle$

Theorem 3.4 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta \in [0,1]$, then

(i) $[\square \Diamond (J_{\alpha,\beta}(A \cup B))]^C = \Diamond \square [J_{\alpha,\beta}(A \cup B)]^C$

(ii) $[\Diamond \square (J_{\alpha,\beta}(A \cup B))]^C = \square\Diamond [J_{\alpha,\beta}(A \cup B)]^C$

(iii) $[\square \Diamond (J_{\alpha,\beta}(A \cap B))]^C = \Diamond \square [J_{\alpha,\beta}(A \cap B)]^C$

(iv) $[\Diamond \square (J_{\alpha,\beta}(A \cap B))]^C = \square\Diamond [J_{\alpha,\beta}(A \cap B)]^C$

(v) $[\square \Diamond (J^*_{\alpha,\beta}(A \cup B))]^C = \Diamond \square [J^*_{\alpha,\beta}(A \cup B)]^C$

(vi) $[\Diamond \square (J^*_{\alpha,\beta}(A \cup B))]^C = \square\Diamond [J^*_{\alpha,\beta}(A \cup B)]^C$

(vii) $[\square \lozenge (J^*_{\alpha,\beta} (A \cap B))]^C = \lozenge \square [J^*_{\alpha,\beta} (A \cap B)]^C$

(viii) $[\lozenge \square (J^*_{\alpha,\beta}(A \cap B))]^C = \square\lozenge [J^*_{\alpha,\beta} (A \cap B)]^C$

Proof (i) Now $J_{\alpha,\beta}(A \cup B) = \{< (\mu_{A\cup B} (x) + \alpha\pi_{A\cup B}(x)), \beta\nu_{A\cup B}(x) >\}$

$\lozenge(J_{\alpha,\beta}(A \cup B)) = \{< 1-\beta\nu_{A\cup B}(x), \beta\nu_{A\cup B}(x) >\}$

$\square \lozenge(J_{\alpha,\beta}(A \cup B))= \{< 1-\beta\nu_{A\cup B}(x), \beta\nu_{A\cup B}(x) >\}$

$[\square \lozenge (J_{\alpha,\beta}(A \cup B))]^C=\{<\beta\nu_{A\cup B}(x), 1-\beta\nu_{A\cup B}(x) >\}$

Again $[J_{\alpha,\beta}(A \cup B)]^C = \{<\beta\nu_{A\cup B}(x), (\mu_{A\cup B} (x) + \alpha\pi_{A\cup B}(x)) >\}$

$\square[J_{\alpha,\beta}(A \cup B)]^C= \{ <\beta\nu_{A\cup B}(x), 1-\beta\nu_{A\cup B}(x) >\}$

$\lozenge \square[J_{\alpha,\beta}(A \cup B)]^C= \{ <\beta\nu_{A\cup B}(x), 1-\beta\nu_{A\cup B}(x) >\}$

Hence $[\square \lozenge (J_{\alpha,\beta}(A \cup B))]^C = \lozenge \square [J_{\alpha,\beta}(A \cup B)]^C$

Similarly (ii) to (viii) can be proved.

Theorem 3.5 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta\in[0,1]$, then

(i) $[\square \lozenge (J_{\alpha,\beta}(A \oplus B))]^C = \lozenge \square [J_{\alpha,\beta}(A \oplus B)]^C$

(ii) $[\lozenge \square (J_{\alpha,\beta}(A \oplus B))]^C = \square\lozenge [J_{\alpha,\beta}(A\oplus B)]^C$

(iii) $[\square \lozenge (J_{\alpha,\beta} (A \otimes B))]^C = \lozenge \square [J_{\alpha,\beta} (A \otimes B)]^C$

(iv) $[\lozenge \square (J_{\alpha,\beta}(A \otimes B))]^C = \square\lozenge [J_{\alpha,\beta} (A\otimes B)]^C$

(v) $[\square \lozenge (J^*_{\alpha,\beta}(A \oplus B))]^C = \lozenge \square [J^*_{\alpha,\beta}(A \oplus B)]^C$

(vi) $[\lozenge \square (J^*_{\alpha,\beta}(A \oplus B))]^C = \square\lozenge [J^*_{\alpha,\beta}(A\oplus B)]^C$

vii) $[\square \lozenge (J^*_{\alpha,\beta} (A \otimes B))]^C = \lozenge \square [J^*_{\alpha,\beta} (A \otimes B)]^C$

(viii) $[\lozenge \square (J^*_{\alpha,\beta}(A \otimes B))]^C = \square\lozenge [J^*_{\alpha,\beta} (A\otimes B)]^C$

Proof Similar to the theorem 3.4.

Theorem 3.6 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta \in[0,1]$, then

(i) $[\square \lozenge (J_{\alpha,\beta}(A -B))]^C = \lozenge \square [J_{\alpha,\beta}(A - B)]^C$

(ii) $[\lozenge \square (J_{\alpha,\beta}(A -B))]^C = \square\lozenge [J_{\alpha,\beta}(A- B)]^C$

(iii) $[\square \lozenge (J_{\alpha,\beta} (A \Delta B))]^C = \lozenge \square [J_{\alpha,\beta} (A \Delta B)]^C$

(iv) $[\lozenge \square (J_{\alpha,\beta}(A \Delta B))]^C = \square\lozenge [J_{\alpha,\beta} (A\Delta B)]^C$

(v) $[\square \lozenge (J^*_{\alpha,\beta}(A -B))]^C = \lozenge \square [J^*_{\alpha,\beta}(A - B)]^C$

(vi) $[\Diamond \Box (J^*_{\alpha,\beta}(A - B))]^C = \Box\Diamond [J^*_{\alpha,\beta}(A - B)]^C$

(vii) $[\Box \Diamond (J^*_{\alpha,\beta} (A \Delta B))]^C = \Diamond \Box [J^*_{\alpha,\beta} (A \Delta B)]^C$

(viii) $[\Diamond \Box (J^*_{\alpha,\beta}(A \Delta B))]^C = \Box\Diamond [J^*_{\alpha,\beta} (A\Delta B)]^C$

Proof (i) Now $J_{\alpha,\beta}(A - B) = \{< (\mu_{A-B} (x) + \alpha\pi_{A-B}(x)), \beta\nu_{A-B}(x) >\}$

$\Diamond (J_{\alpha,\beta}(A - B)) = \{< 1-\beta\nu_{A-B}(x), \beta\nu_{A-B}(x) >\}$

$\Box \Diamond(J_{\alpha,\beta}(A - B)) = \{< 1-\beta\nu_{A-B}(x), \beta\nu_{A-B}(x) >\}$

$[\Box \Diamond (J_{\alpha,\beta}(A - B))]^C = \{ <\beta\nu_{A-B}(x), 1-\beta\nu_{A-B}(x) >\}$

Again $[J_{\alpha,\beta}(A - B)]^C = \{<\beta\nu_{A-B}(x), (\mu_{A-B} (x) + \alpha\pi_{A-B}(x)) >\}$

$\Box[J_{\alpha,\beta}(A - B)]^C = \{ <\beta\nu_{A-B}(x), 1-\beta\nu_{A-B}(x) >\}$

$\Diamond \Box[J_{\alpha,\beta}(A - B)]^C = \{ <\beta\nu_{A-B}(x), 1-\beta\nu_{A-B}(x) >\}$

Hence $[\Box \Diamond (J_{\alpha,\beta}(A - B))]^C = \Diamond \Box [J_{\alpha,\beta}(A - B)]^C$

Similarly (ii) to (viii) can be proved.

Theorem 3.7 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta \in[0,1]$, then

(i) $[\Box \Diamond (J_{\alpha,\beta}(A \ominus B))]^C = \Diamond \Box [J_{\alpha,\beta}(A \ominus B)]^C$

(ii) $[\Diamond \Box (J_{\alpha,\beta}(A \ominus B))]^C = \Box\Diamond [J_{\alpha,\beta}(A\ominus B)]^C$

(iii) $[\Box \Diamond (J_{\alpha,\beta} (A \$ B))]^C = \Diamond \Box [J_{\alpha,\beta} (A \$ B)]^C$

(iv) $[\Diamond \Box (J_{\alpha,\beta}(A \$B))]^C = \Box\Diamond [J_{\alpha,\beta} (A\$ B)]^C$

(v) $[\Box \Diamond (J^*_{\alpha,\beta}(A \ominus B))]^C = \Diamond \Box [J^*_{\alpha,\beta}(A \ominus B)]^C$

(vi) $[\Diamond \Box (J^*_{\alpha,\beta}(A \ominus B))]^C = \Box\Diamond [J^*_{\alpha,\beta}(A\ominus B)]^C$

(vii) $[\Box \Diamond (J^*_{\alpha,\beta} (A \$ B))]^C = \Diamond \Box [J^*_{\alpha,\beta} (A \$ B)]^C$

(viii) $[\Diamond \Box (J^*_{\alpha,\beta}(A \$B))]^C = \Box\Diamond [J^*_{\alpha,\beta} (A\$ B)]^C$

Proof Obvious.

Theorem 3.8 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta \in[0,1]$, then

(i) $[(\Box(J_{\alpha,\beta} (A)) \cup (\Diamond(J_{\alpha,\beta} (B))]^C = \Diamond [J_{\alpha,\beta} (A)]^C\cap\Box[J_{\alpha,\beta} (B)]^C$

(ii) $[(\Diamond (J_{\alpha,\beta} (A)) \cup (\Box (J_{\alpha,\beta} (B))]^C = \Box [J_{\alpha,\beta} (A)]^C\cap\Diamond[J_{\alpha,\beta} (B)]^C$

(iii) $[(\Box(J_{\alpha,\beta} (A)) \cap (\Diamond(J_{\alpha,\beta} (B))]^C = \Diamond [J_{\alpha,\beta} (A)]^C\cup\Box[J_{\alpha,\beta} (B)]^C$

(iv) $[(\Diamond (J_{\alpha,\beta} (A)) \cap (\Box (J_{\alpha,\beta} (B))]^{C} = \Box [J_{\alpha,\beta} (A)]^{C} \cup \Diamond [J_{\alpha,\beta} (B)]^{C}$

(v) $[(\Box(J^*_{\alpha,\beta} (A)) \cup (\Diamond(J^*_{\alpha,\beta} (B))]^{C} = \Diamond [J^*_{\alpha,\beta} (A)]^{C} \cap \Box [J^*_{\alpha,\beta} (B)]^{C}$

(vi) $[(\Diamond (J^*_{\alpha,\beta} (A)) \cup (\Box (J^*_{\alpha,\beta} (B))]^{C} = \Box [J^*_{\alpha,\beta} (A)]^{C} \cap \Diamond [J^*_{\alpha,\beta} (B)]^{C}$

(vii) $[(\Box(J^*_{\alpha,\beta} (A)) \cap (\Diamond(J^*_{\alpha,\beta} (B))]^{C} = \Diamond [J^*_{\alpha,\beta} (A)]^{C} \cup \Box [J^*_{\alpha,\beta}(B)]^{C}$

(viii) $[(\Diamond (J^*_{\alpha,\beta} (A)) \cap (\Box (J^*_{\alpha,\beta} (B))]^{C} = \Box [J^*_{\alpha,\beta}(A)]^{C} \cup \Diamond [J^*_{\alpha,\beta} (B)]^{C}$

Proof (i) Now $[(\Box(J_{\alpha,\beta} (A)) \cup (\Diamond(J_{\alpha,\beta} (B))]$
$= \Box\{< (\mu_A(x) + \alpha\pi_A(x)), \beta\nu_A(x) >\} \cup \Box\{< (\mu_B(x) + \alpha\pi_B(x)), \beta\nu_B(x) >\}$
$= \{< (\mu_A(x) + \alpha\pi_A(x)), 1 - \mu_A(x) + \alpha\pi_A(x) >\} \cup \{< 1 - \beta\nu_B(x), \beta\nu_B(x) >\}$
$= \{< \max(\mu_A(x) + \alpha\pi_A(x), 1 - \beta\nu_B(x)), \min(1 - \mu_A(x) -\alpha\pi_A(x), \beta\nu_B(x))>\}$
Therefore $[(\Box(J_{\alpha,\beta} (A)) \cup (\Diamond(J_{\alpha,\beta} (B))]^{C}$
$= \{<\min(1 - \mu_A(x) - \alpha\pi_A(x), \beta\nu_B(x)), \max(\mu_A(x) + \alpha\pi_A(x), 1 - \beta\nu_B(x)), >\}$

Again $\Diamond [J_{\alpha,\beta} (A)]^{C} \cap \Box [J_{\alpha,\beta} (B)]^{C}$
$= \{\Diamond <\beta\nu_A(x), (\mu_A(x) + \alpha\pi_A(x)) >\} \cap \{\Box <\beta\nu_B(x), (\mu_B(x) + \alpha\pi_B(x)) >\}$
$=\{< 1- \mu_A(x) - \alpha\pi_A(x), \mu_A(x) + \alpha\pi_A(x) >\} \cap \{<\beta\nu_B(x), 1- \beta\nu_B(x) >\}$
$= \{< \min(1 - \mu_A(x) - \alpha\pi_A(x), \beta\nu_B(x)), \max(\mu_A(x) + \alpha\pi_A(x), 1 - \beta\nu_B(x)), >\}$

Hence $[(\Box(J_{\alpha,\beta} (A)) \cup (\Diamond(J_{\alpha,\beta} (B))]^{C} = \Diamond [J_{\alpha,\beta} (A)]^{C} \cap \Box [J_{\alpha,\beta} (B)]^{C}$

Similarly (ii) to (viii) can be proved.

Theorem 3.9 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta \in [0,1]$, then

(i) $[(\Box(J_{\alpha,\beta} (A)) \oplus (\Diamond(J_{\alpha,\beta} (B))]^{C} = \Diamond [J_{\alpha,\beta} (A)]^{C} \otimes \Box [J_{\alpha,\beta} (B)]^{C}$

(ii) $[(\Diamond (J_{\alpha,\beta} (A)) \oplus (\Box (J_{\alpha,\beta} (B))]^{C} = \Box [J_{\alpha,\beta} (A)]^{C} \otimes \Diamond [J_{\alpha,\beta} (B)]^{C}$

(iii) $[(\Box(J_{\alpha,\beta} (A)) \otimes (\Diamond(J_{\alpha,\beta} (B))]^{C} = \Diamond [J_{\alpha,\beta} (A)]^{C} \oplus \Box [J_{\alpha,\beta} (B)]^{C}$

(iv) $[(\Diamond (J_{\alpha,\beta} (A)) \otimes (\Box (J_{\alpha,\beta} (B))]^{C} = \Box [J_{\alpha,\beta} (A)]^{C} \oplus \Diamond [J_{\alpha,\beta} (B)]^{C}$

(v) $[(\Box(J^*_{\alpha,\beta} (A)) \oplus (\Diamond(J^*_{\alpha,\beta} (B))]^{C} = \Diamond [J^*_{\alpha,\beta} (A)]^{C} \otimes \Box [J^*_{\alpha,\beta} (B)]^{C}$

(vi) $[(\Diamond (J^*_{\alpha,\beta} (A)) \oplus (\Box (J^*_{\alpha,\beta} (B))]^{C} = \Box [J^*_{\alpha,\beta} (A)]^{C} \otimes \Diamond [J^*_{\alpha,\beta} (B)]^{C}$

(vii) $[(\Box(J^*_{\alpha,\beta} (A)) \otimes (\Diamond(J^*_{\alpha,\beta} (B))]^{C} = \Diamond [J^*_{\alpha,\beta} (A)]^{C} \oplus \Box [J^*_{\alpha,\beta}(B)]^{C}$

(viii) $[(\Diamond (J^*_{\alpha,\beta} (A)) \otimes (\Box (J^*_{\alpha,\beta} (B))]^C = \Box [J^*_{\alpha,\beta}(A)]^C \oplus \Diamond [J^*_{\alpha,\beta} (B)]^C$

Proof Obvious.

Remark 3.10 Let X be nonempty and A and B be two IFSs of X.Then for $\alpha,\beta \in [0,1]$

(a) $(\Box J_{\alpha,\beta} (A)) \Delta (\Box J_{\alpha,\beta} (B)) \neq \Box J_{\alpha,\beta} (A \Delta B)$
(b) $(\Diamond J_{\alpha,\beta} (A)) \Delta (\Diamond J_{\alpha,\beta}(B)) \neq \Diamond J_{\alpha,\beta}(A \Delta B)$
(c) $(\Box J^*_{\alpha,\beta} (A)) \Delta (\Box J^*_{\alpha,\beta} (B)) \neq \Box J^*_{\alpha,\beta} (A \Delta B)$
(d) $(\Diamond J^*_{\alpha,\beta} (A)) \Delta (\Diamond J^*_{\alpha,\beta} (B)) \neq \Diamond J^*_{\alpha,\beta} (A \Delta B)$
(e) $(\Box J_{\alpha,\beta} (A)) - (\Box J_{\alpha,\beta} (B)) \neq \Box J_{\alpha,\beta} (A - B)$
(f) $(\Diamond J_{\alpha,\beta} (A)) - (\Diamond J_{\alpha,\beta} (B)) \neq \Diamond J_{\alpha,\beta} (A - B)$
(g) $(\Box J^*_{\alpha,\beta} (A)) - (\Box J^*_{\alpha,\beta} (B)) \neq \Box J^*_{\alpha,\beta} (A- B)$
(h) $(\Diamond J^*_{\alpha,\beta} (A)) - (\Diamond J^*_{\alpha,\beta} (B)) \neq \Diamond J^*_{\alpha,\beta}(A- B)$

Example : Let A= < .7, .2, .1> and B= < .6, .3, .1> and $\alpha = .2$, $\beta = .4$.

(a) $(\Box J_{\alpha,\beta} (A)) \Delta (\Box J_{\alpha,\beta} (B))$ = < .38, .62 >
And $\Box J_{\alpha,\beta} (A \Delta B)$ = < .32, .68>

So $(\Box J_{\alpha,\beta} (A)) \Delta (\Box J_{\alpha,\beta} (B)) \neq \Box J_{\alpha,\beta} (A \Delta B)$

Similarly we can show that

(b) $(\Diamond J_{\alpha,\beta} (A)) \Delta (\Diamond J_{\alpha,\beta} (B))$ = < .09, .91 > and $\Diamond J_{\alpha,\beta} (A \Delta B)$ = < .82, .18 >
(c) $(\Box J^*_{\alpha,\beta} (A)) \Delta (\Box J^*_{\alpha,\beta} (B))$ = < .338, .662>and $\Box J^*_{\alpha,\beta} (A \Delta B)$ =< .404, .596>
(d) $(\Diamond J^*_{\alpha,\beta} (A)) \Delta (\Diamond J^*_{\alpha,\beta} (B))$ = < .09, .91 >and $\Diamond J^*_{\alpha,\beta} (A \Delta B)$ = < .82, .18>
(e) $(\Box J_{\alpha,\beta} (A)) - (\Box J_{\alpha,\beta} (B))$ = < .38, .62 >and $\Box J_{\alpha,\beta} (A - B)$ = < .32, .68>
(f) $(\Diamond J_{\alpha,\beta} (A)) - (\Diamond J_{\alpha,\beta} (B))$ = < .09, .91>and $\Diamond J_{\alpha,\beta} (A - B)$ = < .82, .18>
(g) $(\Box J^*_{\alpha,\beta} (A)) - (\Box J^*_{\alpha,\beta} (B))$ = < .338, .662>and $\Box J^*_{\alpha,\beta} (A - B)$ = < .404, .596>
(h) $(\Diamond J^*_{\alpha,\beta} (A)) - (\Diamond J^*_{\alpha,\beta} (B))$ = < .09, .91 >and $\Diamond J^*_{\alpha,\beta} (A - B)$ = < .82, .18>

Conclusion :

Some new properties are established in intuitionistic fuzzy sets with the help of certain operations together with the modal operators. Further investigations are going on and it seems that more new properties and relations may be obtained in future. These will certainly give a new dimension for developing the literature.

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An Interval Type 2 Fuzzy AHP Framework in mHealth Applications for Type 2 Diabetes Mellitus Management

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Abstract: Recent advancements in information technology have led to notable changes in the medical field. This study aims to identify and evaluate the criteria weight for assessing mobile health (mHealth) applications designed for the self-management of T2DM. With the recent doctor-to-patient ratio in our country being (1:1900, as reported by the Times of India in 2024), the development of self-management mHealth applications becomes vital. To improve patient satisfaction, these applications need to be intuitive and user-friendly. The primary objective of this research is to establish the criteria weight for evaluating the usability and effectiveness of T2DM-focused mHealth applications. The Interval Type 2 Fuzzy Analytic Hierarchy Process (IT2F-AHP) is utilized to identify and prioritize these criteria, offering a structured approach to determining the most significant factors that impact user satisfaction and application usability.

Keywords: T2DM , mHealth application, IT2F-AHP.

1. Introduction

Healthcare professionals and organizations looking for cost-effective solutions to provide high-quality patient care may find mobile devices to be advantageous [1,2]. mHealth applications, which utilize mobile technology, offer a range of health benefits to users [3]. As mobile devices become more widespread, the use of mHealth apps that support medical treatments is also on the rise. One significant health concern today is type 2 diabetes mellitus (T2DM), a chronic disease that requires effective management through personalized treatment plans, regular nutritional counseling, blood glucose (BG) monitoring, and medication management [4].

mHealth applications hold potential for helping T2DM patients with self-management and monitoring, but their use can be complicated and time-consuming [5]. With the fast growth of mHealth options, selecting the most suitable app is challenging. Interface design and user experience are key factors in determining app usability, making a thorough usability evaluation essential for choosing the best option .

Usability, as defined by ISO 9241-11 [6], measures how well a product helps users achieve their goals efficiently, effectively, and with satisfaction. In diabetes apps, this refers to how patients, clinicians, or caregivers use the app to manage diabetes. Users expect the app to be time-saving, accurate, and reliable for tasks like tracking blood glucose and carbohydrate intake. Common features include automated data transfer, activity logs, reminders, educational tools, and communication options [7,8] .

1.1 Motivation

Finding the best mHealth app for T2DM monitoring is a complex task. Multi-Criteria Decision-Making (MCDM) methodologies can be applied to address this issue. MCDM is a structured, multi-dimensional approach that helps solve decision-making problems across various fields by considering all relevant factors. It enhances the decision-making process by making it more rational and efficient, thus helping identify the most suitable option among available alternatives [9].

This study aims to apply an IT2F-AHP approach in evaluating mHealth applications for T2DM management. The motivation behind this choice lies in the need to improve the quality of decision-making in selecting mHealth applications by incorporating both expert input and the uncertain nature of user needs. By leveraging IT2F-AHP, this research will provide a more robust and comprehensive evaluation framework, ultimately supporting the development and recommendation of more effective mHealth solutions for diabetes management.

2. Literature Review

Liang and Mendel [10] created an interval type-2 fuzzy logic system, introducing upper and lower membership functions and an inference method for Gaussian primary functions. Mendel and John[11] further developed type-2 fuzzy sets to reduce uncertainties in rule-based systems by proposing a simplified representation and deriving formulas for union, intersection, and complement without using the extension principle. Manoj Kumar's [12] study proposes an IT2FS based AHP framework to identify key factors affecting the sustainability of the Indian tea industry, demonstrating its effectiveness in complex decision-making.

Omer Soner [13] developed a hybrid methodology for multiple-Criteria decision-making in maritime transportation, combining Analytic Hierarchy Process (AHP) and VIKOR techniques within an IT2F framework to improve decision-making under uncertainty.

Gupta's [14] study assesses the usability and effectiveness of various mHealth applications for managing T2DM. It utilizes three multi-dimensional MCDM methods—TOPSIS, VIKOR, and PROMETHEE II—to evaluate five leading T2DM mHealth apps: Glucose Buddy, mySugr, Diabetes: M, Blood Glucose Tracker, and OneTouch Reveal.

Additionally, K. Gupta [15] introduces two hybrid MCDM approaches, CODAS-FAHP and MOORA-FAHP, to appraise these applications based on ten criteria. The FAHP is applied to enhance weight estimation by addressing uncertainties in expert judgments, while the CODAS and MOORA methods are used to rank the applications based on usability and effectiveness.

3. Preliminaries

In this section, we examine and elaborate on the foundational definitions of fuzzy sets and Interval Type-2 Fuzzy Sets (IT2FS)

Definition 1. Fuzzy Sets (FS)

A FS [16] Y within the information universe V can be described as a collection of elements, which can be expressed as:

$$Y= \{(x, \mu(x)|x \epsilon V)\} \quad (1)$$

Here, $\mu(x)$ represents the degree of membership of x in Y such that the values lie between 0 and 1.

Definition 2. Type 2 Fuzzy Set (T2FS)

A T2FS [17] $\tilde{S}$ in Y is a fuzzy set where the membership function is itself a fuzzy set. The membership function is known as type 2 membership function. A T2FS $\tilde{S}$ is defined as :

$$\tilde{S} = \{((x,\mu), \mu_{\tilde{A}}(x,\mu)); \forall x \in Y, \forall u \in J_x \subseteq [0,1]\} \tag{2}$$

Where $0 \leq \mu_{\tilde{A}}(x, \mu) \leq 1$ is the type 2 membership function (secondary), while J_x is the primary membership function.

The T2FS $\tilde{S}$ also can be represented as follows;

$$\tilde{S} = \int_{x\epsilon X}\int_{\mu\epsilon J_x}\mu_{\tilde{A}}(x,\mu)/(x, \mu), J_x \epsilon\ [0,1] \tag{3}$$

where $J_x \subseteq [0,1]$ and $\iint$ denote union over all admissible x and u.

Definition 3. Interval Type 2 Fuzzy Set (IT2FS)

IT2FS [18] are the special case of this defination where $\mu_{\tilde{A}} = \tilde{1}$, $\forall$ x∈Y. For an IT2FS $\tilde{S}$, the following defination holds true if

$$\tilde{S} = \int_{x\epsilon X}\int_{\mu\epsilon J_x} 1/(x, \mu) \tag{4}$$

Definition 4. Trapezoidal Interval Type-2 Fuzzy Set (TIT2FS)

TIT2FS are represented as follows [11] ;

$$\begin{aligned}\tilde{\tilde{S}}_i &= (\tilde{S}_i^u, \tilde{S}_i^l)\\ &= \left(\left(s_{i1}^u, s_{i2}^u, s_{i3}^u, s_{i4}^u; H_1(\tilde{S}_i^u), H_2(\tilde{S}_i^u)\right), \left(s_{i1}^l, s_{i2}^l, s_{i3}^l, s_{i4}^l; H_1(\tilde{S}_i^l), H_2(\tilde{S}_i^l)\right)\right)\end{aligned} \tag{5}$$

where $\tilde{S}_i^u$ and $\tilde{S}_i^l$ are type-1 fuzzy set, $s_{i1}^u, s_{i2}^u, s_{i3}^u, s_{i4}^u, s_{i1}^l, s_{i2}^l, s_{i3}^l$ and s_{i4}^l are the reference point of the IT2FS $\tilde{\tilde{S}}_i$;$H_j(\tilde{S}_i^u)$ shows the membership value of the element $s_{i(j+1)}^u$ in the upper trapezoidal membership function $\tilde{S}_i^u$; $1 \leq \text{j} \leq 2, H_j(\tilde{S}_i^u)$ denotes the membership value of the element $s_{i(j+1)}^l$ in the lower trapezoidal membership function $\tilde{S}_i^l$; $1 \leq \text{j} \leq 2, H_j(\tilde{S}_i^l)$ [19]

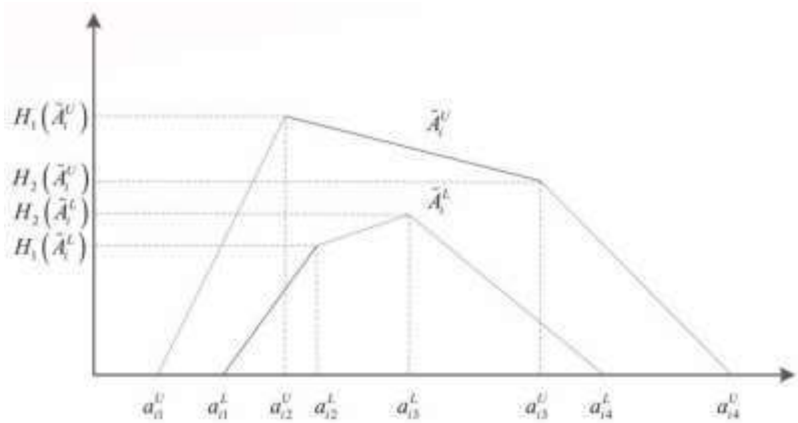

Fig.1 Trapezoidal interval type-2 fuzzy set (TIT2FS)

$H_1(\tilde{S}_i^u) \in [0,1], H_2(\tilde{S}_i^u) \in [0,1], H_1(\tilde{S}_i^l) \in [0,1], H_2(\tilde{S}_i^l) \in [0,1]$ and $1 \leq \text{i} \leq \text{n}$.

Operations on IT2FS:

Let 'K' be a crisp number and consider $\tilde{\tilde{S}}_1, \tilde{\tilde{S}}_2$ are the following IT2FS as;

$$\tilde{\tilde{S}}_1 = \left(s_{11}^u, s_{12}^u, s_{13}^u, s_{14}^u; H_1(\tilde{S}_1^u), H_2(\tilde{S}_1^u)\right), \left(s_{11}^l, s_{12}^l, s_{13}^l, s_{14}^l; H_1(\tilde{S}_1^l), H_2(\tilde{S}_1^l)\right)$$

$$\tilde{\tilde{S}}_2 = \left(s_{21}^u, s_{22}^u, s_{23}^u, s_{24}^u; H_1(\tilde{S}_2^u), H_2(\tilde{S}_2^u)\right), \left(s_{21}^l, s_{22}^l, s_{23}^l, s_{24}^l; H_1(\tilde{S}_2^l), H_2(\tilde{S}_2^l)\right)$$

Chen and Lee [19] outline the arithmetic operations for these numbers as described below:

Addition

$$\begin{aligned}\tilde{\tilde{S}}_1 \oplus \tilde{\tilde{S}}_2 = \Big(& s_{11}^u + s_{21}^u, s_{12}^u + s_{22}^u, s_{13}^u + s_{23}^u, s_{14}^u \\ & + s_{24}^u; min\left(H_1(\tilde{S}_1^u); H_1(\tilde{S}_2^u)\right), min\left(H_2(\tilde{S}_2^u); H_2(\tilde{S}_2^u)\right)\Big), \Big(s_{11}^l \\ & + s_{21}^l, s_{12}^l + s_{22}^l, s_{13}^l + s_{23}^l, s_{14}^l \\ & + s_{24}^l; min\left(H_1(\tilde{S}_1^l); H_1(\tilde{S}_2^l)\right), min\left(H_2(\tilde{S}_2^l); H_2(\tilde{S}_2^l)\right)\Big)\end{aligned} \quad (6)$$

Type equation here.

Substraction

$$\tilde{\tilde{S}}_1 \ominus \tilde{\tilde{S}}_2 = \Big(s_{11}^u - s_{21}^u, s_{12}^u - s_{22}^u, s_{13}^u - s_{23}^u, s_{14}^u - s_{24}^u; min\left(H_1(\tilde{S}_1^u); H_1(\tilde{S}_2^u)\right), min\left(H_2(\tilde{S}_2^u); H_2(\tilde{S}_2^u)\right)\Big), \Big(s_{11}^l - s_{21}^l, s_{12}^l - s_{22}^l, s_{13}^l - s_{23}^l, s_{14}^l - s_{24}^l; min\left(H_1(\tilde{S}_1^l); H_1(\tilde{S}_2^l)\right), min\left(H_2(\tilde{S}_2^l); H_2(\tilde{S}_2^l)\right)\Big) \tag{7}$$

Multiplication

$$\tilde{\tilde{S}}_1 \otimes \tilde{\tilde{S}}_2 = \Big(s_{11}^u \times s_{21}^u, s_{12}^u \times s_{22}^u, s_{13}^u \times s_{23}^u, s_{14}^u \times s_{24}^u; min\left(H_1(\tilde{S}_1^u); H_1(\tilde{S}_2^u)\right), min\left(H_2(\tilde{S}_2^u); H_2(\tilde{S}_2^u)\right)\Big), \Big(s_{11}^l \times s_{21}^l, s_{12}^l \times s_{22}^l, s_{13}^l \times s_{23}^l, s_{14}^l \times s_{24}^l; min\left(H_1(\tilde{S}_1^l); H_1(\tilde{S}_2^l)\right), min\left(H_2(\tilde{S}_2^l); H_2(\tilde{S}_2^l)\right)\Big) \tag{8}$$

Multiplication with crisp number

$$\mathrm{K}\tilde{\tilde{S}}_1 = \Big((K \times s_{11}^u, K \times s_{12}^u, K \times s_{13}^u, K \times s_{14}^u; H_1(\tilde{S}_1^u), H_2(\tilde{S}_1^u)\Big), \Big(K \times s_{11}^l, K \times s_{12}^l, K \times s_{13}^l, K \times s_{14}^l; H_1(\tilde{S}_1^l); H_1(\tilde{S}_2^l)\Big) \tag{9}$$

Division

$$\frac{\tilde{\tilde{\mathrm{s}}}_{\mathrm{ıȷ}}}{\tilde{\tilde{\mathrm{t}}}_{\mathrm{ıȷ}}} = \left(\frac{\mathrm{s}_1^{\mathrm{u}}}{\mathrm{t}_4^{\mathrm{u}}}, \frac{\mathrm{s}_2^{\mathrm{u}}}{\mathrm{t}_3^{\mathrm{u}}}, \frac{\mathrm{s}_3^{\mathrm{u}}}{\mathrm{t}_2^{\mathrm{u}}}, \frac{\mathrm{s}_4^{\mathrm{u}}}{\mathrm{t}_1^{\mathrm{u}}}, \min\left(\mathrm{H}_1^{\mathrm{u}}(\mathrm{s}), \mathrm{H}_1^{\mathrm{u}}(\mathrm{t})\right), \min\left(\mathrm{H}_2^{\mathrm{u}}(\mathrm{s}), \mathrm{H}_2^{\mathrm{u}}(\mathrm{t})\right)\right) \tag{10}$$

$$\left(\frac{\mathrm{s}_1^{\mathrm{u}}}{\mathrm{t}_4^{\mathrm{u}}}, \frac{\mathrm{s}_2^{\mathrm{u}}}{\mathrm{t}_3^{\mathrm{u}}}, \frac{\mathrm{s}_3^{\mathrm{u}}}{\mathrm{t}_2^{\mathrm{u}}}, \frac{\mathrm{s}_4^{\mathrm{u}}}{\mathrm{t}_1^{\mathrm{u}}}, \min(\mathrm{H}_1^{\mathrm{u}}(\mathrm{s}), \mathrm{H}_1^{\mathrm{u}}(\mathrm{t})), \min(\mathrm{H}_2^{\mathrm{u}}(\mathrm{s}), \mathrm{H}_2^{\mathrm{u}}(\mathrm{t})\right)$$

4. Proposed Methodology Fuzzy AHP

Saaty (1980) [20] introduced the AHP for making decisions that involve multiple criteria, and it has been widely used in practice. This method helps decision-makers break down problems into a hierarchy, including goals, factor, sub-factor. Buckley (1985) [18] enhanced Saaty's AHP by adding FS to account for uncertainty. In this paper, we use Buckley's approach to calculate the importance of each criterion.The method's steps are outlined below.

Step 1: A pairwise comparison matrix based on IT2FSs is developed for each criterion in the hierarchical framework. Using TIT2FS scale [21] are shown in Table 1.

$$\tilde{\tilde{A}} = \begin{bmatrix} 1 & \widetilde{\widetilde{a_{12}}} & \cdots & \widetilde{\widetilde{a_{1n}}} \\ 1/\widetilde{\widetilde{a_{21}}} & 1 & \cdots & \widetilde{\widetilde{a_{2n}}} \\ \vdots & \vdots & \ddots & \vdots \\ 1/\widetilde{\widetilde{a_{n1}}} & 1/\widetilde{\widetilde{a_{n2}}} & \cdots & 1 \end{bmatrix} \tag{11}$$

Where,

$$\tilde{\tilde{a}} = \left(\left(a_{11}^u, a_{12}^u, a_{13}^u, a_{14}^u; H_1(a_{12}^u), H_2(a_{13}^u)\right)\right), \left(\left(a_{21}^l, a_{22}^l, a_{23}^l, a_{24}^l; H_1(a_{22}^l), H_2(a_{23}^l)\right)\right)$$

And

$$1/\tilde{\tilde{a}} = \left(\left(a_{11}^u, a_{12}^u, a_{13}^u, a_{14}^u; H_1(a_{12}^u), H_2(a_{13}^u)\right)\right), \left(\left(a_{11}^u, a_{12}^u, a_{13}^u, a_{14}^u; H_1(a_{12}^u), H_2(a_{13}^u)\right)\right)$$

Table 1. IT2F scale of qualitative variables

Qualitative Variables	IT2FS	Reciprocal IT2FS
Equally Important (1)	(1,1,1,1;1,1) (1,1,1,1;0.9,0.9)	(1,1,1,1;1,1) (1,1,1,1;0.9,0.9)
Intermidiate Value (2)	(1,2,2,3;1,1) (1.5,2,2,2.5;0.9,0.9)	(0.33,0.5,0.5,1;1,1) (0.4,0.5,0.5,0.67;0.9,0.9)
Moderately More Important (3)	(2,3,3,4;1,1) (2.5,3,3,3.5;0.9,0.9)	(0.25,0.33,0.33,0.5;1,1) (0.29,0.33,0.33,0.4;0.9,0.9)
Intermidiate Value (4)	(3,4,4,5;1,1) (3.5,4,4,4.5;0.9,0.9)	(0.2,0.25,0.25,0.33;1,1) (0.22,0.25,0.25,0.29;0.9,0.9)
Strongly More Important (5)	(4,5,5,6;1,1) (4.5,5,5,5.5;0.9,0.9)	(0.17,0.2,0.2,0.25;1,1) (0.18,0.2,0.2,0.22;0.9,0.9)
Intermediate Value (6)	(5,6,6,7;1,1) (6.5,7,7,7.5;0.9,0.9)	(0.14,0.17,0.17,0.2;1,1) (0.15,0.17,0.17,0.18;0.9,0.9)
Very Strongly More Important (7)	(6,7,7,8;1,1) (6.5,7,7,7.5;0.9,0.9)	(0.13,0.14,0.14,0.4;1,1) (0.13,0.14,0.14,0.13;0.9,0.9)
Intermediate Value (8)	(7,8,8,9;1,1) (7.5,8,8,8.5;0.9,0.9)	(0.11,0.13,0.13,0.14;1,1) (0.12,0.13,0.13,0.13;0.9,0.9)
Extremely Important (9)	(8,9,9,10;1,1) (8.5,9,9,9.5;0.9,0.9)	(0.1,0.11,0.11,0.13;1,1) (0.11,0.11,0.11,0.12;0.9,0.9

Step 2: The consistency of the fuzzy preference evaluation is evaluated. To determine the consistency ratio (CR) of a matrix, the consistency index (CI) is first calculated as follows.

$$\text{CI}=(\lambda_{max} - m)/(m - 1) \quad (12)$$

Where,$A_w = \lambda_{max} w$. Next, the consistency index (CI) is compared with the random index (RI), which is based on the matrix order (m) and is obtained from Saaty's table (1980). The largest eigenvalue, $\lambda_{max}$of the pairwise judgment matrix A is used. The matrix is consistent if λ_{max} equals m, and it is always greater than or equal to m. A consistency ratio (CR) of 0.1 or less is deemed acceptable. The CR is calculated as follows:

$$CR = CI / RI \tag{13}$$

Step 3: The geometric mean method is used to calculate the fuzzy geometric mean(FGM) as outlined below.

$$\tilde{\tilde{r}}_i = \left[\widetilde{\widetilde{a_{ij}}} \otimes \ldots \otimes \widetilde{\widetilde{a_{ij}}}\right]^{1/n} \tag{14}$$

Where,

$$\sqrt[n]{\widetilde{\widetilde{a_{ij}}}} = \left(\left(\sqrt[n]{a_{ij1}^{u}}, \sqrt[n]{a_{ij2}^{u}}, \sqrt[n]{a_{ij3}^{u}}, \sqrt[n]{a_{ij4}^{u}}; H_1^u(a_{ij}), H_2^u(a_{ij})\right), \left(\sqrt[n]{a_{ij1}^{l}}, \sqrt[n]{a_{ij2}^{l}}, \sqrt[n]{a_{ij3}^{l}}, \sqrt[n]{a_{ij4}^{l}}; H_1^l(a_{ij}), H_2^l(a_{ij})\right)\right)$$

Step 4: The fuzzy weights assigned to each factor are determined through the following process.

$$\widetilde{\widetilde{p_i}} = \tilde{\tilde{r}}_i \otimes \left[\tilde{\tilde{r}}_1 \oplus \cdots \oplus \tilde{\tilde{r}}_i \oplus \cdots \oplus \tilde{\tilde{r}}_n\right]^{-1} \tag{15}$$

The fuzzy weight obtained from the pairwise comparison matrix is known as the local weight. To calculate the global weight for each sub-factor, the local weight is multiplied by the local weight of its parent factor at the higher level.

Step 5: The defuzzified value of A is calculated using the center of area (COA) method (Kahraman et al., 2014) [22], which determines the weights for each key performance indicator.

$$\text{DTtrT} = \frac{\frac{(U_u - L_u)+(\beta_u \cdot m_{1u} - L_u)+(\alpha_u \cdot m_{2u} - L_u)}{4} + L_u + \left[\frac{(U_l - L_l)+(\beta_l \cdot m_{1l} - L_l)+(\alpha_l \cdot m_{2l} - L_l)}{4} + L_l\right]}{2} \tag{16}$$

5. Case Study

This study aims to identify and assign weights to the criteria used to evaluate the usability of top mHealth applications. The IT2F-AHP is chosen for this study because it involves only a few pairwise judgment matrices, making it a simpler and more systematic approach .

The factor and their sub-factor for evaluating mHealth applications were carefully selected. Initially, relevant factor were identified through a review of existing literature [15]. Subsequently, expert opinions were obtained on these factor. As a result, ten factor and twenty-nine sub-factor were chosen for evaluation, as outlined in Table 2.

Table 2. Evaluation factor and related sub-factor for assessing mHealth applications for T2DM.

Factor	Sub-Factor
Learnability (F1)	Familiarity (F1.1) Learning time (F1.2) Minimal Action (F1.3)
Efficiency (F2)	No. of Taps (F2.1) Task Completion Rate (F2.2) Response Time (F2.3) Ease of Use (F2.4) Connection (F2.5)
Memorability (F3)	Saving (F3.1) Retain (F3.2) Remainder (F3.3)
Aesthetic (F4)	Attractive (F4.1) Appeal (F4.2) Organized (F4.3)
Error (F5)	Presence of Error (F5.1)
Navigation (F6)	Search (F6.1) Intutive (F6.2) Involvement (F6.3)
Readability (F7)	Legible (F7.1) Understandable (F7.2)
Cognitive Load (F8)	Essentially (F8.1) Presentation (F8.2)

Provision for Physically Challenged User (F9)	Weak Muscle Control (F9.1) Low Vision (F9.2) Hearing Impairment (F9.3)
Satisfaction (F10)	Provision (F10.1) Finding correct Information (F10.2) Improvement (F10.3) Recommendation (F10.4)

In this study, the beneficial factor are F1, F2, F3, F4, F7, F9, and F10. In contrast, the non-beneficial factor are F5, F6, and F8. The main aim of this approach is to find the best mHealth applications for managing T2DM. We calculated the weights for each factor and sub-factor using the IT2F-AHP.

6. Result Analysis

6.1 Determination of the factor weights

Ten primary factor and twenty-nine sub-factor were identified for evaluating the usability of mHealth applications, based on expert assessments and a comprehensive literature review. Detailed information on each factor and sub-factor was collected through expert feedback. This data was then converted into interval type 2 trapizoidal fuzzy numbers using the fuzzy linguistic scale outlined in Table 1.

The fuzzy assessment matrices for the factor, using interval type 2 trapizoidal fuzzy scale, is presented in Table 3. Fuzzy comparison matrices for all sub-factor are shown in Tables (6-15) [15] . Consistency Index (CI) and Consistency Ratio (CR) were calculated as per Step 2 of the FAHP method to assess the consistency ratio. As all CR values were below 0.1, the pairwise judgment matrix, derived from expert input, is considered consistent and valid for further analysis.

Table 3. Fuzzy assessment matrices comparing the factor in pairs.

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
F1	1	1/(3)	1/(4)	1/(3)	1/(3)	1/(3)	1/(3)	1/(3)	1/(4)	1/(3)
F2	(3)	(1)	(4)	1/(2)	1/(2)	1/(2)	1/(3)	1/(2)	1/(3)	1/(3)
F3	1/(3)	1/(4)	(1)	1/(3)	1/(4)	1/(3)	1/(4)	1/(4)	1/(4)	1/(4)
F4	(4)	(2)	(3)	(1)	(2)	(3)	(3)	(3)	(2)	1/(2)
F5	(3)	(2)	(4)	1/(2)	(1)	(3)	(2)	(3)	(3)	(4)
F6	(3)	(2)	(3)	1/(3)	1/(3)	(1)	(2)	(2)	(2)	(3)
F7	(3)	(3)	(4)	1/(3)	1/(2)	1/(2)	(1)	(3)	(2)	(2)
F8	(3)	(2)	(4)	1/(3)	1/(3)	1/(2)	1/(3)	(1)	(4)	(2)
F9	(4)	(3)	(4)	1/(2)	1/(3)	1/(2)	1/(2)	1/3	(1)	1/(3)
F10	(3)	(3)	(4)	(2)	1/(4)	(3)	1/(2)	(2)	(3)	(1)

The geometric mean of each row is calculated using the values from Table 3. For the FGM ($\tilde{\tilde{r}}_i$) is calculated with equation 14 as follows.

Table 4. FGM of each row

	FGMs
$\tilde{\tilde{r}}_1$	(0.27,0.35,0.35,0.49;1,1) (0.31,0.35,0.35,0.41;0.9,0.9)
$\tilde{\tilde{r}}_2$	(0.37,0.52,0.52,0.85;1,1) (0.44,0.52,0.52,0.65;0.9,0.9)
$\tilde{\tilde{r}}_3$	(0.25,0.31,0.31,0.42;1,1) (0.28,0.31,0.31,0.36;0.9,0.9)
$\tilde{\tilde{r}}_4$	(1.32,2.05,2.05,2.65;1,1) (1.69,2.05,2.05,5.99;0.9,0.9)
$\tilde{\tilde{r}}_5$	(1.18,1.19,1.19,2.70;1,1) (1.55,1.91,1.91,5.65;0.9,0.9)
$\tilde{\tilde{r}}_6$	(0.33,1.47,1.47,2.05;1,1) (1.21,1.47,1.47,3.44;0.9,0.9)
$\tilde{\tilde{r}}_7$	(0.96,1.43,1.43,2.07,;1,1) (1.19,1.43,1.43,3.37;0.9,0.9)
$\tilde{\tilde{r}}_8$	(0.71,0.99,0.99,1.44;1,1) (0.84,0.95,0.95,1.47;0.9,0.9)
$\tilde{\tilde{r}}_9$	(0.63,0.86,0.86,1.29;1,1) (0.74,0.77,0.77,1.29;0.9,0.9)
$\tilde{\tilde{r}}_{10}$	(1.12,1.66,1.66,2.28;1,1) (1.39,1.66,1.66,4.83;0.9,0.9)

Table 5. Fuzzy and Crisp weights of factor

Factor	Fuzzy weights	Crisp Weights
F1	(0.016,0.032,0.032,0.069;1,1) (0.012,0.032,0.032,0.041;0.9,0.9)	0.032
F2	(0.022,0.047,0.047,0.119;1,1) (0.018,0.047,0.047,0.065;0.9,0.9)	0.050
F3	(0.015,0.028,0.028,0.059;1,1) (0.011,0.028,0.028,0.036;0.9,0.9)	0.029
F4	(0.079,0.185,0.185,0.371;1,1) (0.068,0.185.0.185,0.599;0.9,0.9)	0.228
F5	(0.071,0.107,0.107,0.378;1,1) (0.062,0.172,0.172,0.565;0.9,0.9)	0.200
F6	(0.020,0.132,0.132,0.287;1,1) (0.048,0.132,0.132,0.344;0.9,0.9)	0.150
F7	(0.058,0.129,0.129,0.290;1,1) (0.048,0.129,0.129,0.337;0.9,0.9)	0.153
F8	(0.043,0.089,0.089,0.202;1,1) (0.034,0.086,0.086,0.147;0.9,0.9)	0.095
F9	(0.038,0.077,0.077,0.181;1,1) (0.030,0.069,0.069,0.129;0.9,0.9)	0.082
F10	(0.083,0.149,0.149,0.319;1,1) (0.056,0.149,0.149,0.483;0.9,0.9)	0.188

The fuzzy and crips weights of the factor are claculated with the help of equation 15 and 16. Following the same technique, the importance weights of the sub-factor are determined. The table (6-15) presents the expert assessments of the sub-factor in relation to the corresponding factor.

Table 6. The fuzzy judgment matrix for the sub-factor associated with the learnability (F1) criterion.

F1	F1.1	F1.2	F1.3
F1.1	(1)	(3)	(2)
F1.2	(1/3)	(1)	(1/2)
F1.3	(1/2)	(2)	(1)

Table 7. The fuzzy judgment matrix for the sub-factor associated with the efficiency (F2) criterion.

F2	F2.1	F2.2	F2.3	F2.4	F2.5
F2.1	(1)	(1/2)	(1/3)	(1/5)	(2)
F2.2	(2)	(1)	(2)	(1/3)	(3)
F2.3	(3)	(1/2)	(1)	(1/2)	(2)
F2.4	(5)	(3)	(2)	(1)	(5)
F2.5	(1/2)	(1/3)	(1/2)	(1/5)	(1)

Table 8. The fuzzy judgment matrix for the sub-factor associated with the memorability (F3) factor.

F3	F3.1	F3.2	F3.3
F3.1	(1)	(3)	(3)
F3.2	(1/3)	(1)	(3)
F3.3	(1/3)	(1/3)	(1)

Table 9. The fuzzy judgment matrix for the sub-factor related to the aesthetic (F4) factor.

F4	F4.1	F4.2	F4.3
F4.1	(1)	(3)	(3)
F4.2	(1/3)	(1)	(3)
F4.3	(1/3)	(1/3)	(1)

Table 10. The fuzzy judgement comparison matrix for the sub-factor related to the error (F5) factor.

F5	F5.1
F5.1	(1)

Table 11. The fuzzy judgement matrix for the sub-factor related to the navigation (F6) factor.

F6	F6.1	F6.2	F6.3
F6.1	(1)	(1/2)	(3)
F6.2	(2)	(1)	(3)
F6.3	(1/3)	(1/3)	(1)

Table 12. The fuzzy judgment matrix for the sub-factor related to the readability (F7) factor.

F7	F7.1	F7.2
F7.1	(1)	(3)
F7.2	(1/3)	(1)

Table 13. The fuzzy judgment matrix for the sub-factor related to the cognitive load (F8) factor.

F8	F8.1	F8.2
F8.1	(1)	(1/3)
F8.2	(3)	(1)

Table 14. The fuzzy judgment matrix for the sub-factor related to the provision for physically challenged users (F9) factor.

F9	F9.1	F9.2	F9.3
F9.1	(1)	(1/4)	(1/3)
F9.2	(4)	(1)	(2)
F9.3	(3)	(1/2)	(1)

Table 15. The fuzzy judgment matrix for the sub-factor related to the satisfaction (F10) factor.

F10	F10.1	F10.2	F10.3	F10.4
F10.1	(1)	(1/3)	(1/2)	(1/4)
F10.2	(3)	(1)	(3)	(1/3)
F10.3	(2)	(1/3)	(1)	(1/3)
F10.4	(4)	(3)	(3)	(1)

Once the judgment matrices were established, the weights for each factor and its sub-factor were calculated using the Geometric Mean (GM) method. The sub-factor weights were subsequently determined, and their consistency was confirmed. Furthermore, as presented in Table16, the global weights were derived by multiplying the weights by the corresponding sub-factor weights.

Table 16. Weights of factor, sub-factor and global weights.

Factor	***Sub-Factor***	***Factor Weight***	***Sub-factor Weights***	***Global Weights***
(F1)	(F1.1)	0.032	0.548	0.017536
	(F1.2)		0.179	0.005728
	(F1.3)		0.315	0.01008
(F2)	(F2.1)	0.05	0.098	0.0049
	(F2.2)		0.218	0.0109
	(F2.3)		0.201	0.01005
	(F2.4)		0.443	0.02215
	(F2.5)		0.076	0.0038
(F3)	(F3.1)	0.029	0.586	0.016878
	(F3.2)		0.285	0.008265
	(F3.3)		0.140	0.00406
(F4)	(F4.1)	0.228	0.343	0.078204
	(F4.2)		0.152	0.034656
	(F4.3)		0.532	0.1211296
(F5)	(F5.1)	0.2	1.000	0.2
(F6)	(F6.1)	0.15	0.336	0.0504
	(F6.2)		0.513	0.07695
	(F6.3)		0.170	0.0255
(F7)	(F7.1)	0.153	0.742	0.113526
	(F7.2)		0.252	0.038556

(F8)	(F8.1)	0.095	0.252	0.02394
	(F8.2)		0.742	0.074049
(F9)	(F9.1)	0.082	0.126	0.010332
	(F9.2)		0.560	0.04592
	(F9.3)		0.333	0.027906
	(F10.1)		0.097	0.018236
(F10)	(F10.2)	0.188	0.273	0.051324
	(F10.3)		0.144	0.027072
	(F10.4)		0.503	0.094564

7. **Result & Conclusion** In this study, we undertook a detailed multi-criteria usability evaluation of a mHealth application specifically designed for managing T2DM. The evaluation utilized the IT2F-AHP to assess various attributes. The research identified a total of 10 primary factor (factor) and 29 secondary factor (sub-factor) based on an analysis of mHealth application features and expert assessments, which were incorporated into the questionnaire-based evaluation process.

The IT2F-AHP methodology was employed to determine the weights of these factor. The computed crisp weights for each factor are presented above. The analysis reveals that the most significant main factor holds a weight of 0.228, with the next most significant factor holding a weight of 0.200. Among the sub-factor, the most important sub-factor has a weight of 0.11. This is followed by 'Organized' with a weight of 0.12, 'Recommendation' with a weight of 0.09, 'Attractive' with a weight of 0.07, and 'Presentation' with a weight of 0.070. Conversely, the sub-criteria with the lowest weights include 'Connection' with a weight of 0.003, 'Number of Taps' with a weight of 0.004, and 'Learning Time' with a weight of 0.005. These findings highlight the relative significance of various factor and sub-factor in the usability evaluation of the mHealth application.

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A Novel Entropy-Based MCDM Framework under Fermatean Fuzzy Environment

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Abstract

In the realm of decision-making, FFSs (Fermatean fuzzy sets) serve as a potent mechanism for capturing and representing experts judgement and insights. This research introduces a novel entropy measure particularly designed for FFSs, aimed at quantifying the degree of fuzziness inherent in FFSs. To ensure the reliability of this measure, the study establishes that the proposed Fermatean fuzzy entropy adheres to the fundamental axioms required for a valid fuzzy entropy measure. Building on this foundation, a new multicriteria decision-making (MCDM) methodology is developed, utilizing the FF entropy. Within this framework, each FFN (Fermatean fuzzy number) is treated as a distinct piece of indication, and the determination of criteria weights is based on the entropy measure associated with FFSs. Following this, the alternatives are evaluated by computing an aggregated FFN across all criteria, utilizing the FF-weighted power average aggregation operator. This operator incorporates the criteria weights into the aggregation process, ensuring a thorough evaluation. The proposed approach is particularly adept at handling uncertainty in decision-making scenarios, significantly mitigating information loss at the time of the process. The practicality and effectiveness of the approach are ultimately validated through application to real-world instances, demonstrating its robustness and utility in complex decision-making environments.

Keywords: Fermatean fuzzy entropy, Fermatean fuzzy number, Fermatean fuzzy sets, Multicriteria decision-making

1. Introduction

Uncertainty is a fundamental aspect of decision-making in real-world scenarios. Initially, decisions were made using precise values, but this approach failed to account for the complexity and unpredictability present in many situations. To address this, Zadeh[1] introduced the FS (Fuzzy Sets) concept, that uses degrees of membership to model uncertainty. However, as decision-making environments became more intricate, the limitations of Fuzzy sets became evident, prompting the development of more advanced models like Intuitionistic Fuzzy Sets (IFS) and Pythagorean Fuzzy Sets (PFS). These models, though improvements, encountered their own challenges, such as when the sum of membership along with the non-membership degrees in IFS, or the squared sum in PFS, exceeded the value of one. To address these shortcomings, Senapati and Yager[2] introduced the FFS, an extension of both PFS and IFS. FFS offers an improved way to represent uncertain information in decision-making issues by incorporating both membership & non-membership degrees. A crucial concept in decision-making is entropy, which measures uncertainty or disorder within a system. When combined with fuzzy logic, this leads to the concept of fuzzy entropy, which quantifies the uncertainty present within Fuzzy Sets. This idea has been further extended to FFE (Fermatean Fuzzy Entropy), which is used to capture uncertainty in decision-making situations modeled by Fermatean Fuzzy Sets.

Fermatean Fuzzy Entropy is particularly useful in MCDM (Multi-Criteria Decision-Making), where it helps to find out the weights of various criteria by measuring the uncertainty associated with each. To better handle Fermatean fuzzy data, a new FFE measure has been developed. This measure effectively reflects the degree of fuzziness in FFS while satisfying the required axioms of a fuzzy entropy measure.

1.1 Motivation

The motivation for developing this new entropy measure stemmed from the complexities and challenges associated with existing MCDM methods like AHP, SWARA, and CRITIC, which often involve lengthy and intricate processes for determining criteria weights. The goal is to create an approach that not only meets the axiomatic requirements of fuzzy entropy measures but also simplifies the process of determining criteria weights, especially when dealing with large datasets. This new entropy measure enables a more efficient and straightforward way to process Fermatean fuzzy information, making it easier to determine criteria weights without the need for complex calculations.

Furthermore, using FFE to estimate the weights of the criteria, a novel Fermatean fuzzy MCDM approach is devised. The suggested approach is capable of handling unclear information in the Fermatean fuzzy environment as well as decision-making situations having the unknown criterion weight.

1.2 Contribution

To summarize, the following are the main contributions made in this paper:

1. A novel FFE measure is introduced to quantify the degree of fuzziness in FFS. This new measure adheres to the axiomatic necessities of fuzzy entropy measures.

2. A new MCDM technique is suggested, utilizing Fermatean fuzzy entropy. This method addresses decision-making problems having unknown criterion weights and effectively manages uncertainty within the Fermatean fuzzy framework.
3. The feasibility as well as effectiveness of the suggested method have been depicted using numerical examples. The benefits of the new method in handling MCDM problems within a Fermatean fuzzy environment are emphasized.

2. Literature review

The literature (Table 1) reveals one of the popular approaches for addressing multiple competing criteria through entropy measures. In recent years, several studies have been completed, and this section presents an overview of some of these works.

Table 1: Literature on some relevant entropy measures.

Author	Entropy measure expression
Szmidt, E., & Kacprzyk, J. (2001) [3]	E(F) = $\frac{1}{n}\sum_{i=1}^{n}\frac{\max count\ (F_i \cap F_i{}^c)}{\max count\ (F_i \cup F_i{}^c)}$
Huang, Guo-shun. (2007) [4]	E(F) = 1- $\frac{1}{n}\sum_{i=1}^{n}\lvert\mu_F(z_i) - \nu_F(z_i)\rvert$
Huang, Guo-shun. (2007) [4]	E(F) = 1- $\frac{1}{n}\sum_{i=1}^{n}\lvert\mu_F^2(z_i) - \nu_F^2(z_i)\rvert$
Huang, Guo-shun. (2007) [4]	E(F) = 1- $\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\lvert\mu_F^2(z_i) - \nu_F^2(z_i)\rvert^p}$, p ≥ 1
Peng, X., Yuan, H., & Yang, Y. (2017)[5]	E(F) = $\frac{\sum_{i=1}^{n} min\left(\mu_F^2(z_i),\nu_F^2(z_i)\right)}{\sum_{i=1}^{n} max\left(\mu_F^2(z_i),\nu_F^2(z_i)\right)}$
Xue, Wenting, et al. (2018) [6]	E(F) = $\frac{1}{n}\sum_{i=1}^{n}\{1-(\mu_F^2(z_i)+\ \nu_F^2(z_i))\ \lvert\mu_F^2(z_i) - \nu_F^2(z_i)\rvert\}$
Deng, Z., & Wang, J. (2021) [7]	E(F) = 1- $\frac{1}{n}\sum_{i=1}^{n}(\ (\mu_F^3(z_i)-\nu_F^3(z_i))(\mu_F^3(z_i)+\nu_F^3(z_i)))^2$
Mishra, A. R., & Rani, P. (2021)[8]	E(F) = $\frac{1}{n}\sum_{i=1}^{n}\left[1-\sin\left(\frac{(\mu_F^3(z_i)-\nu_F^3(z_i))}{2(1+\pi_F^3(z_i))}\right)\pi\right]$
Alahmadi, Reham A., et al. (2023) [9]	E(F) = 1- $\frac{1}{n}\sum_{i=1}^{n}2(\ \lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert$ - $\lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert^2)$
Alahmadi, Reham A., et al. (2023) [9]	E(F) =1- $\frac{1}{n}\sum_{i=1}^{n}\frac{2\lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert}{1+\lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert^2}$
Alahmadi, Reham A., et al. (2023) [9]	E(F) = 1- $\frac{1}{n}\sum_{i=1}^{n}\min\ (1,\ 2\lvert\ \mu_F^3(z_i)-\nu_F^3(z_i)\rvert\ )$
Alahmadi, Reham A., et al. (2023) [9]	E(F) = 1- $\frac{1}{n}\sum_{i=1}^{n}\frac{2(\ \lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert-\lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert^2)}{1-\lvert\mu_F^3(z_i)-\nu_F^3(z_i)\rvert^2}$

Several studies highlight the significance of entropy measures based on MCDM techniques. Deng and Wang [7] employed FFE to find out the criteria weights within the MCDM framework. Chang, Kuei-Hu, et al. [10] utilized a hybrid FFS and FFE approach to prioritize product failure risks by considering expert cognitive information for more accurate risk ranking. Azadfallah, M. [19] studied the crisp entropy to "determine weights in decision-making. Mishra, Arunodaya Raj, et al.[20] developed an interval-valued Pythagorean fuzzy decision support system (DSS) to analyze blockchain platforms in healthcare supply chains", using entropy-based and PIPRECIA models for weighting criteria under uncertainty.

3. Preliminaries

In this section, we give a brief outline of the key definitions related to Fermatean Fuzzy Sets (FFS).

Definition 1. Let $\alpha = \{A_1, A_2, \ldots, A_n\}$ be a specified set. A FFS β on α is mathematically represented by the following expression:

$$\beta = \{ < A_i, (\mu_\beta(A_i), \nu_\beta(A_i))> \mid A_i \in \alpha>\} \quad (1)$$

Here μ_β: $\alpha \to [0,1]$ and ν_β:$\alpha \to [0,1]$ represent the membership degree and non-membership degree of an element $A_i \in \alpha$ to the set β, correspondingly, this satisfies the condition that

$$0 \leq (\mu_\beta(A_i))^3 + (\nu_\beta(A_i))^3 \leq 1.$$

The degree of indeterminacy of $A_i \in \alpha$ provided as:

$$\pi_\beta(A_i) = \sqrt[3]{1 - {\mu_\beta}^3(A_i) - {\nu_\beta}^3(A_i)} \quad (2)$$

To simplify, Senapati and Yager [2] referred to $(\mu_\beta(A_i), \nu_\beta(A_i))$ as a FFN and defined it as follows:

$$\beta = (\mu_\beta, \nu_\beta), \text{ where } \mu_\beta, \nu_\beta \in [0,1] \text{ and } 0 \leq {\mu_\beta}^3 + {\nu_\beta}^3 \leq 1.$$

Definition 2. Consider $\beta = (\mu_\beta, \nu_\beta)$, $\beta_1 = (\mu_{\beta_1}, \nu_{\beta_1})$, $\beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$. Senapati and Yager [2] specified several operations on these FFNs, which are described below:

1. $\beta_1 \cap \beta_2 = (\min\{\mu_{\beta_1}, \mu_{\beta_2}\}, \max\{\nu_{\beta_1}, \nu_{\beta_2}\})$
2. $\beta_1 \cup \beta_2 = (\max\{\mu_{\beta_1}, \mu_{\beta_2}\}, \min\{\nu_{\beta_1}, \nu_{\beta_2}\})$
3. $\beta^c = (\nu_\beta, \mu_\beta)$

4. $\beta_1 \geq \beta_2$ if and only if $\mu_{\beta_1} \geq \mu_{\beta_2}$ and $\nu_{\beta_1} \leq \nu_{\beta_2}$

Definition 3. To assess the size of two FFNs, Senapati and Yager [2] introduced the score function for an FFN, defined as follows:

Let $\beta = (\mu_\beta, \nu_\beta)$ be an FFN, the score function for β is expressed by

$$S(\beta) = {\mu_\beta}^3 - {\nu_\beta}^3 \ , S(\beta) \in [-1, 1] \qquad (3)$$

Definition 4. Let $\beta = (\mu_\beta, \nu_\beta)$ be an FFN. The accuracy function of β is represented as:

$$Acc(\beta) = {\mu_\beta}^3 + {\nu_\beta}^3 \ , Acc(\beta) \in [0, 1] \qquad (4)$$

Definition 5. Suppose β_1 and β_2 are two FFNs. Let S(β_1) and S(β_2) represent their respective score values, while Acc(β_1) & Acc(β_2) denote their respective accuracy values. Then

1. If S(β_1) > S(β_2), then $\beta_1 > \beta_2$
2. If S(β_1) < S(β_2), then $\beta_1 < \beta_2$
3. If S(β_1) = S(β_2), then
   - If Acc(β_1) > Acc(β_2), then $\beta_1 > \beta_2$
   - If Acc(β_1) = Acc(β_2), then $\beta_1 = \beta_2$
   - If Acc(β_1) < Acc(β_2), then $\beta_1 < \beta_2$

4. Proposed Fermatean Fuzzy entropy measure

The following section details the axiomatic formulation of FFE, as per principles of intuitionistic fuzzy entropy. Using this foundational concepts, we develop a new FFE that fulfills the necessary requirements. The axiomatic formulation of Fermatean fuzzy entropy is outlined below:

Definition 6: Let $\beta_1 = \{ < A_i, (\mu_{\beta_1}(A_i), \nu_{\beta_1}(A_i))> \mid A_i \in Z>\}$ and $\beta_2 = \{ < z_i, (\mu_{\beta_2}(A_i), \nu_{\beta_2}(A_i))> \mid A_i \in Z>\}$ be two Fermatean fuzzy sets (FFS). "A function E: FFSs (Z) → [0,1] is considered an FFE measure if it fulfills the following requirements:

1. E(β) = 0, iff F is a crisp set

2. E(β)= 1, iff $(\mu_\beta(A_i))^3$ = $(\nu_\beta(A_i))^3$ for any $z_i \in Z$

3. E(β_1) ≤ E(β_2) if F_1 is less fuzzy than F_2, that is, for any $A_i \in Z$

   $(\mu_{\beta_1}(A_i))^3 \leq (\mu_{\beta_2}(A_i))^3$ and $(\nu_{\beta_1}(A_i))^3 \geq (\nu_{\beta_2}(A_i))^3$ for $(\mu_{\beta_2}(A_i))^3 \leq (\nu_{\beta_2}(A_i))^3$

   $(\mu_{\beta_1}(A_i))^3 \geq (\mu_{\beta_2}(A_i))^3$ and $(\nu_{\beta_1}(A_i))^3 \leq (\nu_{\beta_2}(A_i))^3$ for $(\mu_{\beta_2}(A_i))^3 \geq (\nu_{\beta_2}(A_i))^3$

4. E(β) = E(β^C)

Fuzzy entropy serves as a key measure for quantifying the level of uncertainty within fuzzy sets. Zadeh[21] initially introduced entropy to assess the fuzziness in FS. Later, Luca and Termini [22] refined and expanded this concept. As per entropy measures for IFS, we propose a new measure for FFS.

Definition 7: Let $\beta = \{ < A_i, (\mu_\beta(A_i), \nu_\beta(A_i))> \mid A_i \in Z >$ "be an FFS in the universe of discourse " A = $\{A_1, A_2, \ldots, A_n\}$. The entropy measure of β has been explained as

$$E(\beta) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{2(\mu_\beta^3(A_i)-\nu_\beta^3(A_i))}{\mu_\beta^3(A_i)+\nu_\beta^3(A_i)+1}\right]^2 \qquad (5)$$

Theorem: The newly proposed entropy measure for FFS meets the criteria specified in Definition 6 for FFE measures.

Proof: Based on equation (1), the entropy measure of a FFN could be represented as follows:

$$E(\beta) = 1 - \left[\frac{2(\mu_\beta^3(A_i)-\nu_\beta^3(A_i))}{\mu_\beta^3(A_i)+\nu_\beta^3(A_i)+1}\right]^2 \quad \forall A_i \in Z \qquad (6)$$

1. When F is a crisp set, namely $\forall A_i \in$ Z, we have β =(1,0) or β =(0,1). Then we can obtain $2|\mu_\beta^3(A_i) - \nu_\beta^3(A_i)|$ = 2, and $(\mu_\beta^3(A_i) + \nu_\beta^3(A_i) + 1)$ = 2. Hence E(β_i) = 0, i.e. E(β) = 0.

   If E(β) =0, we can get $\left[\frac{2(\mu_\beta^3(A_i)-\nu_\beta^3(A_i))}{\mu_\beta^3(A_i)+\nu_\beta^3(A_i)+1}\right]^2$ = 1 as per the equation (1). Thus, for any $A_i \in$ Z, we have $2|\mu_\beta^3(A_i) - \nu_\beta^3(A_i)|$ = 2 and $(\mu_\beta^3(A_i) + \nu_\beta^3(A_i) + 1)$ = 2. Because of 0 ≤

$\mu_\beta^3(A_i) \leq 1$ and $0 \leq \nu_\beta^3(A_i) \leq 1$, so we have $\mu_\beta^3(A_i)$=0, $\nu_\beta^3(A_i)$=1 or $\mu_\beta^3(A_i)$=1, $\nu_\beta^3(A_i)$=0. Hence, F is a crisp set.

2. For any $\beta_i \in$ Z, when $\mu_\beta^3(A_i) = \nu_\beta^3(A_i)$, we have $(\mu_\beta^3(A_i) - \nu_\beta^3(A_i))$ = 0. As per equation (1), we can get E(β) = 1. Hence, we have E(β) = 1 when $\mu_\beta^3(A_i) = \nu_\beta^3(A_i)$.
   If E(β) = 1, for any $A_i \in$ Z, we can get $\left[\frac{2(\mu_\beta^3(A_i)-\nu_\beta^3(A_i))}{\mu_\beta^3(A_i)+\nu_\beta^3(A_i)+1}\right]^2$ = 0 according to equation (1). When $0 < \mu_\beta^3(A_i) \leq 1$ and $0 < \nu_\beta^3(A_i) \leq 1$, if and only if $(\mu_\beta^3(A_i) - \nu_\beta^3(A_i))$ = 0, we have $\left[\frac{2(\mu_\beta^3(A_i)-\nu_\beta^3(A_i))}{\mu_\beta^3(A_i)+\nu_\beta^3(A_i)+1}\right]^2$=0. When $\mu_\beta^3(A_i)$= $\nu_\beta^3(\beta_i)$=0, we get $\left[\frac{2(\mu_\beta^3(A_i)-\nu_\beta^3(A_i))}{\mu_\beta^3(A_i)+\nu_\beta^3(A_i)+1}\right]^2$ = 0. Therefore, we have a conclusion that $\mu_\beta^3(A_i)$= $\nu_\beta^3(A_i)$ $\forall A_i \in$ Z .

3. "For any $A_i \in$ Z, we could form the function f(m, n)=1-$\left[\frac{2(m-n)}{m+n+1}\right]^2$ where m= $\mu_\beta^3(A_i)$ and n= $\nu_\beta^3(A_i)$, m, n ∈[0,1]. Taking the partial derivatives of f(m, n) to m and n, we could obtain

$$\frac{\partial f(m,n)}{\partial m} = -4(4n+2)\,\frac{(m-n)}{(m+n+1)^3}$$
$$\frac{\partial f(m,n)}{\partial n} = 4(4m+2)\,\frac{(m-n)}{(m+n+1)^3}$$

We are able to derive the following conclusion from the preceding equation.
Since m, n ∈[0,1], we can get $\frac{\partial f(m,n)}{\partial m} \geq 0$ and $\frac{\partial f(m,n)}{\partial n} \leq 0$ if m ≤ n. Thus, f(m, n) is strictly monotonically rising with respect to m" and strictly monotonically reducing with respect to n respectively. This conclusion allows us to accomplish that if $(\mu_{\beta_2}(A_i))^3 \leq (\nu_{\beta_2}(A_i))^3$ and $(\mu_{\beta_1}(A_i))^3 \leq (\mu_{\beta_2}(A_i))^3$, $(\nu_{\beta_1}(A_i))^3 \geq (\nu_{\beta_2}(A_i))^3$, "then E($\beta_1$) ≤ E($\beta_2$). Since m, n∈ [0,1], we obtain that $\frac{\partial f(m,n)}{\partial m} \leq 0$ and $\frac{\partial f(m,n)}{\partial n} \geq 0$ when m ≥ n. Hence, f(m, n) is strictly monotonically increasing along with n and strictly monotonously decreasing with m". This result makes it simple to determine if $(\mu_{\beta_2}(A_i))^3 \geq (\nu_{\beta_2}(A_i))^3$ and $(\mu_{\beta_1}(A_i))^3 \geq (\mu_{\beta_2}(A_i))^3$, $(\nu_{\beta_1}(A_i))^3 \leq (\nu_{\beta_2}(A_i))^3$, then E($\beta_1$) ≤ E($\beta_2$).

From the conversation above, we can infer that if $\beta_1 \leq \beta_2$ then E(β_1) ≤ E(β_2).

4. Let $\beta = (\mu_\beta, \nu_\beta)$, then we know that $\beta^c = (\nu_\beta, \mu_\beta)$ $\forall\ \beta_i \in$ Z.

$$E(\beta_i^c) = 1 - \left[\frac{2(\nu_\beta^3(A_i) - \mu_\beta^3(A_i))}{\nu_\beta^3(A_i) + \mu_\beta^3(A_i) + 1}\right]^2$$

$$= 1 - \left[\frac{-2(\mu_\beta^3(A_i) - \nu_\beta^3(A_i))}{\mu_\beta^3(A_i) + \nu_\beta^3(A_i) + 1}\right]^2$$

$$= 1 - \left[\frac{2(\mu_\beta^3(A_i) - \nu_\beta^3(A_i))}{\mu_\beta^3(A_i) + \nu_\beta^3(A_i) + 1}\right]^2$$

$$= E(\beta)$$

So, "for any $A_i \in$ Z, we attain E(β_i^c) = E(β). Hence, E (β^c) = E (β). As such, the suggested technique is an entropy measure for FFS.

The suggested method can be used to measure the degree of fuzziness of FFS", as demonstrated by the following basic example.

Example: Assume that β = (0.55, 0.45) is an FFN. The following formula is used to get the entropy measure of β:

$$E(\beta) = 1 - \left[\frac{2(0.55^3 - 0.45^3)}{0.55^3 + 0.45^3 + 1}\right]^2 = 0.9857$$

5. Proposed FF entropy-based MCDM model

In this research, we suggested a new Fermatean fuzzy MCDM approach that employs FF entropy to find out the weights of the criteria. The methodology is implemented through the following sequential steps.

Let $X = \{X_1, X_2, \ldots, X_n.\}$ indicate a collection of alternatives and $C = \{C_1, C_2, \ldots, C_n.\}$ indicate a set of criteria in a decision problem. In order to represent each alternative's evaluation of criterion C_j (j = 1,2,..,k) in relation to each alternative, X_i (i =1, 2,....,n), experts use FFN.

This evaluation is represented as g_{ij}= (μ_{ij}, ν_{ij}), where μ_{ij} indicates the membership degree and ν_{ij} signifies the non-membership degree.

Step 1. Create the FFDM (Fuzzy Fermatean Decision Matrix). Utilizing the evaluation data of the alternatives, a FFDM G = $(g_{ij})_{n\times k}$ is formulated for MCDM problems. The following is the definition of the Fermatean fuzzy decision matrix.

$$G = (g_{ij})_{n\times k} = \begin{bmatrix} (\mu_{11},\ \nu_{11}) & \cdots & (\mu_{1k},\ \nu_{1k}) \\ \vdots & \ddots & \vdots \\ (\mu_{n1},\ \nu_{n1}) & \cdots & (\mu_{nk},\ \nu_{nk}) \end{bmatrix} \quad (7)$$

Step 2: Find the FFDM that has been normalized.

$$h_{ij} = \begin{cases} g_{ij} = (\mu_{ij},\ \nu_{ij}),\ for\ benefit\ criteria \\ g_{ij}{}^{c} = (\nu_{ij}, \mu_{ij}),\ for\ cost\ criteria \end{cases} \quad (8)$$

Step 3: Equation (9) is used to determine the entropy for each FFN within the normalized FFDM. The entropy is represented as E (h_{ij}).

$$\mathrm{E}(\mathrm{h_{ij}}) = 1 - \left[\frac{2(\mu_{\beta}^{3}(A_{ij}) - \nu_{\beta}^{3}(A_{ij}))}{\mu_{\beta}^{3}(A_{ij}) + \nu_{\beta}^{3}(A_{ij}) + 1}\right]^{2} \quad (9)$$

Step 4: The entropy measure of the FFN had been utilized to calculate each criterion's weight. The following formula can be used to determine the weights for each criterion.

$$w_{c_j} = \frac{1 - \bar{E}(c_j)}{k - \sum_{j=1}^{k} \bar{E}(c_j)} \quad (10)$$

where $\bar{E}(c_j) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}(\mathrm{h_{ij}})$, j=1, 2, …, k

Step 5. Aggregate the FFNs from eqn (8) of the alternative X_i (i =1, 2, …, n) w.r.t criteria C_j (j=1,2,..,k) by using the FF weighted power average aggregation operator (FFWPA).

$$\mathrm{FFWPA}\ (\beta_1, \beta_2, \ldots, \beta_k) = \left[\left(\sum_{j=1}^{k} w_j\, \mu_{\beta_j}^{3}\right)^{\frac{1}{3}},\ \left(\sum_{j=1}^{k} w_j\, \nu_{\beta_j}^{3}\right)^{\frac{1}{3}}\right] \quad (11)$$

Where, w_j Represent the weights of the criteria.

Step 6: After aggregating, compute the scores function of each FFN using the given equation.

$$S(\beta) = \mu_\beta^3 - \nu_\beta^3 \quad , S(\beta) \in [-1, 1] \qquad (12)$$

Step 7: Rank the alternatives as per their score function. In cases where scores are identical, apply the accuracy function to determine the ranking of the alternatives.

6. Numerical illustration

We apply the proposed method to assess the testing capabilities of various institutions, comparing it with the methods of Deng and Wang [7] and Garg et al. [18]. This example, adapted from Reference [18], assumes unknown criterion weights. Given the ongoing complexity and volatility in global epidemic management, different countries have implemented various measures to control the virus. Due to the virus's extended incubation period, effective detection methods, such as enhanced nucleic acid testing, are crucial. Five institutions with nucleic acid testing capabilities were evaluated by experts on three criteria: ξ_1 (time constraints), ξ_2 (accuracy), and ξ_3 (client location flexibility). The set of alternatives, consisting of five testing institutions, is denoted as τ ={ τ_1, τ_2, τ_3, τ_4, τ_5} and the experts used Fermatean fuzzy numbers (FFNs) to describe their evaluations. The resulting Fermatean fuzzy decision matrix has been depicted in Table 2.

Step 1. As per evaluation data provided by the experts, the FFDM has been constructed, as presented in Table 1.

Table 2. Fermatean "fuzzy decision matrix given by an expert.

	C_1	C_2	C_3
T_1	(0.7, 0.4)	(0.6, 0.3)	(0.8,0.3)
T_2	(0.8, 0.6)	(0.7, 0.5)	(0.5, 0.2)
T_3	(0.5, 0.3)	(0.6, 0.8)	(0.6, 0.4)
T_4	(0.7, 0.5)	(0.9, 0.3)	(0.9, 0.4)
T_5	(0.6, 0.1)	(0.4, 0.1)	(0.3, 0.4)"

Step 2. Since this example does not include any criteria with cost values, normalization procedures are not necessary.

Step 3. Equation (9), with results shown in Table 2, is used to calculate the entropy measure for each FFN.

Table 3. Entropy measure of FFN

	C_1	C_2	C_3
T_1	0.842718	0.907521	0.602747875
T_2	0.88263	0.911789	0.957344809
T_3	0.971053	0.88263	0.94359375
T_4	0.911789	0.360729	0.44977198
T_5	0.875159	0.986003	0.995399406

Step 4. The weights for each criterion have been find out by utilizing Equation (10), and the outcomes are as "follows:

$$\omega 1 = 0.367, \omega 2 = 0.321, \omega 3 = 0.311.$$

Step 5: Next, to rank the alternatives, we aggregated the FFNs from Table 2 for each alternative X_i (i =1, 2, …, n) with respect to the criteria C_j (j=1,2,..,k) using the" FFWPA aggregation operator specified in Equation (11). Table 4 presents the findings.

Step 6: After aggregation, the score function for each FFN has been computed by utilizing Eq (12). Table 4 presents the findings.

Step 7: The alternatives are ranked according to their score function, with the rankings provided in Table 4.

Table 4. Ranking the alternatives based on the score function

Alternatives	Aggregated FFNs	Score function	Ranking
T_1	(0.707, 0.343)	0.314132201	2
T_2	(0.696, 0.496)	0.21524357	3
T_3	(0.567, 0.579)	-0.011898943	5
T_4	(0.837, 0.420)	0.512666425	1
T_5	(0.476, 0.274)	0.087742781	4

Table 5. Decision results derived from our proposed approach, Deng, Z. & Wang, J. method [7] & Garg et al.'s approach [18].

	Ranking order	Best alternative
The proposed method	$\tau_4 > \tau_1 > \tau_2 > \tau_5 > \tau_3$	τ_4
Deng, Z., & Wang, J. method [2]	$\tau_4 > \tau_1 > \tau_2 > \tau_5 > \tau_3$	τ_4
Garg et al.'s method [12]	$\tau_4 > \tau_1 > \tau_2 > \tau_5 > \tau_3$	τ_4

Table 4 demonstrates that the ranking of alternatives produced by our new approach aligns with the rankings from Deng and Wang's method [7] and Garg et al.'s approach [18], all of which identify τ_4 as the preferred option. These outcomes show that the suggested approach works well.

7. Conclusion & Future Studies

In this research, a novel entropy measure for Fermatean Fuzzy Sets (FFS) was developed to assess their degree of fuzziness. It was demonstrated that this new measure adheres to the required axioms for FFE. The approach treats each FFN as a source of information, with "the weights of the criteria determined based on the entropy values of these FFNs. This approach effectively addresses MCDM problems with unknown criterion weights and minimizes the loss of crucial information in the decision-making process. As a result, the proposed method offers a new perspective on Fermatean fuzzy MCDM and enhances decision-making accuracy.

For future research, we plan to develop a new Fermatean fuzzy score function" along with the aggregation operators for using in MCDM problems. Additionally, we aim to apply our proposed method to various fields.

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A Survey of Topological Data Analysis in Aviation Industry

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Abstract:

This paper aims to give an overview of topological techniques called Topological Data Analysis (TDA), to study underlying features and interaction in increasingly high dimensional aviation data sets. In this paper we will go through a technique called Persistent Homology (PH) from topological data analysis, which is applied to aviation data analytics. We will discuss the effectiveness of Persistent Homology technique in aviation data set, based on the articles available that discuss about the application of TDA in the context of aviation industry. In order to manage and identify patterns and trends in aviation data and perform accurate aviation analysis to improve airport management, this work is intended to guide future research efforts focused on the utilization of TDA in the aviation industry.

Keywords: Topological Data Analysis, Point Clouds, Persistent Homology, Aviation Industry, Machine Learning.

1. Introduction:

In today's modern science and engineering world more and more data are getting generated day by day. Various kinds of data are produced and processed every day, in the fields of aviation industry, science, government sector, medical, etc. These produced data are meaningless until and unless we extract robust qualitative and quantitative, meaningful information about the structure of data. In aviation industry more data sets are produced everyday than ever before due to the fact that air traffic is growing at an unprecedented rate. In 2019, 46.8 million flights between more than 3,780 airports were operated by 1,478 commercial airlines and their roughly 33,299 in-service aircraft, carrying nearly 61 million tons of cargo and 4.5 billion passengers (1.9 billion international and 2.6 billion domestic) over 48,044 routes (Air Transport Action Group). All of these aviation system elements serve as source of data. The fact is that the obtained data are very high dimensional, noisy, unstructured, has missing information, dirty, messy, spare, which severely restrict our ability to visualize it. The typical method for analyzing data like statistical method, graph-theoretic cannot easily uncover accurate meaningful insight of data. But from the last few years crucial efforts have been made to propose topological method to infer meaningful pattern of the data set and provide robust and efficient structure of it. Topological Data Analysis (TDA) offers powerful approach to identify the shape underlying the data and provides tools to extract topological feature from the data. TDA aims at providing well-founded mathematical, statistical, and algorithmic methods to infer, analyze, and exploit the complex topological and geometric structures underlying data that are often represented as point clouds in Euclidean or more general metric spaces [2]. TDA is based on principle that data has shape and shape has meaning, meaning drives values [3].

1.1 What is Topology?

The word Topology is derived from two Greek words ‘topos’ meaning ‘surface’ and ‘logos’ meaning ‘discourse’ or ‘study’. The study of topology focuses on the inherent qualitative characteristics of space, which go beyond its dimensions, location and structure. More precisely it is a quality based-mathematics. Topology does not have numbers. The intrinsic qualitative properties of rubber band and rubber dough nut can be expanded or bent without tearing or gluing. In topology we study the properties of geometrical objects, which remain same or preserved under continuous deformation of objects such as stretching, twisting but cutting or tearing are not allowed. Thus, a cube and a sphere are equivalent from topological perspective.

Example: From a topological point of view, a coffee mug and a donut are equivalent because they can be continuously deformed from one to the other as shown below:

Fig.1: Transforming a coffee mug to a donut. [4].

The primary concern of topology is shape of the data. In the study of aviation perception, the concept of shape is quite essential. Shape of the data reveals many mysteries and draw attention to the complicated patterns as well as complex connection found in data. In contrast to standard method, hypothesis-free Topological Data Analysis requires set of points i.e point clouds of data with metric defined on it that are unconstrained by any scale, number of neighbours or noise bound.

1.2 Persistent Homology

Persistent Homology (PH) is algebraic topological concept. Persistent homology is one of the most widely applicable tools in the emerging field of computational topology [5]. PH is a powerful tool in TDA for investigating the structure of data [6]. The information about clustering of a given point cloud can be shown by persistence diagram with no a connectivity parameter selected by experts, which is typically required. The structure such as loops and voids which are complicated and not easily visible with other techniques are also can be described by Persistent Homology. A crucial fact that makes persistent homology valuable for application in data analysis is its ability to record topological changes throughout the whole process and store the information in persistence diagrams.

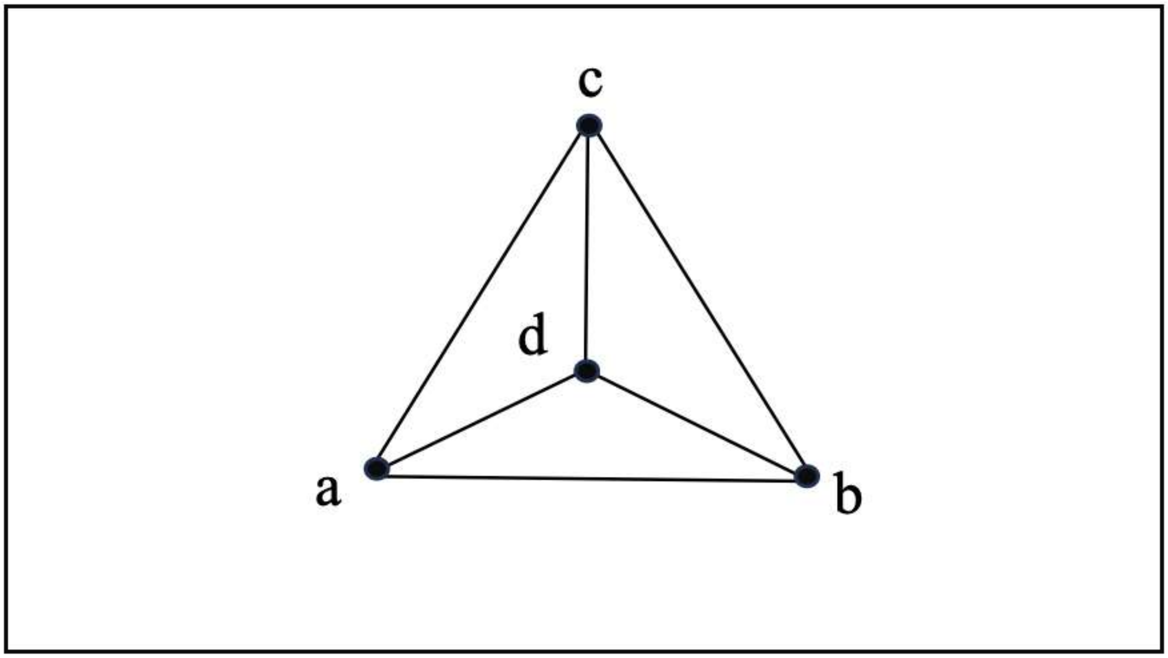

Fig.2: Simplicial complexes modelling a tetrahedron.

1.3 Persistent Homology for Modeling of Data

Persistent Homology is a method which uses topology to model data in order to infer structures and useful information from the data. Persistent homology applies topological space on the top of a collection of data i.e point cloud data with a metric defined on it in order to model the data. Persistent homology acknowledges the topological features such as loop, void and components of data. In this algorithm a set of data points is converted into a family of simplicial complexes by constructing Cech complex or Rips complex. Then homology is applied to this simplicial complex, which reveals the presence of the central hole. These generalized holes form the bases of homology groups and this gives a formal way to characterize different shapes of data. Then by using a Persistence Diagram the persistence of these holes is tracked. In persistence diagram, point that sits near diagonal line ($x = y$ line) corresponds to a hole that disappeared soon after it appeared and on the other hand point that sits far away from the diagonal line corresponds to a hole that disappeared long after it appeared. Therefore, the points closed to this $x = y$ line is noise while the points relatively far from this line are significant.

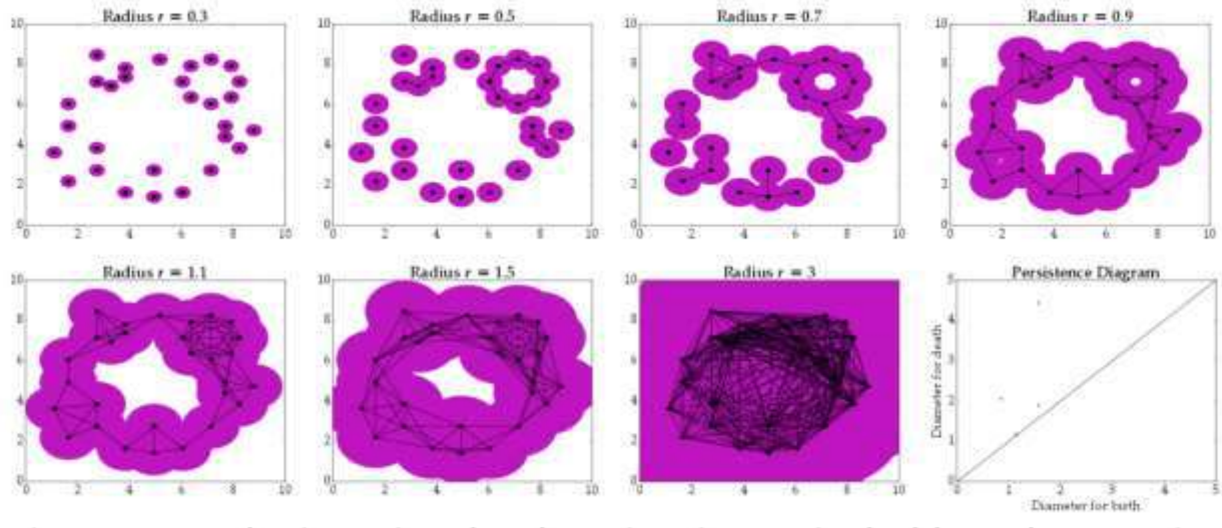

Fig.3: An illustration of investigating a point cloud data set using persistent homology by building the Rips complex, whose edge set is displayed in black on each figure. The persistence diagram drawn at the bottom right gives a summary of the appearance and disappearance of loops in the space as the Rips complex parameter changes. [7].

A more thorough discussion is available in [8], [9] and [10].

1.4 Problems in Aviation Industry

Every day we keep hearing about the problems happening in the aviation industry, but here we will discuss only about the few. Let's start our story with management failure. In 2007, Air India losses worth ₹5.41 billion and Indian Airlines losses worth ₹2.31 billion because neither they were able to make the right decision nor they were skilled enough to manage the problems [11]. On 19 May 2016 at 02:33 Egypt Standard Time (UTC+2), EgyptAir Flight 804 operated by EgyptAir, the Airbus A320 crashed into the Mediterranean Sea, killing all 56 passengers, 3 security personnel, and 7 crew members on board because Egyptian radars were unable to track the aircraft due to its distance from the aircraft [12]. On February 12, 2024, Air India owned by TATA, failed to provide enough wheel chair to passengers who require assistance during embarking or disembarking from the aircraft during their journey [13]. Why did these kinds of failure occur, one of the essential reasons is actually not having enough knowledge about management skills which needs a way to deal with such situation.

Another problem is excessive crowding or increasing number of flights or passengers. It causes inconvenience to both passengers and flight staff or airport staff which includes delaying flight, fuel cost uncertainty, lack of qualified staff, runway shortage which causes major damage to airline companies regarding economy as well as passengers regarding availability of flight in right time at right ticket cost.

The use of topological data analysis techniques to tackle a few numbers of these difficulties will be discussed.

2. Review on Application of Persistent Homology in Aviation Industry

Topological Data Analysis (TDA) is a powerful method which gives us the tool called Persistent Homology (PH) which is capable of determining the actionable insight topological feature from data set by computing number of connected components, number of higher dimensional hole in data and represent those features using persistent diagrams and barcodes [14]. We may look at the identification of these qualities in several sets of data, oriented on TDA, motivated by aviation applications.

In [15], the authors have presented an application of topological methods to UAV geofencing applications. The authors use topological keep-in and keep-out geofences made with α-shapes to examine the capacity and structure of a particular urban airspace in relation to UAV usage. They build three-dimensional keep-in geofences using α-shape building. They discovered that while keep-out geofences yield more conservative findings, these α-shaped keep-in areas offer an upper limit on the quantity of airspace that is available. Since the purpose of keep-out geofences was to create a consistent airspace buffer between unmoving aerial vehicles (UAVs) and stationary barriers like buildings and other infrastructure, they were not designed using α-shapes. As a result, between the two geofences, there is a natural primal-dual relationship, whose trade-offs are investigated using real terrain data from a densely populated area of Seoul, South Korea, as well as simulated urban environments. The authors concluded by noting that in the future, TDA/PH might be used to evaluate the continuities and connectivity of useable urban airspace data sets that have been separated from their keep-in and keep-out partitions.

In [16], the authors have developed an instructive example based on a sample data set that includes aviation geography, airspace, and operational aspects. Throughout the paper, this sample data set is utilized to construct simplicial complexes from data and to introduce simplicial homology. They have conducted case study to explore nerve complexes of topology in runway configurations and declared capacities and utilizing auditory representations of airport surface networks for air traffic control to deduce active taxiway configurations, also they have shown that maximal simplex loss indicates significant major change in airfield topology. The authors have elaborated on features of topology are correspond to specific features of aviation such as robustness in air transportation networks, airspace partitions and airport surface characteristics, additionally discussed the operational and supervisory insights that can be obtained from these topological data science techniques. Furthermore, the authors guarantee that the techniques of TDA/PH could be used to generate origin and destination trajectory of the flight.

In [17], the authors have combined an algebraic-topological approach with a graph-theoretic approach to quantify robustness in the context of random and targeted node removal-type attacks in highly connected and complex networks. Using a network that a canonical graph represents and a probability q that nodes would be eliminated during an attack, this research computes extra conventional graph-theoretic measures, such as network connectivity and a fraction that refers the percentage of nodes within known as major components after the removal of q nodes as a result of an attack. The authors apply fraction, which gives a measurement of the big component size in relation to the entire network, to calculate a robustness R-value. The authors use an algebraic-topological measure, represented as a ratio h_k between the total number of linked components in the network and the number of k-dimensional holes, to describe the topology of the network under node-removal attacks. This ratioh_k uses homology in a condensed manner to measure the impact of a node removal on the intrinsic topology of the network.

In [18], the authors discuss how flight trajectory data can be analyzed using Topological Data Analysis (TDA) to find patterns in the movement of aircraft. They also find common patterns and anomalies in airport operations and traffic, establishes relationships between various variables involved in the spatial and temporal flight trajectory and delays, and aids in identifying the root causes of delays and the development of more efficient mitigation techniques. The analysis's findings demonstrate how various airport groupings, despite their small numbers, adhere to a kind of cluster. Additionally, it makes it possible to identify airports that stand out from the rest of their pre-assigned category. It also aids in determining when an airport is remote and alone inside the network.

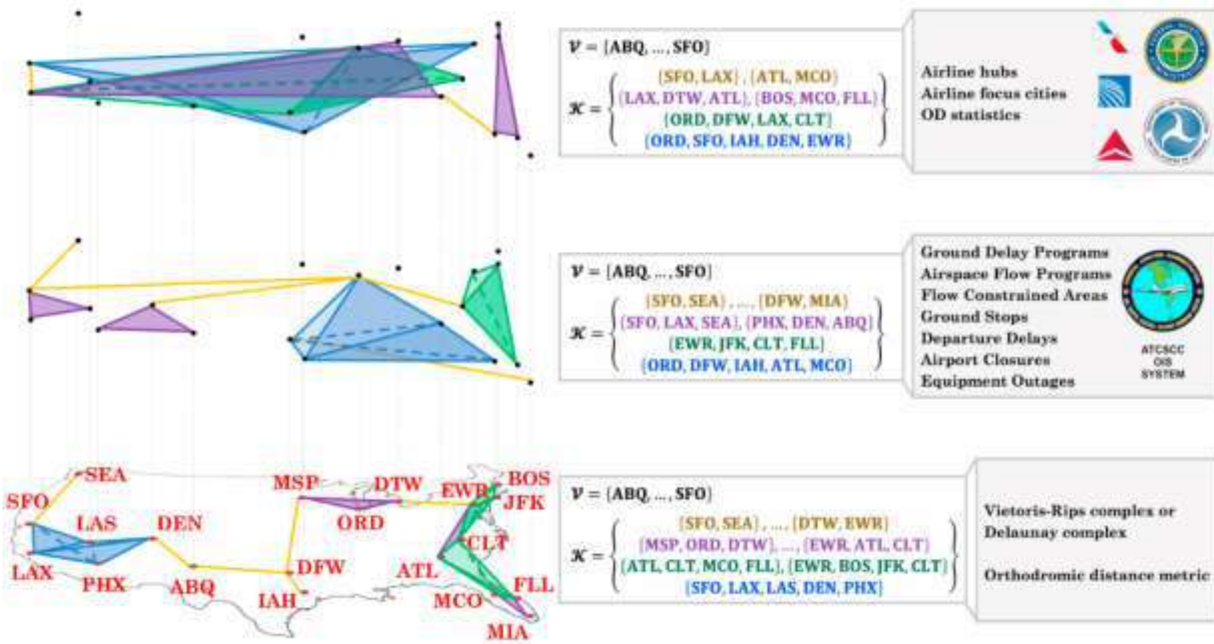

Fig.4: Example of multi-layered simplicial complexes corresponding to different types of aviation data. [16].

3. Conclusion

The primary topic of this survey was management performance enhancement in the aviation industry. The aviation industry's management will produce reliable results if the right data sets and photos are available, making it easier to track aircraft, provide accurate booking information, and offer freebies to tourists. Because the combination of topological data analysis and machine learning is a quick, first approach of data, thorough, also deep understanding, as well as more accurate and efficient, the proposed study will also ensure performances. Topological data analysis and machine learning combine marvelously.

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Generalization of equivalence relation

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Abstract: The “Relation” can be interpreted as the association or dissociation ,interaction or interconnectedness of elements of two set.In the mathematics various form of relations are available .Most commonly used relation are “Pre-order”,”Order-relation”,Partial order realtion ,”Strict ordering relation”,”Binary relation”,”n-array relation”,”Fuzzy Sets” and “Rough Set relation” ,”Equivalence relation”,etc.The form of any relation is special type of the set that connects some other sets in a proper way. In this paper we want to discuss certain type of relation is called partial equivalence relation. This relation is said to be partial reflexive , Spartial symmetric and partial transitive is called partial equivalence relation.

Keywords: partial relation, partial reflexive, partial symmetrc, partial transitive

1.1 Introduction

Mathematical meaning of the word “Relation” can be interpreted as the association or dissociation, interaction or interconnectedness of elements of two sets. In logical point of view the same word “Relation” may be interpreted as the quantification or qualification of propositions. In Math literature various forms of relations are available. Most commonly used relations are “Preorder relation”, “Order relation”, “Partial order relation”, “Strict ordering relation”, “Binary relation”, “n-array Relation”, “Equivalence relation” , “Fuzzy relation” etc. Whatever may be the forms, any relation is a special type of set that connects some other sets in a proper way.

In this paper we discuss certain concepts of the generalization of equivalence relations. A natural question arises, “Does any equivalence relation Incomplete”? Answer to this question cannot be said as a direct No, as there exist many relations those neither satisfy the definition assumptions of Equivalence Relations, such as Reflexivity , symmetric and transitivity nor any other generalization of these relations Although Fuzzy relation describes some of the incomplete equivalence relations, but this description is insufficient. There exist certain relations, which often used in the day-to-day linguistic expressions and need more attention for the application point of view. The illustration expressed in mathematical modeling subsection of this paper narrates some of these situations. The other huddles in the applications of these relations .We categorize these relations and not only defined these relations as an addition to mathematics literature, but also studied some of their properties and define some quantification techniques.

1.2 Comparison of Fuzzy Relation with Partial Equivalence Relation

Fuzzy relation and partial equivalence relation are different concepts. To define any Fuzzy relation, the knowledge of the memberships of individual elements within its domain is necessary. Whereas the definition of partial equivalence relation does not require it essentially.

Some researchers suggest any crisp relation is a particular case of Fuzzy relation. In the same logic, the definition of a crisp reflexive relation can be derived form a fuzzy relation $\mu_R(x,y)$,if $\mu_R(x,x)=1$ for each $x \epsilon X$. The function $\mu_R \epsilon [0,1]$ is called the membership function, and the set X is the domain of the relation . It can be observed that the function $\mu_R(x,x)$ can take any value within the interval [0,1]. When R(x,x) is not equal to one for some $x \epsilon X$, then it will equal to zero and hence named as non-reflexive. However, $\mu_R(x,x)$ may not equal to zero in this case and equal to ϵ. Then some researchers suggested this relation as ϵ- reflexivity. Thus the fuzzy relation is assumed as the generalization of crisp relation. As there are no fix rules for the assignment of membership functional value of a fuzzy relation, therefore, the specification of values for ϵ is not unique; hence we suggest a new name for this non-reflexive relation to *partial reflexive relation.*The formal definitions and certain measures for partial relations are stated in the subsequent sections.

There is another motivation to define such a special relation, which is commonly known as partial relation, due to the following cause. The definition of reflexive relation states that if each element x of X is related with itself, than the relation is said to be reflexive. It may be noticed that the complete associations of one element x with itself is assumed as full or whole. But there are certain cases where some elements of a set may not associate with themselves fully. For example, let the relation be R(Xx X)={(x ,y)/x looks similar with y}where X={0,O,8,B}. Then the character pairs (0,0),(O,O),(B,B),(8,8),(0,O) and (B,8) has some similarity. The first four pairs satisfy reflexive property under the relation "similarity"; whereas the components of fifth and sixth pairs have some degrees of similar appearances within their attributes. But this type of similar appearance can not be taken into consideration further study of reflexive relation and it may be quantified with the fuzzy relation concepts. In a rough estimations one may express that the characters "8" and "B" or "0" and "O" are partially related. In this way there are many instances, which motivated us to define a new type of relation called partial relation. This relation has certain specialty in comparison to other (above mentioned) relation and brief description of which is presented in this paper after the formal definition of the partial relation. We proceed to define partial relation with considering the following illustrations;

Consider a grid of seven segments. Let R_1 and R_2 be the relations defined over this set of segments such that these relations related the end vertices of each segment to design the shape of the character "B" and "P" respectively. If S={ a_{11},a_{12},a_{21},a_{22},a_{31},a_{32}}, be the set of vertices of this grid, than thesetofedgesR_1={(a_{31},a_{21}),(a_{21},a_{11}),(a_{11},a_{12}),(a_{12},a_{22}),(a_{21},a_{22}),(a_{31},a_{32}),(a_{22},a_{32})}and R_2={(a_{31},a_{21}),(a_{21},a_{11}),(a_{11},a_{12}),(a_{12},a_{22}),(a_{21},a_{22})} can be represent "B" and "P" The above example is one of the approaches to measure partial reflexivity of two relations those are derived from the structural representations of two alphabets. This type of measurement may not be uniquely done. Because, one may consider either R_2={(a_{31},a_{11}),(a_{11},a_{12}),(a_{12},a_{22}),(a_{22},a_{21})},R_2={(a_{31},a_{11}),(a_{11},a_{12}),(a_{12},a_{22}),(a_{21},a_{22})}, R_2={(a_{31},a_{21}),(a_{21},a_{11}),(a_{11},a_{12}),(a_{12},a_{22}),(a_{21},a_{22})}orR_2={(a_{31},a_{21},(a_{21},a_{11}),(a_{11},a_{12}),(a_{12},a_{22}),(a_{21},a_{22})} etc., by measuring or counting individual segments used for the design of the geometrical structural points of view. If one will consider the orientations, the directions, or the starting and end the points of individual segments, than it will be more complex to draw any standard conclusion about the partial reflexivity of the relations R_1 and R_2 as partial relationship can be measured in several points of views.

Therefore we are proposing a few algorithms and definitions to measure the partial relationship in general and partial reflexivity, partial symmetric and partial transitive in particular for more than two relations.

1.3.Formal Definitions

A relation R is a subset of the Cartesian product of set S. That means R ⊂SXS. It is said to be reflexive if (x,x)ϵR, for some xϵ S. Any subset of R, is called a **partial relation** of R. In general, *the partial relation* R_{ij} *can bedefined as the intersection of two relations* R_i *and* R_j. Moreover, the partial relation that relates at least two relations in a specific manner can be renamed as partial reflexive, partial symmetry and partial transitive etc.

In the subsequent sections we discuss relationships of partial relations with fuzzy relations are basically, this approach is an attempt to study the generalization of equivalence relation. As equivalence relation plays major rule in decision making problems, and obtaining an equivalence relation is not easy, so, we define some of these partial relations in order to generalize the equivalence relation.

Definition 1. The relation R_1⊂ SXS is said to be partial reflexive with R_2⊂SXS, if (x,x) ϵ $R_{12}=R_1 \cap R_2$ for at least xϵS.

Remark 1. Partial reflexivity preserves the order relation.

Definition 2. The order of partial reflexivity of relation R_1 with R_2 is equal to the ratio of cardinality of $R_1 \cap R_2$ to cardinality of R_1.

From the above example, we can conclude, the relations R_1 and R_2 are partial reflexive as $R_1 \cap R_2 = R_2$, the order of partial reflexivity of R_1 with R_2 is 0.714285, and the order of partial reflexivity of R_2 with R_1 is 1.

Definition 3. Consider a set of horizontal, vertical and slant segments. A subset of this segment set, which represents the shape of a particular character, is called the trace of that character. Thus any trace is a relation.

Definition 4. The relation R_1⊂ SXS and R_2⊂SXS are said to be partial symmetric if (x,y)ϵ$R_1 \cap R_2$ than (y,x)ϵ$R_1 \cap R_2$ for some x and y ϵ S.

Definition 5. Order of partial symmetric nature of two relations R_1 and R_2 is the ratio of *Min*(order of partial reflexivity of relation R_1 with R_2, order of partial reflexivity of R_2 with R_1) with cardinality of ($R_1 \cup R_2$).

Definition 6. Let R_{12} and R_{23} be the partial relation defined on the shape S, such that $(x,y) \epsilon R_1 \cap R_2$ and $(y,z) \epsilon R_2 \cap R_3$ for some x,y and $z \epsilon$ S, than $R_{13} = (R_1 \cap R_3)$ is called the partial transitive relation.

Definition 7. The order of the transitive relation R_{13} is *Min* $\{|R_{12}|/|R_1|, |R_{21}|/|R_2|, |R_{13}|/|R_1|, |R_{31}|/|R_3|, |R_{23}|/|R_2|, |R_{32}|/|R_3|\}$.

1.4 Quantification of partial relation

Partial equivalence relation can be quantified by using certain techniques. The operation on matrices , and the construction of these matrices are obtained by certain rules. The descriptions of these rules are given below If R_{12} is the partial relation of $R_1 = \{(v_1,v_2),(v_2,v_3),(v_3,v_4)\}$ with $R_2 = \{(v_1,v_1),(v_2,v_4),(v_1,v_3)\}$, then, total number of vertices R_1 and R_2 are enlisted in the first column and first row of this matrix, respectively. The elements a_{ij} , of the matrix R_{12}, can be considered in the following manner $a_{ij}=1$, for the ordinates v_i of R_1 and v_j of R_2 are end points of certain edges, that can be formed under the defined relations R_1 and R_2. If there is no such relationship, then, a_{ij} is considered to be zero.

R1=

	v_1	v_2	v_3	v_4	v_3	v_4
v_1	0	1	0	0	1	0
V_2	0	0	1	0	0	1
V_3	0	0	0	1	0	0
V_4	0	0	0	0	0	0

R2=

	v_1	v_2
v_1	1	0
v_2	0	0
v_3	0	0
v_4	0	0

1.5. Comparison of Partial Equivalence Relation with Equivalence Relation

a) Equivalence relation is defined over a single set. That means the reflexive, symmetric and transitive relations are defined on the Cartesian product of a single set. Whereas the partial equivalence is defined either over a single set or over multiple sets, related in a special way.

b) When the domain of a partial relation is considered to be a single set,than the set must be a relation set, otherwise the definition of partial reflexivity property has to be defined.

1.6 Applications of Partial Relation

Case 1: Study of resemblance and similarity

The resemblance or the appearance of two characters can be measured by counting the order of partial reflexivity of one with the other. For which a proper norm is to be considered. The norm may be user defined or simply the cardinality of the intersection of two relations. The following measures may be one of the standards for measuring resemblance of two characters.

$F(R_1,R_2)=Min$(order of partial reflexivity of first character with the second character, order of partial reflexivity of second character with the first character).

Measurement of dissimilarity can be quantified by using the following formulae.

$1-F(R_1,R_2)$.

Case 2 : Study of symmetry

One of the natural paradox is every entity possesses certain similarity and dissimilarity with any other entity. Thus the norms those are used for the measurement of symmetrical properties are object oriented. Hence one measurement procedure may not be appropriate for the measurement of the symmetrical properties of every other object .Thus the application of partial symmetry relation is most efficient for considering a standard for the measurement of symmetry.

Case 3 : Application in Mathematical modeling

Here we illustrate some illustrations to provide a better clarification of the partial symmetric and partial transitive relations.

Illustration 1

(I) Let the relation R C X×Y, Where X is the set of trees, Y is the set of seeds and the relation "R" assigned to the word "produce". The statements (a) Trees produce seeds and (b)seeds produce Trees, imply a symmetric relation. Though this illustration seems that seems that that the relation is complete. But in practical sense it is not. Because, a tree can produce many seed but a single seed can produce only one tree. Therefore a peculiar situation arises. To overcome this ambiguity, the quantification of relation should not be considered, but the statement of the partial relation is to be modified.

(II) If X is the sample set of chemical components, Y is a sample set of electric energy and relation R_1 is assigned to the word "convert" and another relation R_2 is assigned to the word "creates" , then the statement (a) Chemical energy converts to

Electrical energy,(b)Electrical energy creates Chemical ions are related by partial symmetric relation. This also can be quantified by the definition.

Illustration 2

Consider the composite, "Water vapors create clouds and clouds create rain drops". In the crisp sense this yields a transitive relation "Water vapors create rain drops". But if the constraint quality is considered into account than all the water vapor are not used for the creation of cloud and similarly all the cloud molecules are not used for the creation of rain drops . Hence the transitive needs modification for the better consequence. This modification yields the *partial transitive relation.*

Conclusion :-This paper defines order of partial reflexive, partial symmetric and partial transitive relation to give better clarification of the concept. The cases study of certain applications of partial relation and quantification of partial relation can be quantified by using certain techniques. The operations on matrices and construction of these are obtained by certain rules .We use formal definitions to play major role in decision making problems. We define some of these partial relations in order to generalize the equivalence relation.

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Mathematical modeling for climate change mitigation: harnessing tidal and wave energy as alternatives to fossil fuels

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Abstract

At present, climate change is a burning issue which is changed with the rising of atmospheric temperature as well as emissions of greenhouse gases (GHGs) in the environment. A developing country like Bangladesh where economy is gradually raising upwards needs huge supply of power. Now-a-days consumption of fossil fuel is the main source of power generation. But amount of fossil fuel is very limited. In this time renewable energy sources can be the alternatives. Besides, burning of fossil fuels is associated to climate change effects such as rising of atmospheric temperature, emission of greenhouse gases. Due to steadiness of nature and predictable characteristics, tidal energy and wave energy are potential forms of renewable energy. Both have a great deal of potential for producing energy. This study analyzes the advantages of using tidal energy and wave energy instead of fossil fuel. This work develops a dynamic behavior of a five compartmental nonlinear model to illustrate adverse effects of climate change as temperature rising and GHGs' due to fossil fuel consumption. Our main target is to work on the factors contributing to a potential reduction in atmospheric temperature. The analysis has identified several equilibrium points. By thoroughly investigating the

equilibrium points and their stability, the analysis has yielded valuable insights into the dynamics and behavior of the model. The identification of the equilibrium points is crucial for understanding the long-term behavior of the system. After investigating the proposed model analytically and numerically, the numerical simulations have been performed.

Keywords

Mathematical Modeling, Compartmental Models, Fossil fuel, Tidal energy, Wave energy.

Introduction

Bangladesh is endowed with abundant renewable resources that are utilized globally, yet their application within the country remains limited. The land boasts a vast ocean area rich in different power sources as well as including wave energy, Ocean Thermal Energy Conversion (OTEC), and tidal energy. This country faces a critical energy crisis, primarily reliant on fossil fuels, which contribute significantly to greenhouse gas emissions and climate change. With a populace exceeding 160 million, the demand for power keeps to upward thrust, exacerbating environmental degradation and health troubles related to air pollutants. In response, there is a pressing need to discover renewable power alternatives, particularly tidal and wave energy, which can be considerable alongside the country's extensive coastline.

Tidal energy harnesses the gravitational forces exerted by using manner of the moon and solar, ensuing in predictable tidal moves that may be converted into energy. Wave energy is a form of renewable energy that may be harnessed from the motion of the waves. Reza et al. (2024) discussed those renewable elements afford an opportunity for Bangladesh to reduce its carbon footprint at the same time as addressing its energy needs.

By way of transitioning from fossil fuels to tidal and wave energy, Bangladesh cannot most effectively mitigate greenhouse gas emissions but additionally enhance energy protection and sustainability. This model will assess the potential reductions in atmospheric temperature and greenhouse gas emissions by replacing fossil fuel energy with tidal and wave energy, highlighting the feasibility and benefits of these renewable resources in combating climate change.

Haque and Khatun (2018) discussed the vast potential of tidal energy and emphasized it's significant in reducing greenhouse gas emissions and mitigating climate change.. Islam, Mondal and Biswas (2022) specially described the favorable conditions for harnessing wave energy in the Bay of Bengal, particularly during specific seasons. It suggests that wave energy can serve as a substantial alternative to fossil fuels, thus contributing to GHGs emission reductions. Majumder et al. (2024) mentioned both tidal and wave energy technology, assessing their capability contributions to sustainable development in Bangladesh. It emphasizes that transitioning to those renewable resources can assist mitigate climate change by decreasing GHGs emissions. Besides, Samimi et al. (2013) claimed that rapid emissions of GHGs are responsible for climate change, whereas Mandal et al. (2022) briefly described global warming in marine ecosystems.

Objectives

The objectives of the paper are

- ❖ To explain how atmospheric temperature and GHGs can be lessen by using tidal energy and wave energy in place of fossil fuel consumption.
- ❖ To represent the relationship among these dynamical variables.
- ❖ To solve the model both analytically and numerically analysis for representing dynamical behaviors.

Model Formulation

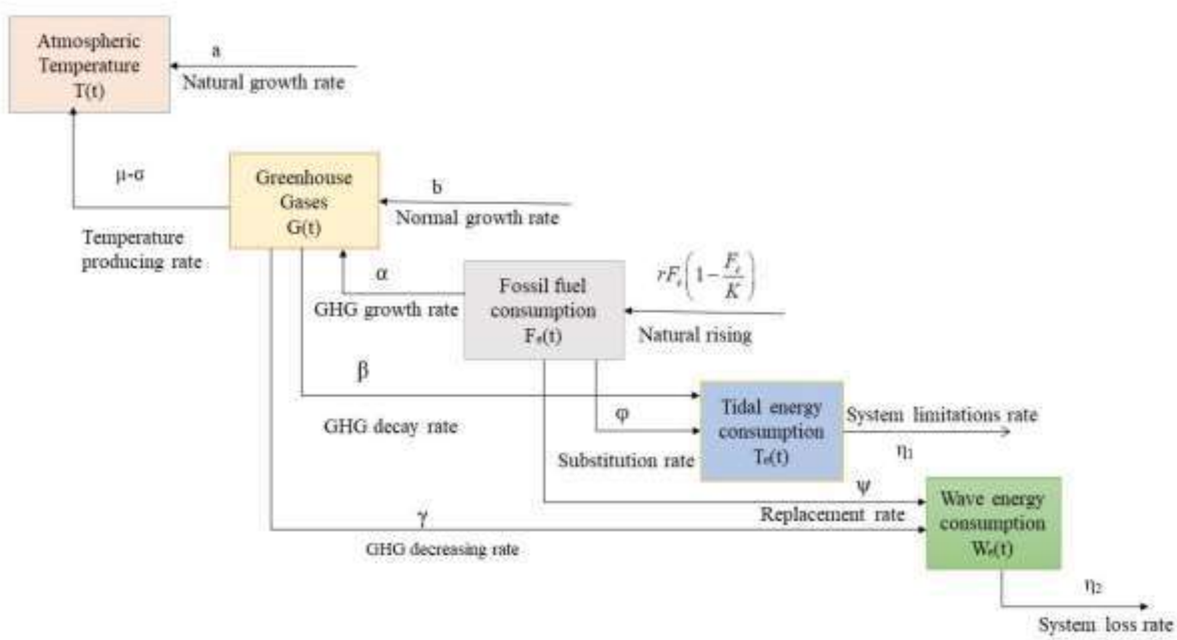

Figure 1: A schematic diagram describing effect of tidal & wave power in reducing use of fossil fuel & decreasing atmospheric temperature & GHGs.

Figure 1 shows the flowchart of the five compartmental which is added to describe the effect of tidal energy consumption and wave energy consumption on atmospheric temperature, greenhouse gases and fossil fuel consumption and their correlations.

According to the flowchart, it leads a set of non-linear ordinary differential equations consisting of five dynamical variables which can be written as the following form:

$$\frac{dT}{dt} = aT + (\mu - \sigma)GT$$

$$\frac{dG}{dt} = bG + \alpha Fe - \beta Te - \gamma We$$

$$\frac{dFe}{dt} = rFe\left(1 - \frac{Fe}{K}\right) - \varphi FeTe - \psi FeWe$$

$$\frac{dTe}{dt} = \varphi FeTe - \eta_1 Te$$

$$\frac{dWe}{dt} = \psi FeWe - \eta_2 We$$

With initial conditions $T_0 = T(0) > 0, G_0 = G(0) > 0, Fe_0 = F_e(0) \geq 0, Te_0 = T_e(0) \geq 0, We_0 = W_e(0) \geq 0$

Here $T(t)$ and $G(t)$ are the atmospheric temperature in degree Celsius and the density of greenhouse gases in the environment, respectively; whereas $F_e(t), T_e(t) \& W_e(t)$ are the consumption of fossil fuel energy, tidal energy and wave energy, respectively, at time t.

Parameters used in the model are described below:

Here a, b, r are the natural growth rate of $T(t), G(t), F_e(t)$, respectively; $(\mu - \sigma)$ be the producing rate of temperature by greenhouse gases; α be the increasing rate of GHGs because of fossil fuel consumption; β, γ be the reduction rate of GHGs due to consumption of tidal energy and wave energy; K be the carrying capacity of fossil fuel consumption; φ, ψ are the rate of replacement of fossil fuel by tidal power and wave power; η_1 be the limitations of tidal energy; η_2 be the system loss of wave energy.

Analytical Analysis

Equilibrium points

The model has equilibrium points at $\frac{dT}{dt}=\frac{dG}{dt}=\frac{dFe}{dt}=\frac{dTe}{dt}=\frac{dWe}{dt}=0$. Solving the equations, we get the equilibrium points as

$E_1\left(0,0,0,0,0\right)$

$$E_2\left(0,\frac{-\left(\eta_2\gamma r+K\alpha\eta_2\varphi-K\gamma r\psi\right)}{Kb\psi^2},\frac{\eta_2}{\psi},0,\frac{-\left(\eta_2 r-Kr\psi\right)}{K\psi^2}\right)$$

$$E_3\left(0,\frac{-K\alpha}{b},K,0,0\right)$$

$$E_4\left(0,\frac{-\left(\beta\eta_1 r+K\alpha\eta_1\varphi-K\beta\varphi r\right)}{Kb\varphi^2},\frac{\eta_1}{\varphi},\frac{-\left(r\left(\eta_1-K\varphi\right)\right)}{K\varphi^2},0\right)$$

Stability Analysis

Jacobian of the model is

$$J=\begin{bmatrix} a+G\left(\mu-\sigma\right) & T\left(\mu-\sigma\right) & 0 & 0 & 0 \\ 0 & b & \alpha & -\beta & -\gamma \\ 0 & 0 & -\varphi Te-\psi We-r\left(\frac{rFe}{K}\right) & -\varphi Fe & -\psi Fe \\ 0 & 0 & \varphi Te & \varphi Fe-\eta_1 & 0 \\ 0 & 0 & \psi We & 0 & \psi Fe-\eta_2 \end{bmatrix}$$

Now, putting the equilibrium point $E_1\left(0,0,0,0,0\right)$ in the Jacobian, we get

$$J_1=\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & \alpha & -\beta & -\gamma \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & -\eta_1 & 0 \\ 0 & 0 & 0 & 0 & -\eta_2 \end{bmatrix}$$

With eigen value λ

$$\begin{vmatrix} a-\lambda & 0 & 0 & 0 & 0 \\ 0 & b-\lambda & \alpha & -\beta & -\gamma \\ 0 & 0 & r-\lambda & 0 & 0 \\ 0 & 0 & 0 & -\eta_1-\lambda & 0 \\ 0 & 0 & 0 & 0 & -\eta_2-\lambda \end{vmatrix} = 0$$

$$-(\eta_1+\lambda)(\eta_2+\lambda)(a-\lambda)(b-\lambda)(\lambda-r)=0$$

Eigenvalues are

$\lambda_1 = a$

$\lambda_2 = b$

$\lambda_3 = r$

$\lambda_4 = -\eta_1$

$\lambda_5 = -\eta_2$

For the dynamical system to be stable, all real parts of the eigenvalues must be negative (if even one eigenvalue has a positive real part, the system becomes unstable).

Here our first eigenvalue is positive, so we have both positive and negative eigenvalues.

Hence the equilibrium point is unstable saddle node.

For second equilibrium, matrix becomes

$$J_2 = \begin{bmatrix} a-\dfrac{\mu(\eta_2\gamma r+K\alpha\eta_2\psi-K\gamma r\psi)}{Kb\psi^2+\dfrac{\sigma(\eta_2\gamma r+K\alpha\eta_2\psi-K\gamma r\psi)}{Kb\psi^2}} & 0 & 0 & 0 & 0 \\ 0 & b & \alpha & -\beta & -\gamma \\ 0 & 0 & \dfrac{\eta_2-Kr\psi}{K\psi}-r\left(\dfrac{\eta_2}{K\psi}-1\right)-\dfrac{\eta_2 r}{K\psi} & \dfrac{-(\eta_2\varphi)}{\psi} & -\eta_2 \\ 0 & 0 & 0 & \dfrac{(\eta_2\varphi)}{\psi}-\eta_1 & 0 \\ 0 & 0 & \dfrac{-(\eta_2 r-Kr\psi)}{K\psi} & 0 & 0 \end{bmatrix}$$

Now

$$\begin{vmatrix} a-\dfrac{\mu(\eta_2\gamma r+K\alpha\eta_2\psi-K\gamma r\psi)}{Kb\psi^2+\dfrac{\sigma(\eta_2\gamma r+K\alpha\eta_2\psi-K\gamma r\psi)}{Kb\psi^2}}-\lambda & 0 & 0 & 0 & 0 \\ 0 & b-\lambda & \alpha & -\beta & -\gamma \\ 0 & 0 & \dfrac{\eta_2-Kr\psi}{K\psi}-r\left(\dfrac{\eta_2}{K\psi}-1\right)-\dfrac{\eta_2 r}{K\psi}-\lambda & \dfrac{-(\eta_2\varphi)}{\psi} & -\eta_2 \\ 0 & 0 & 0 & \dfrac{(\eta_2\varphi)}{\psi}-\eta_1-\lambda & 0 \\ 0 & 0 & \dfrac{-(\eta_2 r-Kr\psi)}{K\psi} & 0 & -\lambda \end{vmatrix}=0$$

Eigenvalues are

$$\lambda_1 = b$$

$$\lambda_2 = \frac{\eta_2\varphi - \eta_1\psi}{\psi}$$

$$\lambda_3 = \frac{-\left(\eta_2 r - \sqrt{\left(\eta_2 r\left(-4K^2\psi^2 + 4\eta_2 K\psi + \eta_2 r\right)\right)}\right)}{2K\psi}$$

$$\lambda_4 = \frac{\left(\eta_2\gamma r\sigma - \eta_2\gamma\mu r + Kab\psi^2 - K\alpha\eta_2\mu\psi + K\alpha\eta_2\psi\sigma + K\gamma\mu r\psi - K\gamma r\psi\sigma\right)}{Kb\psi^2}$$

$$\lambda_5 = \frac{-\left(\eta_2 r + \sqrt{\left(\eta_2 r\left(-4K^2\psi^2 + 4\eta_2 K\psi + \eta_2 r\right)\right)}\right)}{2K\psi}$$

Here λ_1 is positive.

For λ_2 , if $\eta_2\varphi - \eta_1\psi > 0$ then $\lambda_2 > 0$ otherwise it is negative.

λ_3 and λ_4 are more complex, because they both contains square roots, which could indicate either real or complex eigenvalues.

If the discriminant, $\eta_2 r - 4K^2\psi^2 + 4\eta_2 K\psi + \eta_2 r$ for both λ_3 and λ_5 is positive, then λ_3 and λ_5 are real, otherwise imaginary.

The eigenvalues λ_4 depends on the combination of terms in the numerator which seems relatively complex (complicated than other eigenvalues). If it becomes positive, the eigenvalue is positive, otherwise negative, as the denominator must be positive.

For third equilibrium

$$J_3 = \begin{bmatrix} a - \frac{K\alpha\mu}{b} + \frac{K\alpha\sigma}{b} & 0 & 0 & 0 & 0 \\ 0 & b & \alpha & -\beta & -\gamma \\ 0 & 0 & -r & -K\varphi & -K\psi \\ 0 & 0 & 0 & K\varphi - \eta_1 & 0 \\ 0 & 0 & 0 & 0 & K\psi - \eta_2 \end{bmatrix}$$

$$\begin{vmatrix} a-\frac{K\alpha\mu}{b}+\frac{K\alpha\sigma}{b}-\lambda & 0 & 0 & 0 & 0 \\ 0 & b-\lambda & \alpha & -\beta & -\gamma \\ 0 & 0 & -r-\lambda & -K\varphi & -K\psi \\ 0 & 0 & 0 & K\varphi-\eta_1-\lambda & 0 \\ 0 & 0 & 0 & 0 & K\psi-\eta_2-\lambda \end{vmatrix}=0$$

$$-\left(\left(r+\lambda\right)\left(b-\lambda\right)\left(\eta_1-K\varphi+\lambda\right)\left(\eta_2-K\psi+\lambda\right)\left(ab-b\lambda-K\alpha\mu+K\alpha\sigma\right)\right)/b=0$$

Eigenvalues are

$$\lambda_1=b$$
$$\lambda_2=K\varphi-\eta_1$$
$$\lambda_3=K\psi-\eta_2$$
$$\lambda_4=\frac{ab-K\alpha\mu+K\alpha\sigma}{b}$$
$$\lambda_5=-r$$

Here the eigenvalues are both positive and negative. Hence the equilibrium point is unstable saddle node.

For fourth equilibrium

$$J_4=\begin{bmatrix} a-\frac{\mu\left(\beta\eta_1 r+K\alpha\eta_1\alpha-K\beta\varphi r\right)+\sigma\left(\beta\eta_1 r+K\alpha\eta_1\varphi-K\beta\varphi r\right)}{Kb\varphi^2} & 0 & 0 & 0 & 0 \\ 0 & b & \alpha & -\beta & -\gamma \\ 0 & 0 & \frac{r\left(\eta_1-K\varphi\right)}{K\varphi}-r\left(\frac{\eta_1}{K\varphi}-1\right)-\frac{\eta_1 r}{K\varphi} & -\eta_1 & \frac{-\eta_1\psi}{\varphi} \\ 0 & 0 & \frac{-r\left(\eta_1-K\varphi\right)}{K\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\eta_1\psi}{\varphi}-\eta_1 \end{bmatrix}$$

Now

$$\begin{vmatrix} a-\frac{\mu(\beta\eta_1 r+K\alpha\eta_1\alpha-K\beta\varphi r)+\sigma(\beta\eta_1 r+K\alpha\eta_1\varphi-K\beta\varphi r)}{Kb\varphi^2}-\lambda & 0 & 0 & 0 & 0 \\ 0 & b-\lambda & \alpha & -\beta & -\gamma \\ 0 & 0 & \frac{r(\eta_1-K\varphi)}{K\varphi}-r\left(\frac{\eta_1}{K\varphi}-1\right)-\frac{\eta_1 r}{K\varphi}-\lambda & -\eta_1 & \frac{-\eta_1\psi}{\varphi} \\ 0 & 0 & \frac{-r(\eta_1-K\varphi)}{K\varphi} & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \frac{\eta_1\psi}{\varphi}-\eta_1-\lambda \end{vmatrix}=0$$

Eigenvalues are

$$\lambda_1=b$$

$$\lambda_2=\frac{-(\eta_2\varphi-\eta_1\psi)}{\varphi}$$

$$\lambda_3=\frac{-\left(\eta_1 r-\sqrt{\left(\eta_1 r\left(-4K^2\varphi^2+4\eta_1 K\varphi+\eta_1 r\right)\right)}\right)}{2K\varphi}$$

$$\lambda_4=\frac{\left(\eta_1\beta r\sigma-\eta_1\beta\mu r+Kab\varphi^2-K\alpha\eta_1\mu\varphi+K\alpha\eta_1\varphi\sigma+K\beta\mu r\varphi-K\beta r\varphi\sigma\right)}{Kb\varphi^2}$$

$$\lambda_5=\frac{-\left(\eta_1 r+\sqrt{\left(\eta_1 r\left(-4K^2\varphi^2+4\eta_1 K\varphi+\eta_1 r\right)\right)}\right)}{2K\varphi}$$

Here λ_1 is positive.

For λ_2, if $\eta_2\varphi-\eta_1\psi<0$ then $\lambda_2>0$ otherwise it is negative.

λ_3 and λ_4 are more complex, because they both contains square roots, which could indicate either real or complex eigenvalues.

If the discriminant, $\eta_1 r\left(-4K^2\varphi^2+4\eta_1 K\varphi+\eta_1 r\right)$ for both λ_3 and λ_5 is positive, then λ_3 and λ_5 are real, otherwise imaginary.

The eigenvalues λ_4 depends on the combination of terms in the numerator which seems relatively complicated than other eigenvalues. If it becomes positive, the eigenvalue is positive, otherwise negative, as the denominator must be positive.

Numerical Simulation

Parameter values

Parameters	Descriptions	Values
a	Natural rising rate of atmospheric temperature	0.099
μ	Natural rising rate of temperature because of GHGs	0.0025
σ	Reduction rate of temperature due to declination of GHGs	0.003565
b	Natural growth rate of GHGs	0.00095
α	Producing rate of GHGs because of fossil fuel consumption	3.845e-6
β	Declining rate of GHGs by tidal energy consumption	5.984e-6
γ	Decreasing rate of GHGs due to wave energy consumption	5.984e-6
r	Normal growth rate of fossil fuel consumption	0.012
K	Carrying capacity of fossil fuel consumption	955000
φ	Replacement rate of fossil fuels with tidal energy	5e-6
ψ	Substitution rate of fossil fuels with wave energy	5e-6
η_1	System loss of tidal energy	0.001
η_2	Limitations of wave energy	0.001

From figure 2, the rate of change in the temperature and GHGs emission are significantly down after 20 years when the consumption of fossil fuel decreases as well as consumption of tidal energy and wave energy consumption increases. With the increasing rate of renewable energy sources, it decreases GHGs emission which finally decreases atmospheric temperature. In continuation of time, tidal energy and wave energy are useful in not only fulfilling demand of fossil fuel but also reducing impact of climate change. The fossil fuel consumption is replaced by tidal energy and wave energy consumption. It is noticeable that temperature still rising but this is because of natural rate of growth. It is clear that the replacement of fossil fuel with tidal and wave energy is very effective in terms of controlling atmospheric temperature and GHGs emission.

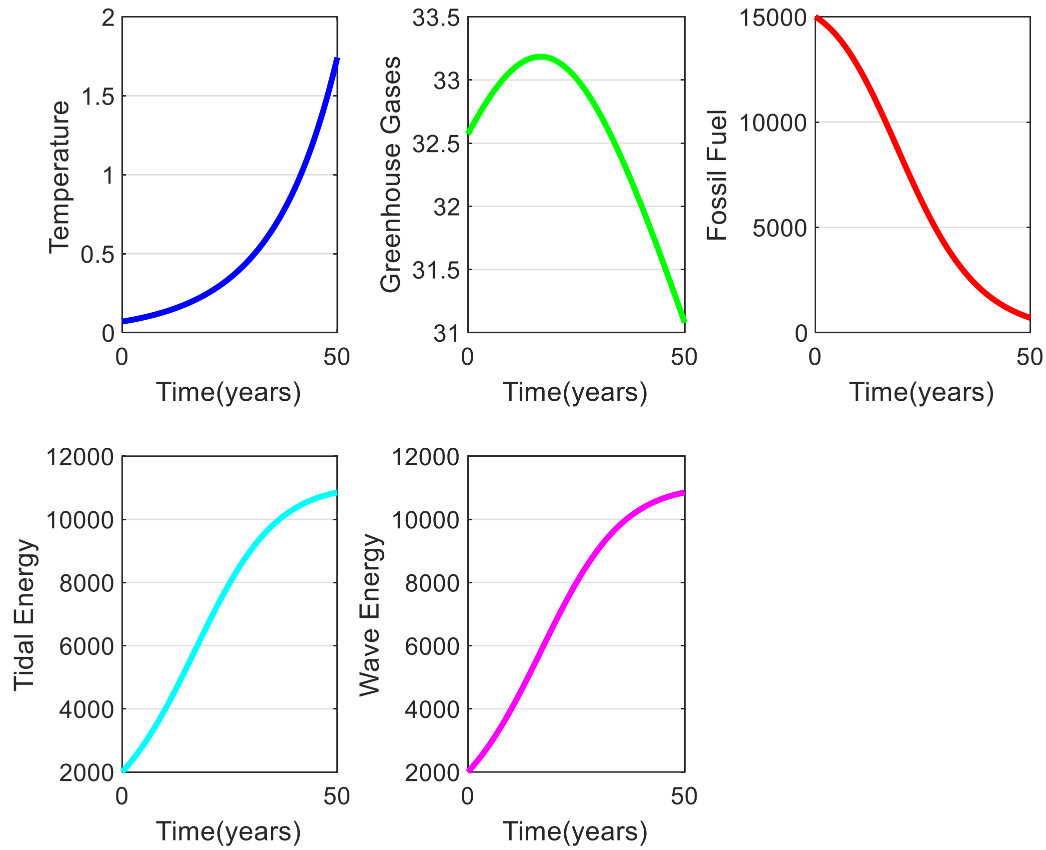

Figure 2: Solution trajectories of temperature, GHGs, fossil fuel, tidal energy and wave energy with respect to time (years).

Conclusion:

In this paper, a mathematical model has been newly formulated to study the dynamical behavior of the atmospheric temperature and GHGs emissions by replacing the consumption of fossil fuel with tidal energy and wave energy. This study is conducted based on a mathematical model with five compartments: Atmospheric temperature T(t), Greenhouse Gas G(t), Consumption of fossil fuel $F_e(t)$, Consumption of tidal energy $T_e(t)$, Consumption of wave energy $W_e(t)$.

This paper studied both mathematical and numerical analysis on the proposed model. In the mathematical analysis, the positivity of the model has been analyzed along with the equilibrium and stability analysis. In the numerical analysis section, the dynamics of the model is visualized with the assistance of MATLAB R2018a.

From the numerical analysis, it is evident that renewable energy can effectively reduce the rise of global temperature as well as GHGs emission that is being happened by fossil fuel consumption. Rate of replacement is another factor. If rate of replacement is lower, temperature and GHGs emission will keep rising up.

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Modeling the environmental benefits of biomass energy over fossil fuels in mitigating climate change in urea manufacturing

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Abstract

Fossil fuels are indispensable to modern life. We cannot think of the norms and expectations of the modern era without using it. However, the convenience and necessity of fossil fuels in our day-to-day activities is undeniable, the environmental consequences of their widespread use can no longer be ignored. Urgent action is needed to transition towards more sustainable energy alternatives to mitigate the detrimental impacts of fossil fuel combustion on the planet's climate system. A total 994 billion cubic feet (BCF) of natural gas was produced in 2019-20 and 5%, was used in fertilizer sector, while all the 7 urea fertilizer factories are dependent on natural gas for feedstock. The combustion of fossil fuels in urea manufacturing releases large amount of carbon dioxide (CO_2), a potent greenhouse gas. Using biomass energy instead of fossil fuels in urea production can significantly reduce greenhouse gas emissions and move towards more sustainable agriculture. This process also reduces production costs and CO_2 emissions compared to conventional urea production from fossil fuels. Our main target is to work on the factors contributing to a potential reduction in atmospheric temperature. We have introduced a five compartmental model in order to diminish the amount of ambient CO2. The analysis has identified multiple equilibrium points. By thoroughly investigating the equilibrium points and their stability, the analysis has yielded valuable insights into the dynamics and behavior of the mathematical model. The identification of these equilibrium points is crucial for understanding the long-term behavior of the system.

Keywords

Mathematical Modeling, GHG emission, Urea fertilizers, Fossil fuel, Biomass energy.

1. Introduction

The worldwide concern over climate change has brought a renewed consciousness on sustainable energy resources to reduce greenhouse gas emissions. Carbon dioxide emissions is a major contributor to rising temperatures and environmental degradation as it is the primary GHGs. In Bangladesh, urea production is a vital industry, contributing substantially to the country's agricultural sector. But, the traditional reliance on fossil fuels for urea production has exacerbated environmental challenges, specifically through carbon dioxide emissions and different harmful byproducts. To cope with this, the usage of biomass energy, a renewable and carbon-neutral alternative, provides a promising solution.

Biomass energy is emerging as a pivotal alternative to fossil fuels, particularly in the context of mitigating climate change. This renewable energy source, derived from organic materials such as agricultural residues, wood, and other biological waste, offers significant environmental benefits. Unlike fossil fuels, biomass is considered carbon-neutral because the carbon dioxide released during its combustion is roughly equal to the amount absorbed by plants during their growth cycle. These characteristic positions biomass as a viable option for reducing greenhouse gas emissions and combating climate change.

Manufacturing of urea—a key component in fertilizers—relies heavily on fossil fuels, transitioning to biomass energy could lead to substantial reductions in carbon emissions. The country's agricultural sector produces ample organic waste that can be converted into biomass energy. By integrating biomass energy into urea manufacturing processes, can reduce reliance on imported fossil fuels. This thesis explores the mathematical modeling of these environmental benefits, focusing on quantifying the potential reductions in atmospheric temperature and carbon emissions by utilizing biomass energy over fossil fuels in urea manufacturing.

Haus, S. (2018) discussed the dynamic climate impact of biomass production and its potential to substitute fossil fuels. Shafiee, M. et al. (2023) highlights the interconnection between biomass practices and climate dynamics, providing insights into sustainable biomass utilization. Zafar, M. W. et al (2021) briefly described the effects of biomass energy consumption on environmental quality, focusing on its role in reducing fossil fuel reliance and GHG emissions. Rathoure, A. K. & Khade, S. M. (2022) discussed various aspects of biomass energy, including its role in climate change mitigation. It emphasizes the importance of transitioning to renewable energy sources to combat environmental pollution. Besides, Lindsey, R. (2024) explains how increased CO_2 levels enhance the greenhouse effect, leading to global temperature rises. It discusses the historical context of CO_2 concentrations and their implications for climate change.

1.1 Objectives

In this study we analyze a five compartmental model with time-varying parameters that may take functional form in time. The objectives of the thesis are:

- To study the dynamics behavior of the environmental factors and agroecosystem factors. Here the environmental factors are atmospheric temperature and concentration of carbon dioxide (CO_2). And the agroecosystem factors are production of urea using fossil fuel & biomass and paddy production.
- To analyze the effect of consumption of fossil fuel while producing urea on temperature and carbon dioxide (CO_2).
- To describe how atmospheric temperature can be reduce by using biomass in urea production instead of fossil fuels.
- To solve the model analytically and numerically by using the ode45 solver in MATLAB programming.

2. Model Formulation

A mathematical model of five compartments with time varying parameter has been formulated to describe the interactions among rising temperature caused by CO_2 which comes from fossil fuel combustion in urea fertilizer production that are used in increasing rice yield. We assume that C(t) denotes the amount of atmospheric carbon dioxide which is rising due to environmental pollution. This has a direct effect on rising atmospheric temperature. T(t) indicates the amount of rising temperature. $F_u(t)$ denotes fossil fuel consumption while $B_u(t)$ denotes the biomass energy consumption and P(t) denotes the amount of rice plants production. The effects of global warming in the ecosystem due to fertilizers is shown in Figure 1. Combustion of fossil increases CO_2 emission in a large amount in the atmosphere. When UV radiation is present, a number of chemical reactions take place among the atmospheric gases. During this period, heat produces and that increase the atmospheric temperature. So here r_2 is considered as the constant rate of increase of CO_2 increasing global warming. Since in present circumstances global temperature is increasing naturally at a rate r_1 and at a rate at η in the cause of CO_2. Using the fossil fuels in many sectors by burning them is one of the most significant facts in increasing greenhouse gas emission like CO_2. Here r_3 is the Consumption rate of fossil fuel while producing urea. We have considered CO_2 is increasing at β rate from burning fossil fuel while urea manufacturing and α is depletion rate of CO_2 through using biomass in urea production instead of fossil fuel. Rice plant absorbs CO_2 for photosynthesis at γ rate. As fossil fuels are natural gas, so these resources are limited in earth. Considering this fact we have taken k as the carrying capacity of fossil fuel and μ is replacement rate of fossil fuel consumption with biomass energy consumption in urea manufacturing. Some limitations in biomass energy consumption are denoted by τ . In paddy production process increasing rice yield through urea from fossil fuels and biomass are at rate of ϕ_1 and ϕ_2. Death rate of rice plant due to rising temperature and due to other environmental reasons is ε.

Parameters:

r_1 = Natural growth rate of atmospheric temperature

η = Rising rate of atmospheric temperature due to CO_2 (Carbon-di-oxide)

σ = The rate of temperature rise from burning fossil fuel

ψ = Temperature reduction rate by using biomass energy

r_2 = Natural increasing rate of CO_2 in the atmosphere

β = Producing rate of CO_2 through using fossil fuel while urea production

α = decreasing rate of CO_2 for using biomass in urea production instead of fossil fuel

γ = Absorbing rate of CO_2 by rice plant

r_3 = Consumption rate of fossil fuel while producing urea

k = carrying capacity of fossil fuel

μ = replacement rate of fossil fuel consumption with biomass energy consumption

r_4 = Rate of biomass energy consumption while urea production

τ = Degradation rate of biomass energy

ϕ_1 = Increasing rate of rice yield through urea from fossil fuels

ϕ_2 = Increasing rate of rice yield through urea from biomass energy

δ = Death rate of rice plant due to other environmental reason

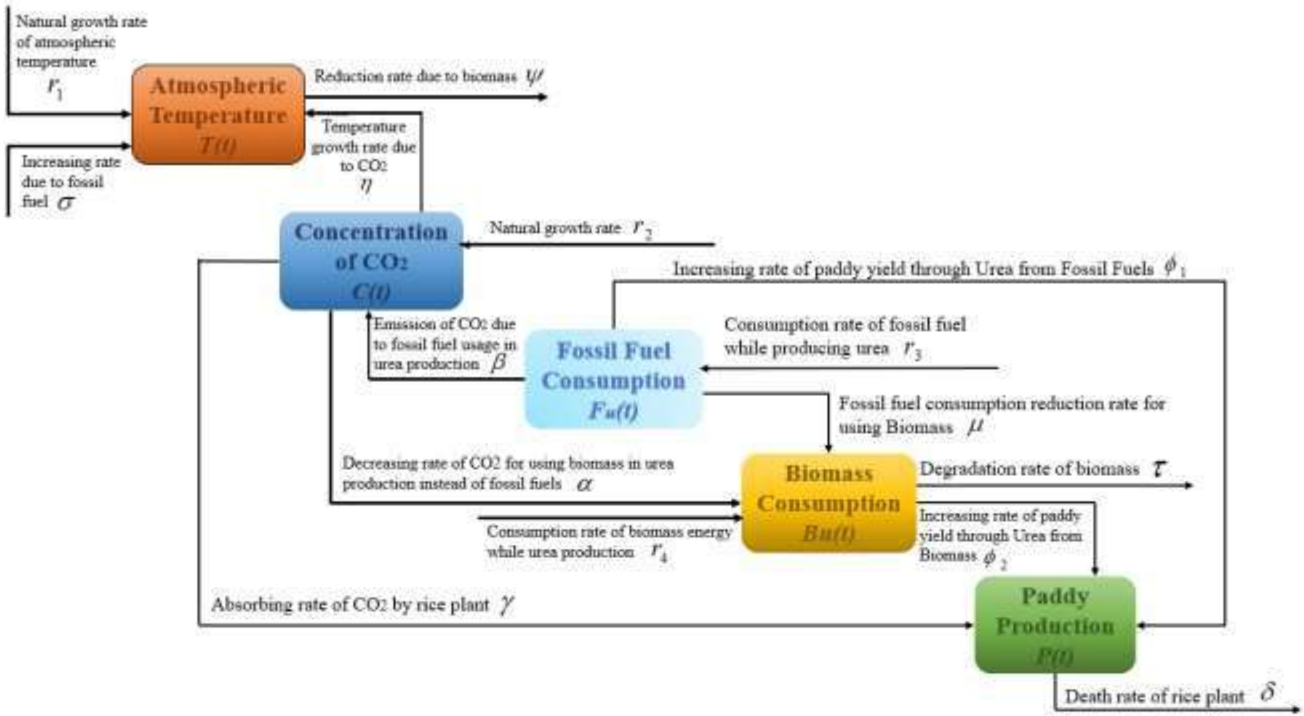

Figure 1: The schematic diagram of the model (1-5) briefly describing the climate change due to the consumption of fossil fuel and biomass in the production of urea fertilizer in paddy fields.

According to the above assumptions, we newly formulate the following mathematical model consisting of a system of NODEs:

$$\frac{dT}{dt} = r_1 T + \eta CT + \sigma F_u - \psi B_u \qquad \text{................ (1)}$$

$$\frac{dC}{dt} = r_2 C + \beta F_u - \alpha B_u - \gamma PC \qquad \text{................ (2)}$$

$$\frac{dF_u}{dt} = r_3 F_u \left(1 - {F_u}/{k}\right) - \mu F_u B_u \qquad \text{................ (3)}$$

$$\frac{dB_u}{dt} = \mu F_u B_u - \tau B_u \qquad \text{................ (4)}$$

$$\frac{dP}{dt} = r_3 F_u \left(1 - {F_u}/{k}\right)\phi_1 + \mu F_u B_u \phi_2 - \delta P \qquad \text{................ (5)}$$

Where T, C, F_u, B_u, P are state variables and $r_1, r_2, r_3, r_4, \alpha, \beta, \gamma, \eta, \sigma, \psi, k, \mu, \tau, \phi_1, \phi_2, \delta$ are parameters.

3. Analytical Analysis

In this section, we test the positivity of the dynamical variables of the system, stability analysis at equilibrium points, and numerical simulation [Dym, C. L. (2004), Banerjee, S. (2021), Biswas et al. (2017), Kabir et al. (2020), Mandal et al. (2020b)].

3.1 Positivity Analysis of the model

Lemma 1: The solutions of the system (1)-(5) are non-negative for all $t \geq 0$.

Proof: To prove lemma 1, we have used the system of equations of the mathematical model.

$$\frac{dT}{dt} = r_1 T + \eta CT + \sigma F_u - \psi B_u \qquad \text{................ (1)}$$

$$\frac{dC}{dt} = r_2 C + \beta F_u - \alpha B_u - \gamma PC \qquad \text{................ (2)}$$

$$\frac{dF_u}{dt} = r_3 F_u \left(1 - {F_u}/{k}\right) - \mu F_u B_u \qquad \text{................ (3)}$$

$$\frac{dB_u}{dt} = \mu F_u B_u - \tau B_u \qquad \text{................ (4)}$$

$$\frac{dP}{dt} = r_3 F_u \left(1 - {F_u}/{k}\right)\phi_1 + \mu F_u B_u \phi_2 - \delta P \qquad \text{................ (5)}$$

The first differential equation of the model describes the change of temperature which is given below,

$$\frac{dT}{dt} = r_1 T + \eta CT + \sigma F - \psi B$$

Taking only linear parts of the above equation, we get

$$\frac{dT}{dt} > r_1T + \sigma F - \psi B$$
$$\Rightarrow \frac{dT}{dt} > T(r_1 + \sigma - \psi)$$
$$\Rightarrow \frac{dT}{T} > (r_1 + \sigma - \psi)dt \qquad \text{..........(6)}$$

Integrating (6) both sides with respect to t, we get

$$\ln T > (r_1 + \sigma - \psi)t + \ln c_1, \quad \text{where } c_1 \text{ is an integrating constant.}$$
$$\therefore T > c_1 e^{(r_1+\sigma-\psi)t} \qquad \text{........... (7)}$$

For the initial condition $T(0) = T_o$ i.e., when t=0, we obtain the following condition from (7)

$$T_0 > c_1 \qquad \text{............ (8)}$$

From (7) and (8) we get,

$$T(t) > T(0)e^{(r_1+\sigma-\psi)t}, \ \forall t \geq 0 \ \text{ and } (r_1 + \sigma - \psi) \in \mathrm{R}$$
$$\therefore e^{(r_1+\sigma-\psi)t} > 0$$
$$\therefore T(t) > 0$$

Therefore, $T(t) > 0$ when $t \to \infty$.

$\therefore T(t)$ is positive for all $t \geq 0$.

Similarly, we obtained $C(t) > 0,\ F_u(t) \geq 0,\ B_u(t) \geq 0, P(t) \geq 0\ \forall t \geq 0$ from Eqs. $(2)-(5)$.

Hence the lemma is completed with $T(t) > 0,\ C(t) > 0,\ F_u(t) \geq 0,\ B_u(t) \geq 0, P(t) \geq 0\ \forall t \geq 0.$

3.2 Equilibrium Points:

The model has equilibrium points at $T^* = C^* = F_u^* = B_u^* = P^* = 0$. Solving the equations, we get four equilibrium points as,

1. $(0,0,0,0,0)$

2. $\left(0, \dfrac{-\beta k}{r_2}, k, 0, 0\right)$
3. $\left(\dfrac{-\psi}{\delta}, \dfrac{\sigma - r_1}{\eta}, 0, 0, \dfrac{r_1}{\gamma}\right)$
4. $\left(0, \dfrac{\delta C}{\mu^2 \delta k r_2 + \gamma A}, \dfrac{\tau - r_4}{\mu}, \dfrac{r_3(r_4 - \tau + \mu k)}{\mu^2 k}, \dfrac{-A}{\mu^2 \delta k}\right)$

where, $A = \phi_1 r_3 r_4{}^2 + \phi_2 r_3 r_4{}^2 + \sigma^2 \phi_1 r_3 + \psi^2 \phi_2 r_3 - 2\sigma\phi_1 r_3 r_4 - 2\psi\phi_2 r_3 r_4$ and
$C = \alpha r_3 r_4 - \alpha\sigma r_3 + \alpha\mu k r_3 + \beta\mu k r_4 - \beta\sigma\mu k$

3.3 Stability Analysis:

Let,

$r_1 T + \eta C T + \sigma F - \psi B = f\left(T, C, F, B, P\right)$

$r_2 C + \beta F - \alpha B - \gamma P C = g\left(T, C, F, B, P\right)$

$r_3 F\left(1 - F/k\right) - \mu F B = h\left(T, C, F, B, P\right)$

$r_4 B + \mu F B - \tau B = q\left(T, C, F, B, P\right)$

$r_3 F\left(1 - F/k\right)\phi_1 + \mu F B \phi_2 - \delta P = m\left(T, C, F, B, P\right)$

The Jacobian matrix is, $J = \begin{pmatrix} f_T & f_C & f_F & f_B & f_P \\ g_T & g_C & g_F & g_B & g_P \\ h_T & h_C & h_F & h_B & h_P \\ q_T & q_C & q_F & q_B & q_P \\ m_T & m_C & m_F & m_B & m_P \end{pmatrix}$

$$= \begin{pmatrix} r_1+\eta C & \eta T & \sigma & -\psi & 0 \\ 0 & r_2-\gamma P & \beta & -\alpha & -\gamma C \\ 0 & 0 & r_3\left(1-{2F}/{k}\right)-\mu B & -\mu F & 0 \\ 0 & 0 & \mu B & r_4+\mu F-\tau & 0 \\ 0 & 0 & r_3\left(1-{2F}/{k}\right)\phi_1+\mu B\phi_2 & \mu F\phi_2 & -\delta \end{pmatrix}$$

i. Stability at $(0,0,0,0,0)$.

$$J^* = \begin{pmatrix} r_1 & 0 & \sigma & -\psi & 0 \\ 0 & r_2 & \beta & -\alpha & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4-\tau & 0 \\ 0 & 0 & r_3\phi_1 & 0 & -\delta \end{pmatrix}$$

Then the characteristic equation is given below,

$$|J-\lambda I| = \begin{pmatrix} r_1-\lambda & 0 & \sigma & -\psi & 0 \\ 0 & r_2-\lambda & \beta & -\alpha & 0 \\ 0 & 0 & r_3-\lambda & 0 & 0 \\ 0 & 0 & 0 & r_4-\tau-\lambda & 0 \\ 0 & 0 & r_3\phi_1 & 0 & -\delta-\lambda \end{pmatrix} = 0$$

Hence the eigenvalues are,

$\lambda_1 = r_2$,

$\lambda_2 = r_3$,

$\lambda_3 = r_4 - \tau$,

$\lambda_4 = r_1 - \sigma$ and

$\lambda_5 = -\delta$.

Here four eigenvalues are positive and one is negative, so the system is unstable for those values.

ii. Stability at $\left(0, \frac{-\beta k}{r_2}, k, 0, 0\right)$.

$$J^* = \begin{pmatrix} r_1 - \frac{\eta\beta k}{r_2} & 0 & \sigma & -\psi & 0 \\ 0 & r_2 & \beta & -\alpha & -\frac{\beta k}{r_2} \\ 0 & 0 & -r_3 & -\mu k & 0 \\ 0 & 0 & 0 & r_4 + \mu k - \tau & 0 \\ 0 & 0 & -r_3\phi_1 & \mu k\phi_2 & -\delta \end{pmatrix}$$

Then the characteristic equation is given below,

$$\left|J - \lambda I\right| = \begin{pmatrix} r_1 - \frac{\eta\beta k}{r_2} - \lambda & 0 & \sigma & -\psi & 0 \\ 0 & r_2 - \lambda & \beta & -\alpha & -\frac{\beta k}{r_2} \\ 0 & 0 & -r_3 - \lambda & -\mu k & 0 \\ 0 & 0 & 0 & r_4 + \mu k - \tau - \lambda & 0 \\ 0 & 0 & -r_3\phi_1 & \mu k\phi_2 & -\delta - \lambda \end{pmatrix} = 0$$

Hence the eigenvalues are,

$\lambda_1 = r_2$

$\lambda_2 = -\delta$

$\lambda_3 = -r_3$

$\lambda_4 = \frac{-\left(\sigma r_2 - r_1 r_2 + \beta k\eta\right)}{r_2}$

$\lambda_5 = r_4 - \tau + \mu k$

It is clear that two eigenvalues are positive and three are negative, so the system is unstable for those values.

iii. Stability at $\left(\frac{-\psi}{\delta}, \frac{\sigma - r_1}{\eta}, 0, 0, \frac{r_1}{\gamma}\right)$.

$$J^* = \begin{pmatrix} \frac{-\eta\sigma}{\delta} & 0 & \sigma & -\psi & 0 \\ 0 & r_2 - r_1 & \beta & -\alpha & \frac{-\gamma(\psi - r_1)}{\eta} \\ 0 & 0 & -r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 - \tau & 0 \\ 0 & 0 & r_3\phi_1 & 0 & \frac{-\delta r_1}{\gamma} \end{pmatrix}$$

Then the characteristic equation is given below,

$$|J - \lambda I| = \begin{pmatrix} \frac{-\eta\sigma}{\delta} - \lambda & 0 & \sigma & -\psi & 0 \\ 0 & r_2 - r_1 - \lambda & \beta & -\alpha & \frac{-\gamma(\psi - r_1)}{\eta} \\ 0 & 0 & -r_3 - \lambda & 0 & 0 \\ 0 & 0 & 0 & r_4 - \tau - \lambda & 0 \\ 0 & 0 & r_3\phi_1 & 0 & \frac{-\delta r_1}{\gamma} - \lambda \end{pmatrix} = 0$$

So, the eigenvalues are given below,

$$\lambda_1 = -r_3$$
$$\lambda_2 = r_4 - \tau$$
$$\lambda_3 = \frac{-\psi\eta}{\delta}$$
$$\lambda_4 = r_2 - r_1$$
$$\lambda_5 = -\left(\frac{\delta r_1}{\gamma} + \sigma\right)$$

Here among the five eigenvalues, if for $\lambda_2 = r_4 - \tau$; $r_4 > \tau$ and for $\lambda_4 = r_2 - r_1$; $r_2 > r_1$ then we obtain two eigenvalues with positive real part. Hence the system is unstable for those values.

Again, for $r_4 < \tau$, λ_2 is negative and for $r_2 < r_1$, λ_4 is negative. Hence the system will be stable.

iv. Stability at $\left(0, \frac{\delta C}{\mu^2 \delta k r_2 + \gamma A}, \frac{\tau - r_4}{\mu}, \frac{r_3(r_4 - \tau + \mu k)}{\mu^2 k}, \frac{-A}{\mu^2 \delta k}\right)$.

$$J^* = \begin{pmatrix} r_1 + \frac{\eta\delta C}{\mu^2\delta k r_2 + \gamma A} & 0 & \sigma & -\psi & 0 \\ 0 & r_2 + \frac{\gamma A}{\mu^2 \delta k} & \beta & -\alpha & \frac{-\gamma\delta C}{\mu^2\delta k r_2 + \gamma A} \\ 0 & 0 & r_3\left(1 - \frac{2(\tau - r_4)}{\mu k}\right) - \frac{\mu r_3(r_4 - \tau + \mu k)}{\mu^2 k} & -(\tau - r_4) & 0 \\ 0 & 0 & \frac{r_3(r_4 - \tau + \mu k)}{\mu k} & r_4 + \frac{(\sigma\tau - r_4\mu)}{\psi} - \tau & 0 \\ \frac{\delta A}{\mu^2 \delta k} & 0 & r_3\left(1 - \frac{2(\tau - r_4)}{\mu k}\right)\phi_1 + \frac{r_3(r_4 - \tau + \mu k)}{\mu k}\phi_2 & (\tau - r_4)\phi_2 & -\delta \end{pmatrix}$$

Then the characteristic equation is given below,

$$|J-\lambda I| = \begin{pmatrix} r_1 + \frac{\eta\delta C}{\mu^2\delta k r_2 + \gamma A} - \lambda & 0 & \sigma & -\psi & 0 \\ 0 & r_2 + \frac{\gamma A}{\mu^2 \delta k} - \lambda & \beta & -\alpha & \frac{-\gamma\delta C}{\mu^2\delta k r_2 + \gamma A} \\ 0 & 0 & r_3\left(1 - \frac{2(\tau - r_4)}{\mu k}\right) - \frac{r_3(r_4 - \tau + \mu k)}{\mu k} - \lambda & -(\tau - r_4) & 0 \\ 0 & 0 & \frac{r_3(r_4 - \tau + \mu k)}{\mu k} & r_4 + \frac{(\sigma\tau - r_4\mu)}{\mu} - \tau - \lambda & 0 \\ \frac{\delta A}{\mu^2\delta k} & 0 & r_3\left(1 - \frac{2(\tau - r_4)}{\mu k}\right)\phi_1 + \frac{r_3(r_4 - \tau + \mu k)}{\mu k}\phi_2 & (\tau - r_4)\phi_2 & -\delta - \lambda \end{pmatrix} = 0$$

Here the eigenvalues are,

$$\lambda_1 = -\delta$$

$$\lambda_2 = \frac{C\eta - \mu^2\psi k r_2 - \mu^2 k r_1 r_2 + A\mu^2\gamma k\eta r_2}{\mu^2 k r_2}$$

$$\lambda_3 = \frac{\sigma k r_2 \mu^2 + r_3\gamma}{\mu^2\sigma k}$$

$$\lambda_4 = -\left(\frac{\mu^2\sigma k r_2 + \gamma A - \gamma\sigma C}{\mu k\gamma}\right)$$

$$\lambda_5 = \frac{r_3 r_4 - \tau r_3 + \left(-r_3(\tau - r_4)\left(r_3 r_4 - \tau r_3 + 4\mu^2 k^2 - 4\tau\mu k + 4\mu k r_4\right)^2\right)}{2\mu k}$$

We can clearly see that here three eigenvalues are positive and two are negative, so the system is unstable for those values.

4. Result and Discussion

Numerical simulation is very important task to represent the interactions among the dynamical variables. Here we use MATLAB (R2018a) to check the feasibility of our analysis concerning stability axioms. Numerical computations have been driven by using a set of parametric values given below:

Table 1: Parametric values of parameters utilized in the model.

Symbol	Values	Symbol	Values	Symbol	Values
r_1	0.001 $°C\ year^{-1}$	β	0.00095	r_4	0.0004 $gm\ day^{-1}\ km^{-2}$
η	0.02 $°C\ year^{-1}$	α	0.00012	τ	0.0002 $gm\ day^{-1}\ km^{-2}$
σ	0.053	γ	0.005 $gm\ day^{-1}\ km^{-2}$	ϕ_1	0.50 $gm\ day^{-1}$
ψ	0.005	r_3	0.06	ϕ_2	0.20 $gm\ day^{-1}$
r_2	0.0119 $gm\ day^{-1}\ km^{-2}$	μ	0.0018	δ	0.000004 $gm\ day^{-1}\ km^{-2}$

Here the initial conditions are $T_0 = 0.25, C_0 = 0.04$, $F_{u0} = 34.35$, $B_{u0} = 20.839$ and $P_0 = 10.701$. Time period is $T = 30$ because of long term management. And weight influences are $r_3 = 0.06$ and $\mu = 0.018$. To examine how well each control technique works, we ran numerical tests in three different scenarios.

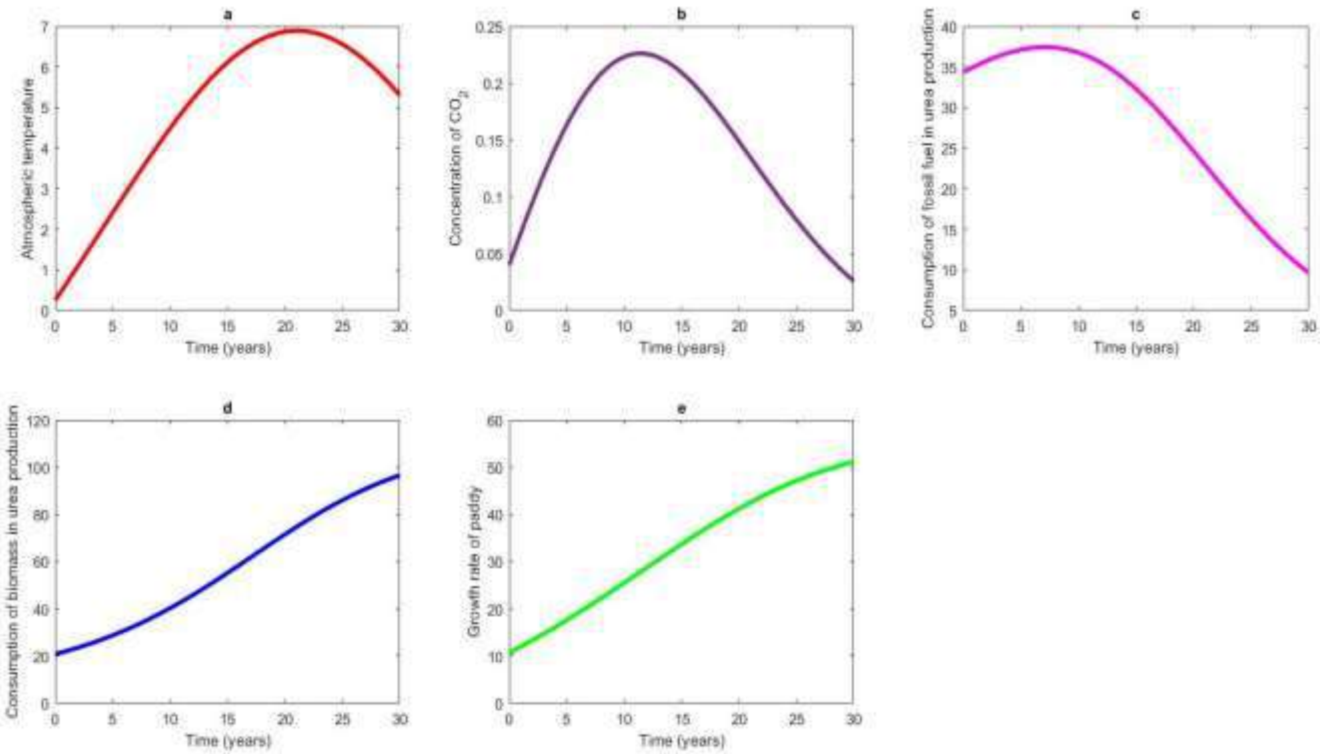

Figure 2: Quantitative simulation of the model.

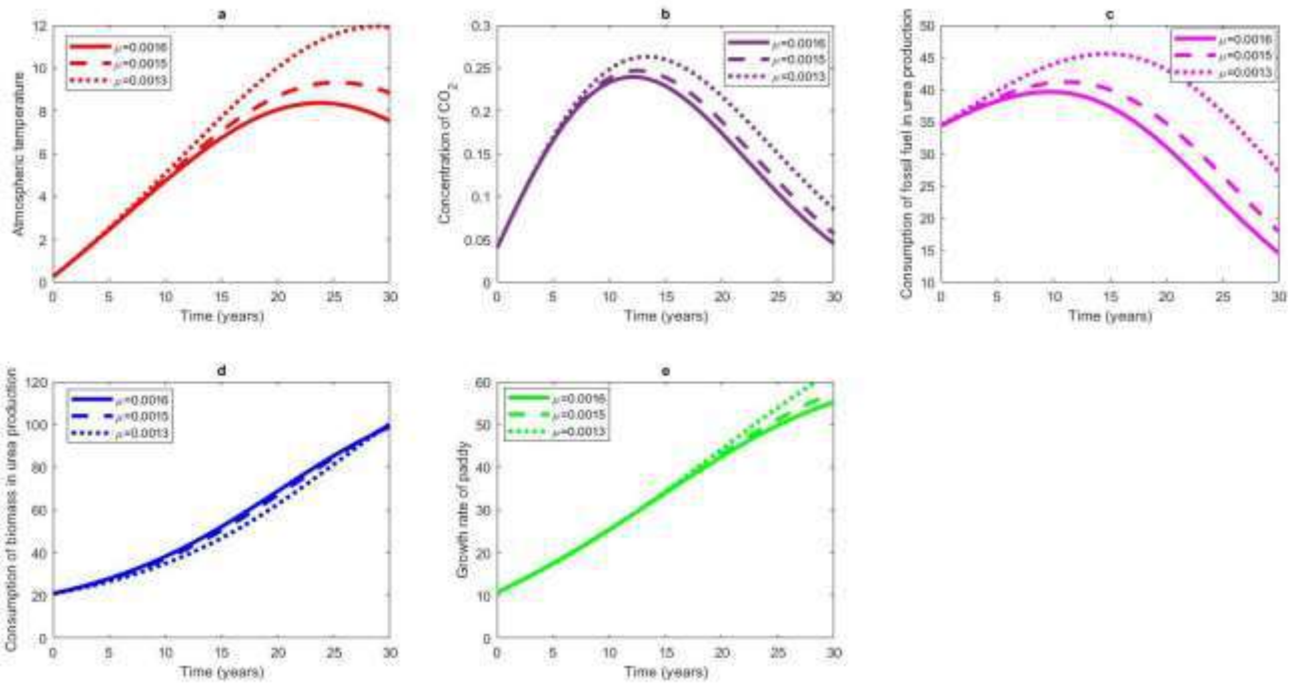

Figure 3: The rate of change of state variables such as temperature and CO_2 level, varies with different values of μ.

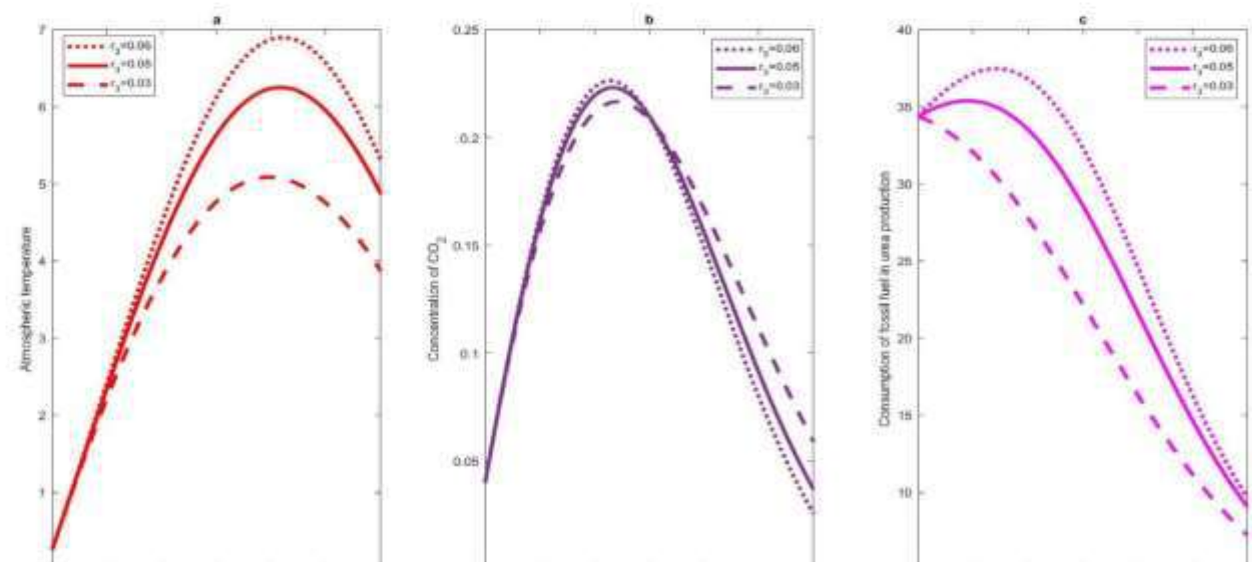

Figure 4: The rate of change of temperature and CO_2 levels, varies with different values of r_3.

Figure 2-4 illustrate the interaction among these dynamic variables over time. As we wanted to replace fossil fuel consumption with biomass energy to show the rate of change of atmospheric temperature caused by CO_2 is shown in Figure 2. We can see that after 10 years, concentration of CO_2 started to decline while the rate of atmospheric temperature begun to decrease from 20 years, because we slower the rate of fossil fuel consumption and enhance biomass energy consumption.

When the replacing rate of fossil fuel with biomass energy increases, it comparatively diminishes the rate of temperature rising as well as the emissions of CO_2 in the environment. Figure 3 represents the reduction rate of CO_2 in the environment when the CO_2 emitting from fossil fuel combustion in urea production decrease. When the emissions of CO_2 decrease due to the biomass replacement, the atmospheric temperature proportionally decreases.

If we reduce the rate of fossil fuel consumption, we can see reduction in temperature rise and concentration levels declination in CO_2. We used this idea and change the value of r_3 which clearly reduces the temperature growth rate and CO_2 concentration level shown in Figure 4.

5. Conclusion

In this paper, a mathematical model has been newly formulated to study the dynamical behavior of the atmospheric temperature and concentration by replacing the consumption of fossil fuel with biomass energy in paddy production. This study is conducted based on a mathematical model with five compartments: Atmospheric temperature $T(t)$, Concentration of CO_2 $C(t)$, Consumption of fossil fuel $F_u(t)$, Consumption of biomass energy $B_u(t)$ and Paddy Production $P(t)$. This paper studied both mathematical and numerical analysis on the proposed model. In the mathematical analysis, the positivity of the model has been analyzed along with the equilibrium and stability analysis. In the numerical analysis section, the dynamics of the model is visualized with the assistance of MATLAB R2018a. This study represents that as the replacement of fossil fuels with biomass energy increases, the rise in temperature and CO_2 emissions in the environment decreases. This study also shows that if we reduce fossil fuel consumption, the atmospheric temperature and CO_2 concentration levels also decreases. Therefore, the study concerns with climate change and presents a relationship among the environmental factors and agroecosystem factors along with mathematical modeling.

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Classes of Rings of Bi-complex Numbers

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Abstract: In this article we explore different types of ring structures that of bi-complex numbers follow and introduce some examples with the help of known involutions on bi-complex numbers.

Keywords: Bi-complex numbers, involution ring, hyper complex number, symmetric ring, nil-symmetric ring, *-symmetric ring.

MSC(AMS 2020): 30G35, 16A48.

1. Introduction

Bi-complex numbers were introduced by Segre[5] in 1892. The space of bi-complex numbers is the first in an infinite sequence of multicomplex spaces. The set of bi-complex numbers is denoted by $B_{\mathbb{C}_2}$. Unlike the quaternions, the multiplication of bi-complex numbers is commutative.

Throughout this paper, all rings are associate with unity. Given a ring R, $\mathrm{nil}(R)$ denote the set of all nilpotent elements of R. The bi-complex numbers forms a commutative ring with zero divisor w.r.t the usual operations. For comprehensive insights into bi-complex numbers, the work in [1,2,3,4,6,8,9,10,11,12] is recommended

Definition 1.1. The set of bi-complex numbers, denoted by $B_{\mathbb{C}_2}$, is the set

$$B_{\mathbb{C}_2} := \{z = z_1 + z_2 i_2 \mid z_1, z_2 \in C(i_1)\},$$

where i_1 and i_2 are imaginary units. i.e., $i_1 i_2 = i_2 i_1, i_1^2 = i_2^2 = -1$, and $i_1 \neq i_2$ and $C(i_1)$ is the set of complex numbers with the imaginary unit i_1. Since there are two imaginary

units inside $B_{\mathbb{C}_2}$, we will distinguish between the two sets of complex numbers as $C(i_1)$ and $C(i_2)$.
The set of bi-complex numbers form a ring with respect to the sum and product defined as follows: if $\mathrm{Z} = z_1 + z_2 i_2$ and $W = w_1 + w_2 i_2 \in B_{\mathbb{C}_2}$ then
$Z + W = (z_1 + z_2 i_2) + (w_1 + w_2 i_2) = (z_1 + w_1) + i_2(z_2 + w_2)$.
$Z \cdot W = (z_1 + z_2 i_2) \cdot (w_1 + w_2 i_2) = (z_1 w_1 - z_2 w_2) + i_2(z_1 w_2 + w_1 z_2)$.
Then $\left(B_{C_2}, +, .\right)$ is a commutative ring with zero divisors.
If z is a bi-complex number, then by writing down explicitly the real and imaginary parts of the components of z, we then obtain the following represantation in terms of real components:

$$z = x_1 + x_2 i_1 + x_3 i_2 + x_4 i_1 i_2$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $k = i_1 i_2$ is called a hyperbolic unit becacuse $k^2 = 1$. using the hyperbolic unit, we can define the set of hyperbolic numbers

$$\mathbb{D}(k) = \{x + ky: x, y \in \mathbb{R}\}.$$

Equipped with the bi-complex addition and multiplication, the set of hyperbolic numbers becomes a commutative subring of the set of bi-complex numbers.

Definition 1.2. An element of the ring is called idempotent element if the multiplication of the element with itself gives the same element.
Besides 0 and 1, there are exactly two nontrivial idempotent elements in $B_{\mathbb{C}_2}$ defined as $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$.
Note: $e_1 + e_2 = 1$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$.
Here, e_1 and e_2 are zero divisor of the ring $\left(B_{\mathbb{C}_2}, +, .\right)$.
Every bi-complex number $z = z + w i_2$ can be written in a unique way as:

$$Z = z_1 e_1 + z_2 e_2$$

where $z_1 = z - i_1 w$ and $z_2 = z + i_1 w$ are elements of $C(i_1)$.

Definition 1.3.
Let $z = z_1 + i_2 z_2$ be any bi-complex numbers. Then there are three conjugations for bicomplex numbers in analogy with the usual complex conjugation:
(1) $(z_1 + i_2 z_2)^{*1} = \overline{z_1} + i_2 \overline{z_2}$, for all $z_1, z_2 \in C(i_1)$ and $\overline{z_1}$ and $\overline{z_2}$ are complex conjugate on $C(i_1)$.
(2) $(z_1 + i_2 z_2)^{*2} = (z_1 - i_2 z_2)$, for all $z_1, z_2 \in C(i_1)$ and $\overline{z_1}$ and $\overline{z_2}$ are complex conjugate on $C(i_1)$.
(3) $(z_1 + i_2 z_2) = (\overline{z_1} - i_2 \overline{z_2})$, for all $z_1, z_2 \in C(i_1)$ and $\overline{z_1}$ and $\overline{z_2}$ are complex conjugate on $C(i_1)$.

Definition 1.4.[7]
A function $\phi: B_{\mathbb{C}_2} \to B_{\mathbb{C}_2}$ is said to be an involution on $B_{\mathbb{C}_2}$, if it satisfies the following properties:
(1) $\phi(\phi(z)) = z$ for any $z \in B_{\mathbb{C}_2}$.
(2) $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2$ and $\phi(\lambda z) = \lambda\phi(z)$ for any $z, z_1, z_2 \in B_{\mathbb{C}_2}$ and $\lambda \in \mathbb{R}$.
(3) $\phi(z_1 z_2) = \phi(z_1)\phi(z_2)$ for any $z, z_1, z_2 \in B_{\mathbb{C}_2}$.

Note: Here, $*1, *2$ and $\sim$ are involutions on $B_{\mathbb{C}_2}$.
For a ring $R, M_n(R)$ and $T_n(R)$ denote the $n \times n$ full matrix ring and the upper triangular ring over R, respectively.
If R is a ring, then

$$\mathrm{nil}(T_n(R)) = \begin{bmatrix} \mathrm{nil}(R) & R & R & \dots & R \\ 0 & \mathrm{nil}(R) & R & \dots & R \\ 0 & 0 & \mathrm{nil}(R) & \dots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{nil}(R) \end{bmatrix}.$$

Definition 1.5. A Ring R is called reduced ring if it has no non-trivial nilpotent element.

Example 1.6. $(\mathbb{Z}, +, .)$ has no non trivial nilpotent element. So, it is a reduced ring.

Definition 1.7.[16] A Ring R is called symmetric ring if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$.
Definition 1.8.[15] An element a in a ring R is called a nilpotent element if $a^n = 0$ for some $n \in \mathbb{N}$.
Definition 1.9.[14] *A* ring R is called right(left) nil-symmetric if whenever, for every $a, b \in \mathrm{nil}(R)$ and for every $c \in R$, $abc = 0 (cab = 0)$ implies $acb = 0. A$ ring R is called nil-symmetric if it is both right and left nil-symmetric ring.

Note: Every symmetric ring is nil-symmetric but the converse is not always true.

Example 1.10. For a reduced ring $R, T_2(R)$ is a nil-symmetric ring which is not a symmetric ring. This can be verified as follows:
Let $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in \mathrm{nil}(T_2(R))$ and $\begin{bmatrix} c & d \\ 0 & e \end{bmatrix} \in T_2(R)$.
If $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & e \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
Then $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & e \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence, $T_2(R)$ is a right nil-symmetric ring. Similarly it can be shown that $T_2(R)$ is a left nil-symmetric ring. But $T_2(R)$ is not symmetric ring.
Since,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Whereas

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition 1.11.[13] A ring R with involution $*$ is called $* -$ symmetric if for any elements $a, b, c \in R, abc = 0$ implies $acb^* = 0$.

Definition 1.12.[13] A ring R with involution $*$ is called $* -$ a reversible ring if ab $= 0$ implies $ba^* = 0$.

2. Main results:

In this section we state and prove the results of this article

Theorem 2.1. $\left(B_{\mathbb{C}_2}, +, .\right)$ is a reduced ring.
Proof: We already know that $\left(B_{\mathbb{C}_2}, +, .\right)$ is a commutative ring with unity. Now to show that $\left(B_{\mathbb{C}_2}, +, .\right)$ is a reduced ring. We have to prove that it has no non-trivial nilpotent element. If possible $\left(B_{\mathbb{C}_2}, +, .\right)$ has a non-trivial nilpotent element say, z.where z can be written as

$$z = \mu_1 z_1 + \mu_2 z_2, \text{ where } \mu_1, \mu_2 \in C(i_1).$$

Therefore,

$$\begin{aligned} & z^n = 0 \\ & \Longrightarrow (\mu_1 z_1 + \mu_2 z_2)^n = 0 \\ & \Longrightarrow (\mu_1)^n z_1 + (\mu_2)^n z_2 = 0 \\ & \Longrightarrow (\mu_1)^n = 0 \text{ and } (\mu_2)^n = 0 \\ & \text{i.e., } z = 0. \end{aligned}$$

So, $\left(B_{\mathbb{C}_2}, +, .\right)$ is a reduced ring.
Therefore, nil$\left(B_{\mathbb{C}_2}, +, .\right) = \{0\}$.

Example 2.2. $T_2\left(B_{\mathbb{C}_2}\right) = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in B_{\mathbb{C}_2}\right\}$, where,

$$\text{nil}\left(T_2\left(B_{\mathbb{C}_2}\right)\right) = \left\{\begin{bmatrix} \text{nil}\left(B_{\mathbb{C}_2}\right) & a \\ 0 & \text{nil}\left(B_{\mathbb{C}_2}\right) \end{bmatrix} : a \in B_{\mathbb{C}_2}\right\} = \left\{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in B_{\mathbb{C}_2}\right\}.$$

Let

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in \text{nil}\left(T_2\left(B_{\mathbb{C}_2}\right) \text{ and } \begin{bmatrix} c & d \\ 0 & e \end{bmatrix} \in T_2\left(B_{\mathbb{C}_2}\right).$$

If

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & e \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & e \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $T_2\left(B_{\mathbb{C}_2}\right)$ is a right nil-symmetric ring. similarly, it can be shown that $T_2\left(B_{\mathbb{C}_2}\right)$ is a left nil-symmetric ring. i.e., $T_2\left(B_{\mathbb{C}_2}\right)$ is a nil-symmetric ring. But it is not a symmetric ring. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \in T_2\left(B_{\mathbb{C}_2}\right)$.

Here,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

But

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Remark 2.3. Every hyperbolic number corresponds to the symmetric 2×2 matrices with equal diagonal entries. i.e., $x + i_1 i_2 y \leftrightarrow \begin{bmatrix} x & y \\ y & x \end{bmatrix}$.

This correspondence is an isomorphism because the operations of addition and multiplication in $\mathbb{D}$ corresponds to the usual matrix operations. Then, we can think every hyperbolic number as a symmetric matrix with equal diagonal entries. Let $\mathbb{D} = \{x + i_1 i_2 y : x, y \in \mathbb{R}\} = \left\{\begin{bmatrix} x & y \\ y & x \end{bmatrix} : x, y \in R\right\}$ is also a reduced ring. Since it also a subring of $\left(B_{\mathbb{C}_2}, +, .\right)$. and it has no non-trivial nilpotent element.

Lemma 2.4. $\mathbb{S}$ is also a symmetric ring.

Proof: Let

$$\text{A} = \begin{bmatrix} a_1 + kb_1 & c_1 + kd_1 \\ 0 & a_1 + kb_1 \end{bmatrix}, B = \begin{bmatrix} a_2 + kb_2 & c_2 + kd_2 \\ 0 & a_2 + kb_2 \end{bmatrix}, C = \begin{bmatrix} a_3 + kb_3 & c_3 + kd_3 \\ 0 & a_3 + kb_3 \end{bmatrix} \in \mathbb{S}.$$

Let $ABC = \mathbb{O}$

$$\Rightarrow \begin{bmatrix} \mathrm{a}_1 + kb_1 & \mathrm{c}_1 + kd_1 \\ 0 & \mathrm{a}_1 + kb_1 \end{bmatrix} \begin{bmatrix} \mathrm{a}_2 + kb_2 & \mathrm{c}_2 + kd_2 \\ 0 & \mathrm{a}_2 + kb_2 \end{bmatrix} \begin{bmatrix} \mathrm{a}_3 + kb_3 & \mathrm{c}_3 + kd_3 \\ 0 & \mathrm{a}_3 + kb_3 \end{bmatrix} = \mathbb{O}$$

$$\Rightarrow \begin{bmatrix} (a_1a_2 + b_1b_2) + k(a_1b_2 + b_1a_2) & (a_1c_2 + b_1d_2 + c_1a_2 + d_1b_2) + \\ 0 & k(a_1d_2 + b_1c_2 + c_1b_2 + d_1a_2) \\ 0 & (a_1a_2 + b_1b_2) + k(a_1b_2 + b_1a_2) \end{bmatrix} \begin{bmatrix} a_3 + kb_3 & c_3 + kd_3 \\ 0 & a_3 + kb_3 \end{bmatrix} = \mathbb{O}$$

$$\Rightarrow \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ 0 & \mathrm{A}_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where,

$$A_{11} = (a_1a_2a_3 + b_1b_2a_3 + a_1b_2b_3 + b_1a_2b_3) + k(a_1b_2a_3 + b_1a_2a_3 + a_1a_2b_3 + b_1b_2b_3) = 0$$

$$A_{12} = (a_1a_2c_3 + b_1b_2c_3 + a_1b_2d_3 + b_1a_2d_3 + a_1c_2a_3 + b_1d_2a_3 + c_1a_2a_3 + d_1b_2a_3 + a_1d_2b_3 + b_1c_2b_3 + c_1b_2b_3 + d_1a_2b_3) + (a_1a_2d_3 + b_1b_2d_3 + a_1b_2c_3 + b_1a_2c_3 + a_1c_2b_3 + b_1d_2b_3 + c_1a_2b_3 + d_1b_2b_3 + a_1d_2a_3 + b_1c_2a_3 + c_1b_2a_3 + d_1a_2a_3) = 0.$$

Then,

$$ACB = \begin{bmatrix} \mathrm{a}_1 + kb_1 & \mathrm{c}_1 + kd_1 \\ 0 & \mathrm{a}_1 + kb_1 \end{bmatrix} \begin{bmatrix} \mathrm{a}_3 + kb_3 & \mathrm{c}_3 + kd_3 \\ 0 & \mathrm{a}_3 + kb_3 \end{bmatrix} \begin{bmatrix} \mathrm{a}_2 + kb_2 & \mathrm{c}_2 + kd_2 \\ 0 & \mathrm{a}_2 + kb_2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{a}_1 + kb_1 & \mathrm{c}_1 + kd_1 \\ 0 & \mathrm{a}_1 + kb_1 \end{bmatrix} \begin{bmatrix} \mathrm{a}_3 + kb_3 & \mathrm{c}_3 + kd_3 \\ 0 & \mathrm{a}_3 + kb_3 \end{bmatrix} \begin{bmatrix} \mathrm{a}_2 + kb_2 & \mathrm{c}_2 + kd_2 \\ 0 & \mathrm{a}_2 + kb_2 \end{bmatrix}$$

$$= \begin{bmatrix} (a_1a_3 + b_1b_3 + k(a_1b_3 + b_1a_3) & (a_1c_3 + b_1d_3 + c_1a_3 + d_1b_3) + k(a_1d_3 + b_1c_3 + c_1b_3 + a_3d_1) \\ 0 & (a_1a_3 + b_1b_3 + k(a_1b_3 + b_1a_3) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{a}_2 + kb_2 & c_2 + kd_2 \\ 0 & \mathrm{a}_2 + kb_2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ 0 & \mathrm{A}_{11} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, $\mathbb{S}$ is also a symmetric ring.

Definition 2.5. Quotient Space: $(B_{\mathbb{C}_2}, +, .)$ is a ring and $I_1 = \{\mu_1 e_1 : z = \mu_1 e_1 + \mu_2 e_2 \in B_{\mathbb{C}_2}\}$ is an ideal generated by e_1. Then the quotient ring of $B_{\mathbb{C}_2}$ by I_1 is defined by

$$\frac{B_{\mathbb{C}_2}}{I_1} = \{z + I_1 : z \in B_{\mathbb{C}_2}\}$$
$$= \{\mu_1 e_1 + \mu_2 e_2 + I_1 : z = \mu_1 e_1 + \mu_2 e_2 \in B_{\mathbb{C}_2}\} = \{\mu_2 e_2 : \mu_2 \in C(i_1)\}.$$

Proposition 2.6. $\frac{Bc_2}{I_1}$ is a reduced ring.
Proof. Let $(\mu_2 e_2 + I_1)^n = I_1$
$\Longrightarrow (\mu_2 e_2)^n + I_1 = I_1 \Longrightarrow \mu_2^n e_2^n + I_1 = I_1 \Longrightarrow \mu_2^n e_2^n \in I_1 \Longrightarrow \mu_2^n e_2^n = \mu_1 e_1$
$\Longrightarrow \mu_2^n e_2 - \mu_1 e_1 = 0$
$\Longrightarrow \mu_2^n = 0$ and $\mu_1 = 0$
$\Longrightarrow \mu_2 = 0$
$\Longrightarrow (\mu_2 e_2 + I_1) = I_1$.
i.e., $\frac{BC_2}{I_1}$ has no non-trivial nilpotent element.
$\mathbb{D} = \{x + i_1 i_2 y : x, y \in B_{\mathbb{C}_2}\}$ is a subring of $B_{\mathbb{C}_2}$.

We define the involution on $\mathbb{D}$ by
$z^* = \tilde{z}$, for all $z \in \mathbb{D}$, where $\bar{z}$ is the conjugate with respect to ij.

Example 2.7. The ring $\mathbb{D}$ with the involution $*$ forms $a *$ - symmetric ring.
Proof: Let, $z_1 = x_1' + i_1\, i_2\, x_1''$, $z_2 = x_2' + i_1\, i_2 x_2''$ and $z_3 = x_3' + i_1 i_2 x_3'' \in \mathbb{D}$.
If, $z_1\, z_2\, z_3 = 0$
$\Rightarrow (x_1{}' + i_1 i_2 x_1{}'')(x_2{}' + i_1 i_2 x_2{}'')(x_3{}' + i_1 i_2 x_3'') = 0$
$\Rightarrow (x_1{}'x_2{}' + x_1''x_2'') + i_1 i_2 (x_1'x_2'' + x_1''x_2')(x_3{}' + i_1 i_2 x_3'') = 0$
$\Rightarrow (x_1{}'x_2{}'x_3{}' + x_1''x_2''x_3' + x_1'x_2''x_3'' + x_1''x_2'x_3'') + i_1 i_2 (x_1'x_2{}'x_3'' + x_1{}''x_2''x_3{}'' + x_1'x_2''x_3' + x_1''x_2'x_3') = 0$

$\Rightarrow x_1{}'x_2{}'x_3{}' + x_1''x_2''x_3{}' + x_1{}'x_2''x_3'' + x_1''x_2{}'x_3'' = 0$ and $x_1{}'x_2{}'x_3{}'' + x_1''x_2''x_3'' + x_1{}'x_2''x_3{}' + x_1''x_2{}'x_3{}' = 0$.
Then $z_1 z_3 z_2{}^*$
$= (x_1' + i_1 i_2 x_1{}'')(x_3{}' + i_1 i_2 x_3'')(x_2' + i_1 i_2 x_2{}'')^*$
$= (x_1' + i_1 i_2 x_1'')(x_3' + i_1 i_2 x_3'')(x_2' + i_1 i_2 x_2'')$
$= \{(x_1'x_3' + x_1''x_3'') + i_1 i_2 (x_1'x_3'' + x_1''x_3')\}(x_2' + i_1 i_2 x_2'')$
$= x_1'x_2'x_3' + x_1''x_2''x_3' + x_1'x_2''x_3'' + x_1''x_2'x_3''$ and $x_1'x_2'x_3'' + x_1''x_2''x_3'' + x_1'x_2''x_3' + x_1''x_2'x_3'$
$= 0$.
So, $\mathbb{D}$ is a *-symmetric ring under the involution *.

Example 2.8. $(B_{\mathbb{C}_2}, +, .)$ is not $*$ reversible ring with respect to $i_1 i_2$-conjugation but $(\mathbb{D}, +, .)$ forms a $*$ reversible ring with respect to $i_1 i_2$ - conjugation.

Proof. Let $z = x_1 + i_1 i_2 x_2$ and $w = w_1 + i_1 i_2 w_2 \in \mathbb{D}$.
If $zw = 0 \Longrightarrow (x_1 + i_1 i_2 x_2)(w_1 + i_1 i_2 w_2) = 0$
$\Longrightarrow (x_1 w_1 + x_2 w_2) + i_1 i_2 (x_1 w_2 + x_2 w_1) = 0$

$(x_1w_1 + x_2w_2) = 0$ and $x_1w_2 + x_2w_1 = 0$.
Then
$wz^* = (w_1 + i_1i_2w_2)(x_1 + i_1i_2x_2)^* = (w_1 + i_1i_2w_2)(x_1 + i_1i_2x_2) = (w_1x_1 + w_2x_2) + i_1i_2(w_1x_2 + x_1w_2) = 0$.

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