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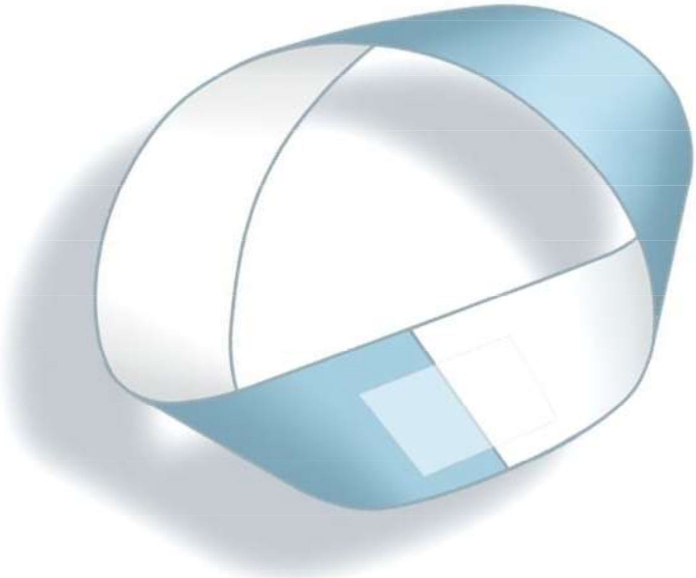

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Email :information.tms@gmail.com

Editor - in - charge
Prof. Binod Chandra Tripathy
Dept. of Mathematics, Tripura University
Suryamaninagar-799022, Agartala, India
Email: tripathybe@yahoo.com, tripathybc@gmail.com

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Interval-Valued Neutrosophic Hesitant Fuzzy Rough Approximation Operators and Its Application in Medical Science

Anjan Mukherjee[1] and Rakhal Das[2]

[1]Department of Mathematics, Tripura University Agartala -799022; Tripura, INDIA;
e-mail: [1]mukherjee123anjan@gmail.com
[2]Department of Mathematics, The ICFAI University Triura, Agartala -799210; Tripura, INDIA;
e-mail [2] rakhaldas95@gmail.com

*Correspondence:Anjan Mukherjee, mukherjee123anjan@gmail.com

***Abstract*:**In this paper, we propose the notion of interval- valued neutrosophic hesitant fuzzy rough set, by combining internal- valued neutrosophic hesitant fuzzy set and rough set. The combination of interval- valued neutrosophic hesitant fuzzy set and rough set is a powerful tool for dealing with uncertainty, granularity and incompleteness of knowledge in information systems. We present definitions and some basic properties of the proposed model.Finally, we give a general approach which is applied to a decision making problem in disease diagnoses, and demonstrate the effectiveness of the approach by a numerical example.

***Keywords*:** Interval- valued neutrosophic hesitant set, internal-valued neutrosophic hesitant relation, interval-valued neutrosophic hesitant fuzzy rough set, decision making method.

I. Introduction

The notion of rough set theory has been proposed by Pawlak in 1982 [6] and the theory of fuzzy set proposed by Zadeh in 1965 [15]. They are generalizations of the classical set theory. Rough set theory is a mathematical approach concerning uncertainty that comes from noisy, inexact or incomplete informations. In Zadeh's fuzzy set theory, the membership function play the important role, whereas in Pawlak's rough set theory, equivalence classes of a set are the significant part for the upper and lower approximations of the set. As a generalization of fuzzy sets and intuitionistic fuzzy set [1], the concept of neutrosophic set (Nset) was introduced by Smarandache [7] in 1999. The concept of neutrosophic set handles indeterminate data whereas fuzzy set theory and intuitionistic fuzzy set theory failed when the relation is indeterminate. Neutrosophic set is described by three functions: true-membership function, indeterminacy-membership function and falsity-membership function that are connected independently. The neutrosophic set theory has been very successful in several areas, such as medical diagnosis, database, topology and decision making problem [4], [14]. While the neutrosophic set is an important tool for handing the indeterminate and inconsistent data, the theory of rough set is a powerful mathematics tool to deal with incompleteness.

Without a particular description, it is hard to use the Nset in real scientific and different domain. Wang et al. [10] proposed single valued neutrosophic set (SVNsets) by simplifying Nsets. SVNsets can also be considered as an extension of intuitionistic fuzzy sets, in which the three membership functions are unrelated and their function values belong to the unit closed interval. As another generalization of fuzzy sets, the hesitant fuzzy set (HF) was defined by Torra [8], which allows its membership function to have a set of possible values. Hesitant fuzzy set is also important concept used to deal with imperfect information [12]. By combining the advantages of the SVNset and HFset, Ye [14] introduced the notion of single valued neutrosophic hesitant fuzzy set (SVNHFset) which allows its membership function to have sets of possible values, which are called truth, indeterminacy, and falsity membership hesitant functions and discussed some properties of SVNHFset to solve multiple attribute decision making problems. In addition, many researchers have studied hesitant fuzzy decision making problems by utilizing plenty of classical decision making tools. Among them, since the rough set approach owns advantages inattribute selection, we aim to deal with the situation by virtue of the rough set theory.
In this paper, we apply rough set model to decision making involving interval-valued neutrosophic hesitant fuzzy sets. Moreover, we also propose an illustrative example to interpret the basic principal and method of the application of the rough set model in interval-valued neutrosophic hesitant fuzzy decision making.

Section 2, recalls some basic concepts of rough sets, interval-valued neutrosophic hesitant fuzzy sets. In Section 3, we present rough set model based on IVNHF relation over two universes and examine some properties of this model. In Section 4, we establish a general approach to decision making based on IVNHF rough set over two universes. Section 5, illustrates the principal steps of the proposed decision method by a numerical example. Finally, in Section 6, we conclude the paper with a summary and outlook for further research.

2. PRELIMINARIES

In this section we recall some basic notions and properties which will be used in this paper.

Definition 2.1 [6] Let U be a non-empty finite universe. R be an equivalence relation on U. We use U/R to denote the family of all equivalence classes of R and $[x]_R$ to denote an equivalence classes of R containing the element $x \in U$. The pair (U, R) is called an approximation space. For any $X \subseteq U$ define the lower & upper approximation of X as

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$$
$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$$

The pair $(\underline{R}(X), \overline{R}(X))$ is refer to as the rough set of X.

Definition 2.2. [8] Let X be a fixed set, a hesitate fuzzy set (HFset) A on X is in determines of function $h_A(x)$ that when applied to X returns a subset of [0, 1] i.e,

$A=\{<x, h_A(x)>: x \in X\}$.

Where $h_A(x)$ is a set of some different values in [0, 1], representing possible membership degrees of the element $x \in X$ to A. $h_A(x)$ is a hesitant fuzzy element.

Definition 2.3. [11] Let X be a fixed set and let int[0, 1] be the set of all closed subset of [0,1]. A interval-valued hesitant fuzzy set.(IVHF set) A on X is defined by

$A=\{<x, h_A(x)>: x \in X\}$ where $h_A(x): X \to \text{int}[0, 1]$ denotes all possible interval-valued membership degree of the elements $x \in X$ to A. $h_A(x)$ is an interval valued hesitant fuzzy element.

Definition 2.4. [9] Let X be a space of points (objects). An interval valued neutrosophic set (IVN set). A in X is characterized by a truth membership $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. For each point $x \in X$ we have

$T_A(x) = [T_A^-(x), T_A^+(x)]$, $I_A(x) = [I_A^-(x), I_A^+(x)]$, $F_A(x) = [F_A^-(x), F_A^+(x)] \subseteq [0, 1]$&

$0 \leq T_A^+(x) + I_A^+(x) + F_A^+(x) \leq 3, x \in X$

Definition 2.5. [4] Let X be a non-empty fixed set, a single valued neutrosophic hesitant fuzzy set (SVNHF set) on X is defined as

$A = \{<x, \hat{T}(x), \hat{I}(x), \hat{F}(x)>: x \in X\}$

Where $\hat{T}(x) = \{\alpha : \alpha \in \overline{T}(x)\}$, $\hat{I}(x) = \{\beta : \beta \in \overline{I}(x)\}$, $\hat{F}(x) = \{\gamma : \gamma \in \overline{F}(x)\}$

Are three sets with value in [0, 1], representing truth, indeterminacy and falsity membership hesitant degree of the element $x \in X$with $\alpha \in [0, 1]$, $\beta \in [0, 1]$ & $\gamma \in [0, 1]$ and

$0 \leq sup\ \alpha + sup\ \beta + sup\ \gamma \leq 3$.

Definition 2.6. [12, 13] Let U be a non-empty and finite universe. A Hesitant fuzzy relation R over U is a hesitant fuzzy subset that R∈ HF(U×U) where R = $\{(x, y), h_R(x,y): (x, y) \in U \times U$

$h_R(x,y)$ is a set of values in [0, 1]. It is denoted the possible membership degrees is the relationships between x and y.

Definition 2.7. [14] Let U and V be two non-empty finite universe. A single valued neutrosophic hesitant fuzzy set (SVNHF set) sub set R of the universe U×V is called a SVNHF relation from U to V, namely R is given by

$$R = \{<(x, y), \hat{T}_R(x, y), \hat{I}_R(x, y), \hat{F}_R(x, y)>: (x,y) \in U \times V\}$$

Where, $\hat{T}_R$, $\hat{I}_R$, $\hat{F}_R$: $U \times V \rightarrow [0,1]$ are triple sets of some values in [0, 1]. It denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant degrees of the relationship between x and y with the condition $0 \leq \alpha, \beta, \gamma \geq 1$ and $0 \leq \alpha^+ + \beta^+ + \gamma^+ \leq 3$ where for all $(x,y) \in U \times V$ where for all $(x,y) \in U \times V$, $\alpha \in \hat{T}_R(x, y)$, $\beta \in \hat{I}_R(x, y)$, $\gamma \in \hat{F}_R(x, y)$, $\alpha^+ \in T_R^+(x, y) = \cup_{\alpha \in \hat{T}_R(x,y)} \max\{\alpha\}$, $\beta^+ \in I_R^+(x, y) = \cup_{\beta \in \hat{I}_R(x,y)} \max\{\beta\}$, $\gamma^+ \in F_R^+(x, y) = \cup_{\gamma \in \hat{F}_R(x,y)} \max\{\gamma\}$

3. Interval-Valued Neutrosophic Hesitant Fuzzy Rough Sets

Definition 3.1. Let U and V be two non-empty and finite universe and P be an interval-valued neutrosophic hesitant fuzzy (IVNHF) relation from U to V. The triple (U, V, R) is called IVNHF approximation space. For any A∈ IVNHF(V) the lower and upper approximation of A with respect to (U, V, R) denoted by $\underline{R}(A) \& \overline{R}(A)$ are two IVNHF sets of U defined as

$$\overline{R} = \{<x, \hat{T}_{\overline{R}}(x), \hat{I}_{\overline{R}}(x), \hat{F}_{\overline{R}}(x)>: x \in X\}$$

$$\underline{R} = \{<x, \hat{T}_{\underline{R}}(x), \hat{I}_{\underline{R}}(x), \hat{F}_{\underline{R}}(x)>: x \in X\}$$

$$\hat{T}_{\underline{R}(A)}(x) = \wedge_{y \in V}\{\hat{F}_R(x,y) \vee \hat{T}_A(y)\}$$

$$\hat{F}_R(x,y) = [\hat{F}_R^-(x,y), \hat{F}_R^+(x,y)]$$

$$\hat{F}_R^-(x,y) = \inf \hat{F}_R(x,y),\ \hat{F}_R^+(x,y) = \sup \hat{F}_R(x,y)$$

$$\hat{T}_A(y) = \left[\hat{T}_A^-(y), \hat{T}_A^+(y)\right] = [\inf \hat{T}_A(y), \sup \hat{T}_A(y)]$$

$$\hat{I}_{\underline{R}(A)}(x) = \vee_{y \in V}\{\hat{I}_R^c(x,y) \wedge \hat{I}_A(y)\} = \vee_{y \in V}\{1 - \hat{I}_R(x,y) \wedge \hat{I}_A(y)\}$$

$$\hat{I}_R(x,y) = [\hat{I}_R^-(x,y), \hat{I}_R^+(x,y)] = [\inf \hat{I}_R(x,y), \sup \hat{I}_R(x,y)]$$

$$\hat{I}_A(\mathrm{y}) = \left[\hat{I}_A^-(\mathrm{y}), \hat{I}_A^+(\mathrm{y})\right] = [\mathrm{inf}\hat{I}_A(\mathrm{y}), \mathrm{sup}\hat{I}_A(\mathrm{y})]$$

$\hat{F}_{\underline{R}(A)}(x) = \vee_{y\in V}\{\hat{T}_R(x,y) \wedge \hat{F}_A(\mathrm{y})\}$

$\hat{T}_R(x,y) = [\hat{T}_R^-(x,y), \hat{T}_R^+(x,y)] = [\inf \hat{T}_R(x,y), \sup \hat{T}_R(x,y)]$

$$\hat{F}_A(\mathrm{y}) = \left[\hat{F}_A^-(\mathrm{y}), \hat{F}_A^+(\mathrm{y})\right] = [\mathrm{inf}\hat{F}_A(\mathrm{y}), \mathrm{sup}\hat{F}_A(\mathrm{y})]$$

Now, $\hat{T}_{\overline{R}(A)}(x) = \vee_{y\in V}\{\hat{T}_R(x,y) \wedge \hat{T}_A(\mathrm{y})\}$

$\hat{I}_{\overline{R}(A)}(x) = \wedge_{y\in V}\{\hat{I}_R(x,y) \vee \hat{I}_A(\mathrm{y})\}$

$\hat{F}_{\overline{R}(A)}(x) = \wedge_{y\in V}\{\hat{F}_R(x,y) \vee \hat{F}_A(\mathrm{y})\}$

The pair ($\underline{R}(A)$, $\overline{R}(A)$) is called the IVNHF rough set of A with respect to (U, V, R) and $\underline{R}(A)$, $\overline{R}(A)$:IVNHF(V)→IVNHF(U) are referred to as lower and upper IVNHF rough approximation operators.

Definition 3.2. The IVNHF relation R from U to V is said to be serial if for each $x \in U$, There exists a $y \in V$ such that $\hat{T}_R(x,y) = [1,1]$ and $\hat{I}_R(x,y) = \hat{F}_R(x,y) = [0,0]$.

R is said to be reflexive on U if $\hat{T}_R(x,y) = [1,1]$ and $\hat{I}_R(x,y) = \hat{F}_R(x,y) = [0,0]$ for all $x \in V$. R is said to be symmetric IVNHF relation on V if $\hat{T}_R(x,y) = \hat{T}_R(y,x)$, $\hat{I}_R(x,y) = \hat{I}_R(y,x)and\hat{F}_R(x,y) = \hat{F}_R(y,x)$ for all $x, y \in V$. R is said to be transitive IVNHF relation on U if $\vee_{y\in V}\{\hat{T}_R(x,y) \wedge \hat{T}_R(y,z)\} \leq \hat{T}_R(x,z)$, $\wedge_{y\in V}\{\hat{I}_R(x,y) \vee \hat{I}_R(y,z)\} \leq \hat{I}_R(x,z)$, and$\wedge_{y\in V}\{\hat{F}_R(x,y) \vee \hat{F}_R(y,z)\} \leq \hat{F}_R(x,z)$ for all $x, z \in U$.

Let $l(\hat{T}_A(\mathrm{x}))$, $l(\hat{I}_A(\mathrm{x}))$, $l(\hat{F}_A(\mathrm{x}))$ stands for the number of intervals in $\hat{T}_A(\mathrm{x})$, $\hat{I}_A(\mathrm{x})$ and $\hat{F}_A(\mathrm{x})$ respectively.

Then alternatively R is transitive IVNHF relation if, $\vee_{y\in V}\{\hat{T}_R^{\delta(s)}(x,y) \wedge \hat{T}_R^{\delta(s)}(y,z)\} \leq \hat{T}_R^{\delta(s)}(x,z) 1 \leq s \leq k$, $\wedge_{y\in V}\{\hat{I}_R^{\delta(t)}(x,y) \vee \hat{I}_R^{\delta(t)}(y,z)\} \leq \hat{I}_R^{\delta(t)}(x,z), 1 \leq t \leq mand\wedge_{y\in V}\{\hat{F}_R^{\delta(p)}(x,y) \vee \hat{F}_R^{\delta(p)}(y,z)\} \leq \hat{F}_R^{\delta(p)}(x,z), 1 \leq P \leq n$

Where $\hat{T}_R^{\delta(s)}$denotes the s largest interval in $\hat{T}$, $\hat{I}_R^{\delta(t)}$denotes the t largest interval in $\hat{I}$ and $\hat{F}_R^{\delta(p)}$denotes the p largest interval in $\hat{F}$.

$K = \max\{l\left(\hat{T}_R(x,y)\right), l\left(\hat{T}_R(y,z)\right), l\left(\hat{T}_R(x,z)\right)\}$

$m = \max\{l\left(\hat{I}_R(x,y)\right), l\left(\hat{I}_R(y,z)\right), l\left(\hat{I}_R(x,z)\right)\}$

$p = \max\{l\left(\hat{F}_R(x,y)\right), l\left(\hat{F}_R(y,z)\right), l\left(\hat{F}_R(x,z)\right)\}$

Definition 3.3. From the definition 3.2 we can write that the definition 3.1 implies the following

$\hat{T}_{\underline{R}(A)}(x) = [\wedge_{y\in V}\{\,(\hat{F}_R^{\delta(s)}(x,y)) \vee \hat{T}_A^{\delta(s)}(y)\}/1 \le s \le \max\{l\left(\hat{F}_R(x,y)\right), l(\hat{T}_A(\mathrm{y}))\}]$

$\hat{I}_{\underline{R}(A)}(x) = [\vee_{y\in V}\{[1,1] - \hat{I}_R^{\delta(t)}(x,y) \wedge \hat{I}_A^{\delta(t)}(y)\}/1 \le t \le \max\{l\left(1 - \hat{I}_R(x,y)\right), l(\hat{I}_A(\mathrm{y}))\}]$

$\hat{F}_{\underline{R}(A)}(x) = [\vee_{y\in V}\{\hat{T}_R^{\delta(p)}(x,y) \wedge \hat{F}_A^{\delta(p)}(y)\}/1 \le p \le \max\{l\left(\hat{T}_R(x,y)\right), l(\hat{F}_A(\mathrm{y}))\}]$

$\hat{T}_{\overline{R}(A)}(x) = [\vee_{y\in V}\{\hat{T}_R^{\delta(s)}(x,y) \wedge \hat{T}_A^{\delta(s)}(y)/1 \le s \le \max\{l\left(\hat{T}_R(x,y)\right), l(\hat{I}_A(\mathrm{y}))\}]$

$\hat{I}_{\overline{R}(A)}(x) \;= [\wedge_{y\in V}\{\,(\hat{I}_R^{\delta(t)}(x,y)) \vee \hat{I}_A^{\delta(t)}(y)\}/1 \le t \le \max\{l\left(\hat{I}_R(x,y)\right), l(\hat{I}_A(\mathrm{y}))\}]$

$\hat{F}_{\overline{R}(A)}(x) \;= [\wedge_{y\in V}\{\,(\hat{F}_R^{\delta(p)}(x,y)) \vee \hat{F}_A^{\delta(p)}(y)\}/1 \le p \le \max\{l\left(\hat{F}_R(x,y)\right), l(\hat{F}_A(\mathrm{y}))\}]$.

Where l stands for number of intervals in hesitant fuzzy elements

Definition 3.3. Let U be a non-empty and finite universe of discourse. Denote

$K = \max\{l\left(\hat{T}_A(x)\right), l\left(\hat{T}_B(x)\right)\}$

$m = \max\{l\left(\hat{I}_A(x)\right), l\left(\hat{I}_B(x)\right)\}$

$p = \max\left\{l\left(\hat{F}_A(x)\right), l\left(\hat{F}_B(x)\right)\right\}, \forall\, A, B \in$IVNHF(V), A is said to be a IVNHF subset of B, if $\hat{T}_A(y) \subseteq \hat{T}_B(y)$, $\hat{I}_A(y) \supseteq \hat{I}_B(y) \& \hat{F}_A(y) \supseteq \hat{F}_B(y)$.

$\Rightarrow \hat{T}_A^{\delta(s)}(y) \subseteq \hat{T}_B^{\delta(s)}(y)$, $\hat{I}_A^{\delta(t)}(y) \supseteq \hat{I}_B^{\delta(t)}(y) \& \hat{F}_A^{\delta(p)}(y) \supseteq \hat{F}_B^{\delta(p)}(y)$ with $1 \le s \le k, 1 \le t \le m$and $1 \le p \le n$, then $A \subseteq B$.

Theorem 1. Let (U, V, R) be a IVNHF approximation space over two universes. Then the lower and upper IVNHF rough approximation operators satisfy the following properties for all A, B∈IVNHF(V)

1) $\underline{R}(A^c) = (\overline{R}(A)^c$
2) $\overline{R}\left(\overline{A}^c\right) = (\underline{R}(A)^c$
3) If $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$
4) If $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$
5) $\underline{R}(A \cap B)$= $\underline{R}(A) \cap \underline{R}(B)$
6) $\overline{R}(A \cup B)$=$\overline{R}(A) \cup \overline{R}(B)$
7) $\underline{R}(U)$= U
8) $\overline{R}(\emptyset) = \emptyset$

Proof: 1) For all $x \in V$, we have $\hat{T}_{\underline{R}(A)}(x)$ = $[\wedge_{y\in V}\{\hat{F}_R(x,y)) \vee \hat{T}_A(y)\}$

Therefore, $\hat{T}_{\underline{R}(A^c)}(x)$ = $[\wedge_{y\in V}\{\hat{F}_R(x,y)) \vee \hat{T}_{A^c}(y)\}$

= $[\wedge_{y\in V}\{\hat{F}_R(x,y)) \vee ([1,1] - \hat{T}_A(y))\}$

= $[\wedge_{y\in V}\{\hat{F}_R(x,y)) \vee ([1,1] - \hat{T}_A(y))\}$

= $[\wedge_{y\in V}\{\hat{F}_R(x,y)) \vee ([(1 - \hat{T}_A^{+}(y)), (1 - \hat{T}_A^{-}(y))]\}$

$\hat{T}_A(y)$=$[\hat{T}_A^{-}(y),\ \ \hat{T}_A^{+}(y)]$&$[1,1] - \hat{T}_A(y) = [(1 - \hat{T}_A^{+}(y)), (1 - \hat{T}_A^{-}(y))]$=$[\hat{F}_A^{-}(y),\ \ \hat{F}_A^{+}(y)]$ = $\hat{F}_A(y)$

($\hat{T}_A^{+}(y)$=$1 - \hat{F}_A^{-}(y)$)&($\hat{F}_A^{+}(y)$=$1 - \hat{T}_A^{-}(y)$)

= $[\wedge_{y\in V}\{\hat{F}_R(x,y)) \vee \hat{F}_A(y)\}$

= $\wedge_{y\in V}\{(\hat{F}_R^{\delta(p)}(x,y)) \vee \hat{F}_A^{\delta(p)}(y)\}$ = $\hat{F}_{\overline{R}(A)}(x)$=$\hat{T}_{\overline{R}(A^c)}(x)$

So, $\underline{R}(A^c) = (\overline{R}(A)^c$

Similarly, we have $\overline{R}(A^c) = (\underline{R}(A)^c$.

3) Since $A \subseteq B$ we have the following $\hat{T}_A^{\delta(s)}(y) \subseteq \hat{T}_B^{\delta(s)}(y)$, $\hat{I}_A^{\delta(t)}(y) \supseteq \hat{I}_B^{\delta(t)}(y) \& \hat{F}_A^{\delta(p)}(y) \supseteq \hat{F}_B^{\delta(p)}(y)$ with $1 \leq s \leq k, 1 \leq t \leq m$ and $1 \leq p \leq n$ for all $y \in V$.

It follows that $\wedge_{y \in V}\{\,(\hat{F}_R^{\delta(s)}(x,y)) \vee \hat{T}_A^{\delta(s)}(y)\} \subseteq \wedge_{y \in V}\{\,(\hat{F}_R^{\delta(s)}(x,y)) \vee \hat{T}_B^{\delta(s)}(y)\}$,

$\vee_{y \in V}\{[1,1] - \hat{I}_R^{\delta(t)}(x,y) \wedge \hat{I}_A^{\delta(t)}(y)\} \supseteq \vee_{y \in V}\{[1,1] - \hat{I}_R^{\delta(t)}(x,y) \wedge \hat{I}_B^{\delta(t)}(y)\} \&$

$$\bigvee_{y \in V}\{\hat{T}_R^{\delta(p)}(x,y) \wedge \hat{F}_A^{\delta(p)}(y)\} \supseteq \bigvee_{y \in V}\{\hat{T}_R^{\delta(p)}(x,y) \wedge \hat{F}_B^{\delta(p)}(y)\}$$

Hence, for each x$\in V$ we have $\hat{T}_{\underline{R}(A)}(x) \subseteq \hat{T}_{\underline{R}(B)}(x)$, $\hat{I}_{\underline{R}(A)}(x) \supseteq \hat{I}_{\underline{R}(B)}(x)$ and $\hat{F}_{\underline{R}(A)}(x) \supseteq \hat{F}_{\underline{R}(B)}(x)$

Hence, $\underline{R}(A) \subseteq \underline{R}(B)$.

4) Similarly, if $A \subseteq B$ then $\overline{R}(A) \subseteq \overline{R}(B)$

5) For all $x \in V$ we have,

$\hat{T}_{\underline{R}(A \cap B)}(x) = [\wedge_{y \in V}\{\,\hat{F}_R(x,y)) \vee \hat{T}_{A \cap B}(y)\} = \wedge_{y \in V}\{\,(\hat{F}_R^{\delta(s)}(x,y)) \vee (\hat{T}_A^{\delta(s)}(y) \wedge \hat{T}_B^{\delta(s)}(y)\}$,

where s=1, 2, …., k

$(\wedge_{y \in V}[\{\,(\hat{F}_R^{\delta(s)}(x,y)) \vee \hat{T}_A^{\delta(s)}(y)\}]) \wedge (\wedge_{y \in V}[\{\,(\hat{F}_R^{\delta(s)}(x,y)) \vee \hat{T}_B^{\delta(s)}(y)\}]$s=1, 2, …., k

$= \hat{T}_{\underline{R}(A)}(x) \wedge \hat{T}_{\underline{R}(B)}(x)$.

Similarly, $\hat{I}_{\underline{R}(A \cap B)}(x) = \hat{I}_{\underline{R}(A)}(x) \wedge \hat{I}_{\underline{R}(B)}(x) \& \hat{F}_{\underline{R}(A \cap B)}(x) = \hat{F}_{\underline{R}(A)}(x) \wedge \hat{F}_{\underline{R}(B)}(x)$.

Thus, $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$.

(7) & (8) straight forward.

Definition 3.4. Let N = $\{\hat{T}, \hat{I}, \hat{F}\}$ be a IVNHF set then the score function can be defined as

S(N) = $\frac{1}{3}\{\frac{1}{l_{\hat{T}}}\sum\alpha + \frac{1}{l_{\hat{I}}}\sum(1-\beta) + \frac{1}{l_{\hat{F}}}\sum(1-\gamma)\}$, where $l_{\hat{T}}$, $l_{\hat{I}}$, $l_{\hat{F}}$ are the numbers of intervals on $\hat{T}$, $\hat{I}$ and $\hat{F}$ respectively in *N*.

Definition 3.5. Let $\underline{R}$ and $\overline{R}$ be two IVNHF sets in U, we defined the sum of $\underline{R}$ and $\overline{R}$ as

$\underline{R} \oplus \overline{R}$ = {<y_j, $\hat{T}_{\underline{R}}(y_j)$⊕ $\hat{T}_{\overline{R}}(y_j)$, $\hat{I}_{\underline{R}}(y_j)$ ⊕ $\hat{I}_{\overline{R}}(y_j)$, $\hat{F}_{\underline{R}}(y_j)$ ⊕ $\hat{F}_{\overline{R}}(y_j)$>: $y_j \in V$}

= {y_j, ($\hat{T}_{\underline{R}}(y_j)$+$\hat{T}_{\overline{R}}(y_j) - \hat{T}_{\underline{R}}(y_j).\hat{T}_{\overline{R}}(y_j)$), ($\hat{I}_{\underline{R}}(y_j)\,.\,\hat{I}_{\overline{R}}(y_j)$), ($\hat{F}_{\underline{R}}(y_j)\,.\,\hat{F}_{\overline{R}}(y_j)$)}

4. Application in Medical Science

In this section we will apply IVNHF rough set model on two universes to medical diagnosis problems U={x_1, x_2, ...,x_m} be a disease set. Let R∈IVNHFR(U×V) be an IVNHF relation from U to V.

For any (x_j, y_i)∈ U × V, $\hat{T}_R$(x_j, y_i) represents the true membership degree of relationship between symptom x_j($x_j \in U$) and the disease y_i($y_i \in V$). $\hat{I}_R$(x_j, y_i) represents the indeterminacy membership degree of relationship between symptom x_j($x_j \in U$) and the disease y_i($y_i \in V$). $\hat{F}_R$(x_j, y_i) represents the falsity membership degree of relationship between symptom x_j($x_j \in U$) and the disease y_i($y_i \in V$), which are evaluated by several doctors in advance. In clinical practice a patient can visit different doctors and may get different diagnosis to decrease the risk of miss diagnosis, we should carefully consider all doctors comments. In that case for any patient set A who has some symptoms in the universe U i.e. A = $\{< x, T_A(x_i), I_A(x_i), F_A(x_i) >: x \in U\}$.

Definition 4.1. IVNHF set is an effective tool to process the uncertain, inconsistent and hesitant information. Assume X is a finite set which contains at least one element. Then an IVNHF set A on X is described as

A = $\{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$,

where $T_A(x) = \{\alpha: \alpha \in T_A(x)\}$, $\alpha = [\alpha^-, \alpha^+] \subseteq [0, 1]$.

$I_A(x) = \{\beta: \beta \in I_A(x)\}$, $\beta = [\beta^-, \beta^+] \subseteq [0, 1]$.

$F_A(x) = \{\gamma: \gamma \in F_A(x)\}$, $\gamma = [\gamma^-, \gamma^+] \subseteq [0, 1]$.

$$0 \leq \sup \alpha + \sup \beta + \sup \gamma \leq 3$$

Definition 4.2. Let A = $\{[T_A^-(x), T_A^+(x)], [I_A^-(x), I_A^+(x)], [F_A^-(x), F_A^+(x)]\}$ be a IVNHF set then the compliment of A is denoted by A^c and is defined by

$$A^c = \bigcup_{\substack{\alpha_A \in T_A \\ \beta_A \in I_A \\ \gamma_A \in F_A}} \{[\gamma^-, \gamma^+], [1-\beta^+, 1-\beta^-], [\alpha^-, \alpha^+]\}$$

Definition 4.3. Let A = $\{[T_A^-(x), T_A^+(x)]$, $[I_A^-(x), I_A^+(x)]$, $[F_A^-(x), F_A^+(x)]\}$

B = $\{[T_B^-(x), T_B^+(x)]$, $[I_B^-(x), I_B^+(x)]$, $[F_B^-(x), F_B^+(x)]\}$ be two IVNHF sets

(a) Then the intersection of A and B is

$$A \cap B = \bigcup_{\substack{\alpha_A \in T_A \\ \beta_A \in I_A \\ \gamma_A \in F_A \\ \alpha_B \in T_B \\ \beta_B \in T_B \\ \gamma \in F_B}} \{[\wedge(\alpha_A^-, \alpha_B^-), \wedge(\alpha_A^+, \alpha_B^+)], [\vee(\beta_A^-, \beta_B^-), \vee(\alpha_A^+, \alpha_B^+)], [\vee(\gamma_A^-, \gamma_B^-), \vee(\gamma_A^+, \gamma_B^+)]$$

(b) The union $A \cup B$ of two IVNHF sets A and B is

$$A \cup B = \bigcup_{\substack{\alpha_A \in T_A \\ \beta_A \in I_A \\ \gamma_A \in F_A \\ \alpha_B \in T_B \\ \beta_B \in T_B \\ \gamma \in F_B}} \{[\vee(\alpha_A^-, \alpha_B^-), \vee(\alpha_A^+, \alpha_B^+)], [\wedge(\beta_A^-, \beta_B^-), \wedge(\alpha_A^+, \alpha_B^+)], [\wedge(\gamma_A^-, \gamma_B^-), \wedge(\gamma_A^+, \gamma_B^+)]$$

The number of values in different IVNHF intervals might be different i.e. $l_{T_A}(x_i) \neq l_{T_B}(x_i)$, $l_{I_A}(x_i) \neq l_{I_B}(x_i)$ and $l_{F_A}(x_i) \neq l_{F_B}(x_i)$

Let,

$$l_T(x_i) = \max\{l_{T_A}(x_i), l_{T_B}(x_i)\}$$

$l_I(x_i) = \max\{l_{I_A}(x_i), l_{I_B}(x_i)\}andl_F(x_i) = \max\{l_{F_A}(x_i), l_{F_B}(x_i)\}$ for each $x_i \in X$. We can make them have the same number of intervals through adding some to the IVNHF intervals which has less member of intervals. The section of this operation mainly depends on the decision maker's risk preferences. He/she may add the minimum of the truth membership intervals degree and maximum value of indeterminacy membership intervals degree and falsity membership intervals degree or add the maximum of the truth membership intervals degree and minimum value of indeterminacy membership intervals degree and falsity membership intervals degree.

According to the pessimistic-principal, if $l_{T_A}(x_i) < l_{T_B}(x_i)$ then least interval of $T_A(x_i)$ or $T_B(x_i)$ will be added to $T_A(x_i)$. If $l_{I_A}(x_i) < l_{I_B}(x_i)$, then largest interval of $I_A(x_i) or\ T_B(x_i)$ will be interval to $I_A(x_i)$ for $x_i \in X$. Similarly $l_{F_A}(x_i) < l_{F_B}(x_i)$ then the largest interval of $l_{F_A}(x_i)$ or $l_{F_B}(x_i)$ will be interval in $F_A(x_i)$ for $x_i \in X$.

Now the problem is that a decision maker needs to make reasonable decision about how to finding what kind of the disease y_i patient A is suffering from.

5. Now we present an approach to the decision making for this kind of problems by using IVNHF rough set theory over two universes.

Algorithm

1. By definition (3.1) we calculate the lower and upper approximations $\underline{R}(A)$and$\overline{R}(A)$ of A.
2. By definition (3.5) we calculate $\underline{R}(A)\oplus \overline{R}(A)$.
3. On the basis of definition (3.4) the score function of IVNHF interval where calculated denote λ_j =S($\underline{R}\oplus \overline{R}$)=S($T_{\underline{R}}(y_j)\oplus T_{\overline{R}}(y_j)$, $I_{\underline{R}}(y_j) \oplus I_{\overline{R}}(y_j)$, $F_{\underline{R}}(y_j) \oplus F_{\overline{R}}(y_j)$).
4. The optimal decision is to select $y_l$if $\lambda_l = max\lambda_j; j = 1, 2, 3, \dots, v$. We conclude that patient A is suffering from the disease y_l.

Table -1

Symptoms Characteristics for the considered diagnosis

R	y_1	y_2
x_1	{([0.6, 0.8], [0.2, 0.3], [0.1, 0.3]), ([0.2, 0.4], [0.1, 0.3], [0.3, 0.5]), ([0.2, 0.5], [0.6, 0.9], [0.1, 0.2])}	{([0.2, 0.5], [0.3, 0.6]), ([0.1, 0.4], [0.3, 0.5]), ([0.4, 0.7])}
x_2	{([0.5, 0.7], [0.2, 0.4], [0.2, 0.5]), ([0.1, 0.3]), ([0.4, 0.6], [0.3, 0.4])}	{([0.6, 0.8]), ([0.2, 0.4], [0.4, 0.6]), ([0.4, 0.6])}
x_3	{([0.4, 0.6], [0.2, 0.5], [0.5, 0.7]), ([0.1, 0.4]),([0.3, 0.5], [0.4, 0.7])}	{([0.5, 0.6], [0.1, 0.3]),([0.2, 0.4], [0.1, 0.3], [0.4, 0.6]) , ([0.2, 0.5], [0.1, 0.3] , [0.6, 0.7])}
x_4	{([0.3, 0.5], [0.4, 0.6], [0.5, 0.8]), ([0.1, 0.6]), ([0.3, 0.5], [0.4, 0.6])}	{([0.5, 0.8], [0.7, 0.9]), ([0.2, 0.4], [0.3, 0.5]), ([0.8, 0.9], [0.4, 0.7])}

x_5	{([0.1, 0.2], [0.1, 0.3], [0.3, 0.5]), ([0.4, 0.7], [0.6, 0.8], [0.2, 0.3]), (([0.8, 0.9], [0.3, 0.4], [0.1, 0.3])}	{([0.4, 0.6], [0.1, 0.2], [0.3, 0.5]), ([0.6, 0.7], [0.5, 0.7]), ([0.5, 0.7], [0.1, 0.3], [0.2, 0.6])}

R	y_3	y_4
x_1	{([0.1, 0.5], [0.7, 0.8]), ([0.5, 0.7], [0.2, 0.4]), ([0.3, 0.5], [0.2, 0.4])}	{([0.2, 0.5], [0.7, 0.8], [0.1, 0.4]), ([0.1, 0.3]), ([0.6, 0.8]), ([0.2, 0.5])}
x_2	{([0.2, 0.4], [0.6, 0.8]), ([0.1, 0.3]), ([0.6, 0.7]), ([0.3, 0.5], [0.1, 0.4], [0.5, 0.7])}	{([0.3, 0.5], [0.1, 0.3], [0.4, 0.6]), ([0.4, 0.6], [0.2, 0.4], [0.4, 0.7]), ([0.8, 0.9], [0.2, 0.4], [0.1, 0.3])}
x_3	{([0.2, 0.5], [0.8, 0.9]), ([0.1, 0.4], [0.9, 0.1]),([0.5, 0.7], [0.6, 0.8])}	{([0.8, 0.9], [0.3, 0.5], [0.1, 0.2]), ([0.4, 0.6], [0.1, 0.4] , [0.3, 0.5]), ([0.6, 0.8] , [0.4, 0.6])}
x_4	{([0.2, 0.4], [0.3, 0.6], [0.6, 0.8]), ([0.1, 0.3], [0.2, 0.4]), ([0.4, 0.7], [0.2, 0.5])}	{([0.5, 0.7], [0.4, 0.6], [0.3, 0.5]), ([0.4, 0.7]), ([0.8, 1], [0.1, 0.5])}
x_5	{([0.9, 1], [0.1, 0.3], [0.4, 0.6]), ([0.1, 0.4]), ([0.4, 0.6], [0.3, 0.5])}	{([0.1, 0.3], [0.2, 0.4]), ([0.1, 0.4], [0.5, 0.7], [0.3, 0.5]), ([0.1, 0.2], [0.4, 0.7])}

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be five symptoms, where x_i stands for “Headache”, “Nausea”, “Stomach pain”, “Vomiting”, “Temperature “and the Universe $V = \{y_1, y_2, y_3, y_4\}$ be four diseases, where y_i stands for “Hepatitis”, “Peptic ulcer”, “Malaria”, “Typhoid”.

Let R be IVNHF relation from U to V. R is a medical knowledge statistic date of the relationship of the symptom $(x_i \in U)$ and the disease $(y_i \in V)$, We assume that A represented a patient and the symptoms of Patient A are described by IVNHF set on the universe U.

Let A = {(y_1, ([0.2, 0.4], [0.3, 0.5]), ([0.2, 0.4]), ([0.1, 0.3], [0.4, 0.6], [0.1, 0.4])), (y_2, ([0.1, 0.4], [0.4, 0.7]), ([0.5, 0.7]), ([0.1, 0.3], [0.6, 0.8]), (y_3, ([0.1, 0.3], [0.3, 0.5]), ([0.8, 0.9]), ([0.2, 0.4], [0.5, 0.7]), (y_4, ([0.2, 0.4], [0.1, 0.3], [0.4, 0.6]), ([0.1, 0.2], [0.5, 0.7], [0.1, 0.3]), ([0.4, 0.6], [0.3, 0.5])}.

First, we calculate the lower and upper approximations $\underline{R}(A)$ and $\overline{R}(A)$ as follows:

$\underline{R}(A)$= {<y_1, ([0.2, 0.5], [0.3, 0.5], [0.2, 0.4]), ([0.2, 0.4], [0.2, 0.4], [0.4, 0.7]), ([0.1, 0.3], [0.4, 0.6], [0.1, 0.4])>, <y_2, ([0.2, 0.5], [0.4, 0.7], [0.1, 0.4]), ([0.5, 0.7], [0.5, 0.7], [0.5, 0.7]), ([0.1, 0.3], [0.6, 0.8], [0.1, 0.3])>, <y_3, ([0.3, 0.5], [0.3, 0.5], [0.1, 0.3]), ([0.7, 0.9], [0.8, 0.9], [0.8, 0.9]), ([0.2, 0.4], [0.5, 0.7], [0.2, 0.4])><y_4, ([0.2, 0.4], [0.1, 0.3], [0.4, 0.6]), ([0.1, 0.2], [0.5, 0.7], [0.2, 0.4]), ([0.4, 0.6], [0.3, 0.5], [0.3, 0.5])>}.

$\overline{R}(A)$= {<y_1, ([0.2, 0.4], [0.3, 0.5], [0.2, 0.4]), ([0.2, 0.4], [0.2, 0.4], [0.2, 0.4]), ([0.2, 0.5], [0.4, 0.6], [0.1, 0.4])>, <y_2, ([0.1, 0.4], [0.4, 0.7], [0.1, 0.4]), ([0.5, 0.7], [0.5, 0.7], [0.5, 0.7]), ([0.2, 0.5], [0.4, 0.7], [0.1, 0.3])>, <y_3, ([0.1, 0.3], [0.3, 0.5], [0.1, 0.3]), ([0.5, 0.9], [0.8, 0.9], [0.2, 0.4]), ([0.3, 0.5], [0.5, 0.7], [0.2, 0.4])><y_4, ([0.2, 0.4], [0.1, 0.3], [0.4, 0.6]), ([0.1, 0.3], [0.5, 0.7], [0.1, 0.3]), ([0.4, 0.6], [0.3, 0.5], [0.3, 0.5])>}.

Then we have

$\underline{R}(A)\oplus\overline{R}(A)$ = {<y_1, ([0.2, 0.86], [0.09, 0.25], [0.04, 0.16]), ([0.24, 0.76], [0.04, 0.16], [0.08, 0.28]), ([0.15, 0.78], [0.16, 0.36], [0.01, 0.16])>, <y_2, ([0.1, 0.88], [0.16, 0.49], [0.1, 0.16]), ([0.75, 0.91], [0.25, 0.47], [0.25, 0.49]), ([0.28, 0.65], [0.24, 0.56], [0.01, 0.09])>, <y_3, ([0.37, 0.65], [0.09, 0.25], [0.01, 0.12]), ([0.94, 0.99], [0.64, 0.81], [0.16, 0.36]), ([0.44, 0.07], [0.25, 0.49], [0.04, 0.16])><y_4, ([0.36, 0.64], [0.01, 0.09], [0.16, 0.36]), ([0.19, 0.44], [0.25, 0.49], [0.02, 0.12]), ([0.64, 0.84], [0.09, 0.25], [0.09, 0.25])>}.

Calculation of S($\underline{R}(A)\oplus\overline{R}(A)$)($y_1$)

$=\frac{1}{3}\{\frac{1}{l_T}\sum\alpha+\frac{1}{l_I}\sum(\hat{I}-\beta)+\frac{1}{l_F}\sum(\hat{I}-\gamma)\}$.

Where l_T, l_I, &l_F respectively in ($\underline{R}(A)\oplus\overline{R}(A)$) ($y_1$) and $\alpha=[\alpha^-,\alpha^+],\beta=[\beta^-,\beta^+],\gamma=[\gamma^-,\gamma^+]\hat{I}=[1,1]$

$=\frac{1}{3}\{\frac{1}{3}([0.2,0.86]+[0.24,0.76]+[0.15,0.78])+\frac{1}{3}([0.75,0.91]+[0.84,0.96]+[0.64,0.84])\frac{1}{3}([0.84,0.96]+[0.72,0.92]+[0.84,0.99])\}$

$=\frac{1}{3}\{\frac{1}{3}[0.59,2.40]+\frac{1}{3}[2.23,2.71]+\frac{1}{3}[2.40,2.87]\}$

$=\frac{1}{9}[5.22,7.98]$.

Take the average of the lower & upper values of the intervals.

$\frac{1}{9}(\frac{13.20}{2}) = \frac{1}{9}(6.6) = 0.7422$

Similarly for the others.

Thus, by definition, we obtain the score functions of IVNHF $\underline{R}(A) \oplus \overline{R}(A)$ as follows:

$S(\underline{R}(A) \oplus \overline{R}(A))(y_1) = 0.7422$

$S(\underline{R}(A) \oplus \overline{R}(A))(y_2) = 0.6872$

$S(\underline{R}(A) \oplus \overline{R}(A))(y_3) = 0.7066$

$S(\underline{R}(A) \oplus \overline{R}(A))(y_4) = 0.7184$

It is clear that the score function $S(\underline{R}(A) \oplus \overline{R}(A))(y_1) = 0.7422$. Hence the final decision is select y_1. We can conclude the patients A is suffering from Hepatitis.

6. CONCLUSION

In this paper, we have presented the concept interval- valued neutrosophic hesitant fuzzy rough sets which are a combination of two powerful topics: interval-valued neutrosophic, hesitant and rough sets.

We define IVNHF rough approximation operators in term of IVNHF relations. Properties of upper and lower IVNHF rough approximation operators are also investigated. Finally, we develop a general framework for dealing with uncertainty decision-making by using the IVNHF rough sets over two universes. A medical diagnosis problem is also shown to indicate the principle steps of the decision methodology. In future, we will mainly focus on investigating uncertain measures and knowledge reductions of the IVNHF rough sets.

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Ethics approval: Data has been collected from reliable sources. We follow all the ethical rules.

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REFERENCES

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems20(1968)87-96.
[2] K. M. Alsager, N. O. Alshehri, Single Valued Neutrosophic Hesitant Fuzzy Rough Set and Its Application, World Academy of Science, Engineering and Technology International Journal of Computer and Information Engineering Vol:13, No:2, 2019,99-108.
[3] S. Broumi, F. Smarandache, M. Dhar, Rough neutrosophic sets, Ital. J. Pure. Appl. Math. 32(2014).
[4] C. F. Liu, Y. S. Luo, New aggregation operators of single-valued neutrosophic hesitant fuzzy set and their application in multi-attributedecision making, Pattern Anal. Appls. (2017)1-11.
[5] J. Pang, J. Wang, X. Wu, Multi-valued neutrosophic sets and power aggregation operators with their applications in multi-criteria groupdecision-making problems, Int. J. Comput. Int. Sys. 8(2015)345-363.
[6] Z. Pawlak, Rough sets, International journal of computer and Information Sciences 11(1982), 341-356.
[7] F. Smarandache, A unifying field in logics, Neutrosophic logic, Multi. Valued Logic 8(1999)489-503.
[8] V. Torra, Hesitan fuzzy sets, Int J. Intell. Syst. 25(2010)529-539.
[9] H. Wang, P. Madiraju, Interval-neutrosophic sets, J.Mech.1(2004)274-277.
[10] H. Wang, F. Smarandache, Y. Zhng, R. Sunderraman, Single-valued neutrosophic sets, Multispace and Multistructure 4(2010)410-413.
[11] WangJian-qiang, WuJia-ting, Wang Jing ,Zhang Hong-yu , Chen Xiao-hong, Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems, Information Sciences 288 (2014) 55-72.
[12] M. M. Xia and Z. S. Xu, Hesitant fuzzy information aggregation indecision making, Int. J. Approx. Reason. 52 (2011), 395- 407.
[13] X.B. Yang, X.N, Song, Y.S. Qi, JY. Yang, Constructive andaxiomatic approaches to hesitant fuzzy rough set. Soft Computing18(2014)1067-1077.
[14] J. Ye, Multi-attribute decision making method under a single valued neutrosophic hesitant fuzzy environment, J. Intell. Syst.24(1)(2015)23-34.
[15] L. A. Zadeh, Fuzzy sets, Information and control 8(1965), 338-353.

Impact of Inter Specific Competition Between Two Predator Species Fighting for a Single Prey Following Crowley-Martin Functional Response

Debadatta Adak*[a], Shekhar Das[b], Sristy Roy[c], Rima Sinha[d]
Department of Mathematics, Maharaja Bir Bikram University, Agartala, Tripura – 799004, India
Emails: [a]dev.adak.math95325@gmail.com, [b]shekhar2018agt@gmail.com,
[c]roysristy1999@gmail.com, [d]rimasinha.agt@gmail.com
* Corresponding author. Email Id: dev.adak.math95325@gmail.com

Abstract

So far various mathematical models have been proposed by the researchers to analyze, explain and predict different complex dynamics observed in prey-predator interactions in natural ecology. In classical Lotka-Volterra or Rosenwig-MacArthur type mathematical models it is assumed that the prey population can grow maximum to environment's carrying capacity following a logistic type growth. But the predator's growth has no such upper bound. Leslie in 1948 first proposed that predator's growth, being completely dependent on availability of prey, must not be unbounded. In this view Leslie and Gower (1960) considered the logistic growth of predator population taking prey density as the upper bound of the growth of predator population. Another important aspect of any prey-predator system is the per capita feeding rate of predator depending on prey abundance i.e., functional response. Two widely used functional responses are Holling Type – II (1959) and Beddington-DeAngelis (1975) functional response. Holling Type – II functional response relates predator's Searching and Handling Time for each captured prey. Along with Searching and Handling Time, Beddington-DeAngelis also considered the intra specific competition of predator species. However, Crowley and Martin (1989) in their "preemption" model modified the Beddington-DeAngelis functional response by removing the assumption that prey handling time and predator's intra-specific competition time are mutually exclusive. In this work we have proposed a deterministic mathematical model involving two predator populations, following Leslie-Gower type growth and Crowley-Martin functional response, competing for one prey species. We have analyzed the effect of competition on the predator species and observed that the impact of competition may cause the periodic fluctuations of predator species observed in nature.

Keywords: Prey-Predator Model, Leslie-Gower, Crowley-Martin, Inter Specific Competition, Stability Analysis, Hopf Bifurcation.

Mathematics Subject Classification: 34D, 34H, 90A, 92B.

1 Introduction

So far various mathematical models have been proposed by the researchers to analyze, explain and predict different complex dynamics observed in prey-predator interactions in natural ecology.In classical ***Lotka-Volterra*** or ***Rosenwig-MacArthur*** type mathematical models it is assumed that the prey population can grow maximum to environment's carrying capacity following a logistic type growth. But the predator's growth has no such upper bound. **Leslie[1]** in 1948 first proposed that predator's growth, being completely dependent on availability of prey, must not be unbounded. In this view he considered the logistic growth of predator population taking prey density as the upper bound of the growth of predator population **[2]**.

Let us assume that $x(t)$ be the population density of prey and $y(t)$ be the population density of predator at any time t. Then the Leslie-Gower type prey-predator model following **[2]** can be considered as:

$$\left.\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right) - f(x,y), \\ \frac{dy}{dt} &= y\left(r_1 - \frac{\kappa_1 y}{x}\right). \end{aligned}\right] \qquad (1)$$

Here, r is the maximum per capita growth rate of prey population, K is the environment's carrying capacity towards prey population, r_1 represents maximum per capita growth rate of predator, κ_1 denotes the number of prey required to support one predator at equilibrium. It should be noted that in (1) it is assumed that the environment's carrying capacity towards predator depending on prey population density and represented as $\frac{r_1 x}{\kappa_1}$. The term $f(x,y)$ is the function that represents predator's functional response, i.e., the description of a predator's instantaneous, per capita feeding rate depending on prey abundance **[3]**. Different functions have been used by the researchers as the predator's functional response in ecological prey-predator systems. Followings are the two most widely used forms of functional response:

A. Holling Type-II Functional Response:

It is defined as $f(x,y) = \frac{a_1 xy}{1+b_1 x}$, where a_1 and b_1 respectively describe the effects of attack rate and handling time per prey **[4]**. The basic concept of the Holling Type II functional response is that the average feeding rate of a predator depends exclusively on the time which predator spends for searching for prey and the time predator takes to process each captured prey item (i.e., handling time).

B. Beddington - DeAngelis Functional Response:

In this case we consider $f(x,y)=\frac{a_1xy}{1+b_1x+c_1y}$, where the parameter c_1 represents the magnitude of the intraspecific competition among the predators **[5, 6]**. Beddington-DeAngelis functional response states that if there are two or more predators in a predator population, they will compete with each other for available prey. Hence their functional response will depend not only on prey allocate time to handling time, but also on the time required for competing with other predators, Here the prey handling time and predator's intra-specific competition time is considered to be mutually exclusive.

Crowley and Martin **[7]** in 1989 modified the Beddington-DeAngelis functional response by removing the assumption that prey handling time and predator's intra-specific competition time are mutually exclusive. They defined their model as ''preemption'' model, considering intra-specific competition among predators irrespective of whether a particular predator individual is currently handling prey or searching for prey. The Crowley-Martin functional response is formulated as,

$$f(x,y)=\frac{a_1xy}{(1+b_1x)(1+c_1y)}. \tag{2}$$

A significant difference between the functional responses proposed by Crowley–Martin and the Beddington–DeAngelis, is that the later one predicts that the impact of the feeding rate of the predator on system dynamics is much less when the prey abundance is high. Whereas, Crowley–Martin considers that the impact of feeding rate as quite significant even with the prey abundance **[8]**.

Various researchers have used Crowley-Martin functional response to model various biological phenomena. Zhou & Cui**[9]**, Xu**[10]**, formulated and analyzed global stability of Human Immunodeficiency Virus (HIV) models, where the rate of infection follows Crowley-Martin functional response. Kang et al**[11]**, studied a diffusive and delayed virus dynamics model with Crowley-Martin functional response. Kumari & Mohan**[12]** proposed and analyzed a tri-trophic food chain model incorporating Crowley-Martin functional response. Yin et al **[13]** investigated pattern formation in a prey-predator system by modifying the Leslie – Gower type functional response with Crowley – Martin functional response and diffusion. How does the asymptotic behavior of a Leslie-Gower type predator prey model changes when modified with Crowley-Martin functional response has been studied by Li **[14]**. Positivity of the system solutions and qualitative analysis of the Crowley-Martin prey-predator models have been studied by Zhou **[15,16].** The effect of harvesting in a Crowley-Martin prey – predator system is investigated by Sivasamy et al **[17].** Liao et al proposed a phytoplankton-zooplankton system incorporating the Crowley-Martin functional response and delay. Non-autonomous stochastic study of a Crowley-Martin type prey-predator model have been performed by

Zhang et al **[18]**and Xu **[19]**. Rana **[20]** proposed a discrete time Crowley–Martinprey-predator modeland analyzed the bifurcations and chaos control. A similar type model was investigated by Liu **[21]** incorporating two-time delays.

In nature it is often observed that more than one predator is competing for the same prey. For example, in African Savanna Wildebeests are the preferred prey of lions, hyenas, cheetahs and wild dogs. Both of Sharks and Dolphins in coasts of Hawaii and North-Eastern Australia feed on a variety of Teleost fish **[22]**. In Central Panama, insectivorous birds and insectivorous lizards both feed on arthropod insect prey **[23]**. Whenever, in any ecological system, more than one predator species are preferring one type of prey, it is evident that they will compete with each other over the available prey resource. Therefore, inter-specific competition will occur among the different predator species. In this study we propose and study (analytically and numerically) a three dimensional deterministic mathematical model with the help of nonlinear differential equations to analyze the impact of competition on a prey-predator system where two predators are competing for one single prey.

2 Mathematical model

In view of the aforementioned biological phenomena, we modify the model system (1) considering the following assumptions:

Assumption 1: Two predator species $y(t)$ and $z(t)$ compete with each other over one particular prey $x(t)$.

Assumption 2: In absence of predators, prey population grows logistically with intrinsic growth rate r and carrying capacity K.

Assumption 3: Both the predators are specialist predators and follows Leslie-Gower type growth (as specified in (1)) and Crowley-Martin functional response (as defined in (2)) with respective growth rates r_1 and r_2.

Assumption 4: Carrying capacity of both the predators depend on the prey abundance and are denoted by the terms $\frac{r_1 x}{\kappa_1}$ and $\frac{r_2 x}{\kappa_2}$ respectively.

Assumption 5: The predator populations compete with each other for available common prey resources. Let, m_1 and m_2 are respectively be the effect of competition of $z(t)$ on $y(t)$ and vice versa.

Assumption 6: All the parameters are considered positive.

Next, we draw the schema diagram based on these assumptions:

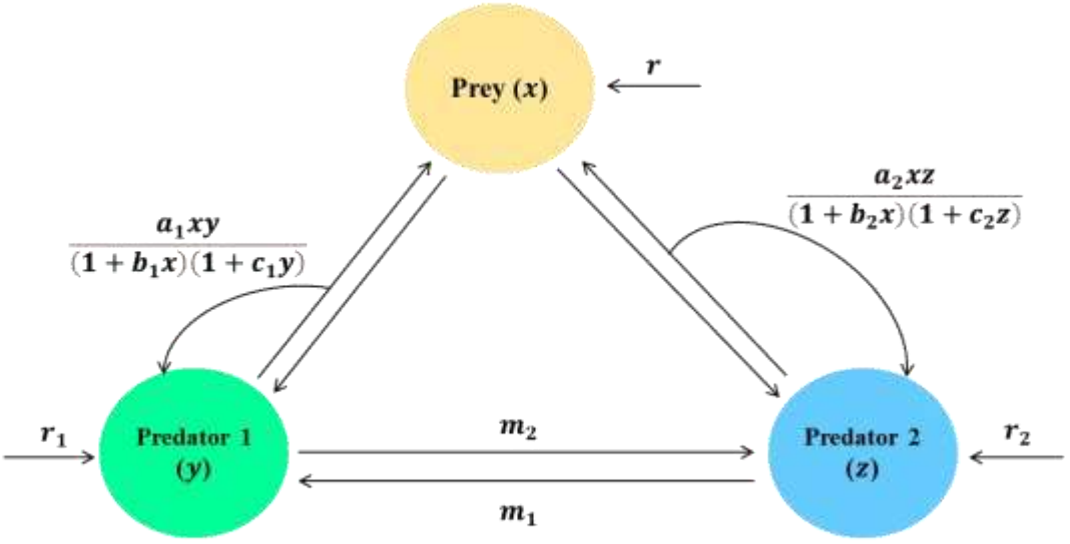

Fig. 1.*Schema diagram with respect to the Assumption 1 to Assumption 6*

Finally in accordance to the schema diagram we propose the following deterministic model using nonlinear ordinary differential equations modifying the model system (1).

$$\left.\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{a_1xy}{(1+b_1x)(1+c_1y)}-\frac{a_2xz}{(1+b_2x)(1+c_2z)},\\
\frac{dy}{dt} &= y\left(r_1-\frac{\kappa_1 y}{x}\right)-m_1yz, \quad if\ x\neq 0,\\
\frac{dz}{dt} &= z\left(r_2-\frac{\kappa_2 z}{x}\right)-m_2yz, \quad if\ x\neq 0,\\
\frac{dy}{dt} &= \frac{dz}{dt}=0\ if\ x=0.
\end{aligned}\right\} \tag{3}$$

Initial Condition:$x(0)>0, y(0)\geq 0, z(0)\geq 0.$ (4)

3 Analytical results

In this section we derive conditions of positivity and boundedness of the solutions of the model system (3).

3.1 Positivity and boundedness of the solutions

Proposition 3.1 *All the solutions of the model system (3) are positively invariant and ultimately bounded in the region* Γ *for sufficiently large* t *where* Γ*is given by:*

$$\Gamma = \left\{(x,y,z) \in \mathbb{R}_+^3 : \ 0 < x(t) \le K, 0 < \ y(t) \le \frac{Kr_1}{\kappa_1}, 0 < z(t) \le \frac{Kr_2}{\kappa_2}\right\}.$$

Proof. First, we show that all the solutions of the system (3) starting with initial conditions (4) are positive, using a lemma proposed by **Nagumo [24].**

Lemma 3.1*Consider a system* $\dot{X} = F(X)$ *where* $F(X) = [F_1(X), F_2(X), \dots, F_n(X)]$, $X \in \mathbb{R}^n$*with initial condition* $X(0) = X_0 \in \mathbb{R}^n$. *If for* $X_i = 0, i = 1,2,\dots,n$ *we get* $F_i(X)|_{X_i=0} \ge 0,$ *then any solution of* $\dot{X} = F(X)$ *with given initial condition, say,* $X(t) = X(t;X_0)$ *will be positive i.e.,* $X(t) \in \mathbb{R}_+^n$

From model system (3), one can easily see that $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$ when $x = y = z = 0$. Hence following Lemma 3.1 we conclude that all solutions of model system (3) is positively invariant in $\mathbb{R}_+^3$.Again, to establish the boundedness of the solutions of (3) first we state the following lemma proposed by **Chen [25]**:

Lemma 3.2*If* $a,b > 0$ *and* $\frac{dX}{dt} \le (or \ \ge) X(t)(a - bX(t))$ *with* $X(0) > 0$, *then* $\limsup_{t\to\infty} X(t) \le \frac{a}{b}\left(or \ \liminf_{t\to\infty} X(t) \ge \frac{a}{b}\right)$.

From the first equation of (3) following Lemma 3.2 and (4) we obtain

$$\frac{dx}{dt} \le rx\left(1 - \frac{x}{K}\right) \Rightarrow \limsup_{t\to\infty} x(t) \le K. \tag{5}$$

Again, using (5) in light of Lemma 3.2, the second equation of (3) yields

$$\frac{dy}{dt} \le y\left(r_1 - \frac{\kappa_1 y}{K}\right) \Rightarrow \limsup_{t\to\infty} y(t) \le \frac{Kr_1}{\kappa_1}. \tag{6}$$

Finally, from the third equation of (3) and applying (5) & Lemma 3.2 one can easily calculate,

$$\frac{dz}{dt} \le z\left(r_2 - \frac{\kappa_2 z}{K}\right) \Rightarrow \limsup_{t\to\infty} z(t) \le \frac{Kr_2}{\kappa_2}.$$

Hence the proposition is proved.

3.3 Equilibrium points and their existence conditions

Model system (3) has the following four significant equilibrium points.

I. The predation-free equilibrium is given by $E_1(K,0{,}0)$ which always exists.

II. There exists one equilibrium of the form$E_{12}(\bar{x},\bar{y},0)$ where, $\bar{y}=\frac{r_1\bar{x}}{\kappa_1}$ and $\bar{x}$ is given by the positive root of the cubic equation, $A_1\bar{x}^3+A_2\bar{x}^2+A_3\bar{x}+A_4=0$, where, $A_1=\frac{rr_1b_1c_1}{K\kappa_1}>0, A_2=\frac{rb_1}{K}+\frac{rr_1c_1}{K\kappa_1}-\frac{rr_1b_1c_1}{\kappa_1}, A_3=\frac{r_1a_1}{\kappa_1}+\frac{r}{K}-rb_1-\frac{rr_1c_1}{\kappa_1}, A_4=-r<0$. Clearly, there always exists at least one feasible E_{12}.

III. Another equilibrium exists of the form $E_{13}(\hat{x},0,\hat{z})$ where, $\hat{z}=\frac{r_2\hat{x}}{\kappa_2}$ and $\hat{x}$ is given by the positive root of the cubic equation, $B_1\hat{x}^3+B_2\hat{x}^2+B_3\hat{x}+B_4=0$, where, $B_1=\frac{rr_2b_2c_2}{K\kappa_2}>0, B_2=\frac{rb_2}{K}+\frac{rr_2c_2}{K\kappa_2}-\frac{rr_2b_2c_2}{\kappa_2}, B_3=\frac{r_2a_2}{\kappa_2}+\frac{r}{K}-rb_2-\frac{rr_2c_2}{\kappa_2}, B_4=-r<0$. One can easily check that there always exists at least one feasible E_{13}.

IV. The coexistence equilibrium is denoted by $E^*(x^*,y^*,z^*)$, where

$$y^*=\frac{(\kappa_2r_1-m_1r_2x^*)x^*}{\kappa_1\kappa_2-m_1m_2x^{*2}},\ z^*=\frac{(\kappa_1r_2-m_2r_1x^*)x^*}{\kappa_1\kappa_2-m_1m_2x^{*2}}.$$

Here x^* is the positive root of the following equation:

$$Q_8x^{*7}+Q_7x^{*6}+Q_6x^{*5}+Q_5x^{*4}+Q_4x^{*3}+Q_3x^{*2}+Q_2x^*+Q_1=0,$$

where,

$$Q_1=P_7\kappa_1\kappa_2>0, Q_2=P_7P_4+P_8\kappa_1\kappa_2-a_1K\kappa_2r_1-a_2\kappa_1r_2K,$$
$$Q_3=P_7P_5+P_8P_4+P_9\kappa_1\kappa_2+a_1m_1r_2K+a_2m_2r_1K,$$
$$Q_4=P_7P_6+P_8P_5+P_9P_4+P_{10}\kappa_1\kappa_2,$$
$$Q_5=P_8P_6+P_9P_5+P_{10}P_4+P_{11}\kappa_1\kappa_2,$$
$$Q_6=P_9P_6+P_{10}P_5+P_{11}P_4, Q_7=P_{10}P_6+P_{11}P_5, Q_8=P_{11}P_6<0,$$
$$P_1=b_1\kappa_1\kappa_2+c_1\kappa_2r_1>0, P_2=-m_1m_2-c_1m_1r_2+b_1c_1\kappa_2r_1,$$
$$P_3=-b_1m_1m_2-b_1c_1m_1r_2<0, P_4=b_2\kappa_1\kappa_2+c_2\kappa_1r_2>0,$$
$$P_5=-m_1m_2-c_2m_2r_1+b_2c_2\kappa_1r_2, P_6=-b_2m_1m_2-b_2c_2m_2r_1<0,$$
$$P_7=rK\kappa_1\kappa_2>0, P_8=rKP_1-r\kappa_1\kappa_2, P_9=rKP_2-rP_1,$$
$$P_{10}=rKP_3-rP_2, P_{11}=-rP_3>0.$$

Clearly one positive x^* always exists. Associated with positive x^* a feasible E^* will exist if the following conditions are satisfied by the system parameters:

$$\begin{aligned} &\text{A. Either, } x^* > \max\left\{\frac{\kappa_2 r_1}{m_1 r_2}, \frac{\kappa_1 r_2}{m_2 r_1}, \sqrt{\frac{\kappa_1 \kappa_2}{m_1 m_2}}\right\} \\ &\text{B. Or, } 0 < x^* < \min\left\{\frac{\kappa_2 r_1}{m_1 r_2}, \frac{\kappa_1 r_2}{m_2 r_1}, \sqrt{\frac{\kappa_1 \kappa_2}{m_1 m_2}}\right\}. \end{aligned} \tag{7}$$

Next, we discuss the local stability of the equilibrium points.

Proposition 3.1. *The equilibrium E_1 is always unstable.*

Proof. The eigen values of the Jacobian matrix associated with the equilibrium E_1 are given by: $-r < 0, r_1 > 0, r_2 > 0$. Hence E_1 is always unstable. This proves the proposition.

Proposition 3.2. *The equilibrium E_{12} is locally asymptotically stable if the following conditions hold,*

$$\bar{x} > \frac{r_2 \kappa_1}{m_2 r_1}, \qquad \Delta_1 = \frac{2r\bar{x}}{K} + \frac{a_1 \bar{y}}{(1 + b_1 \bar{x})^2 (1 + c_1 \bar{y})} - r > 0.$$

Proof. The characteristic equation of the Jacobian matrix associated with the equilibrium E_{12} is given by:

$$(a_{33} - \lambda)[\lambda^2 + X_1 \lambda + X_2] = 0,$$

where

$X_1 = -(a_{11} + a_{22}), X_2 = a_{11}a_{22} - a_{12}a_{21},$

$a_{11} = r - \frac{2r\bar{x}}{K} - \frac{a_1 \bar{y}}{(1+b_1\bar{x})^2(1+c_1\bar{y})}, a_{12} = -\frac{a_1 \bar{x}}{(1+b_1\bar{x})(1+c_1\bar{y})^2} < 0,$

$a_{21} = \frac{\kappa_1 \bar{y}^2}{\bar{x}^2} > 0, a_{22} = -r_1 < 0, a_{33} = r_2 - \frac{m_2 r_1 \bar{x}}{\kappa_1}.$

Following Routh Hurwitz criteria E_{12} will be locally asymptotically stable if and only if,

$$a_{11} < 0 \Rightarrow \Delta_1 > 0 \text{ and } a_{33} < 0 \Rightarrow \bar{x} > \frac{r_2 \kappa_1}{m_2 r_1}.$$

Hence the proposition is proved.

Proposition 3.3. *The equilibrium E_{13} is locally asymptotically stable if the following conditions hold,*

$$\hat{x} > \frac{r_1\kappa_2}{m_1 r_2}, \qquad \Delta_2 = \frac{2r\hat{x}}{K} + \frac{a_2\hat{z}}{(1+b_2\hat{x})^2(1+c_2\hat{z})} - r > 0.$$

Proof. he characteristic equation of the Jacobian matrix associated with the equilibrium E_{12} is given by:

$$(b_{22}-\lambda)[\lambda^2 + Y_1\lambda + Y_2] = 0,$$

where,$Y_1 = -(b_{11}+b_{33}), Y_2 = b_{11}b_{33} - b_{13}b_{31}$,

$$b_{11} = r - \frac{2r\hat{x}}{K} - \frac{a_2\hat{z}}{(1+b_2\hat{x})^2(1+c_2\hat{z})}, b_{13} = -\frac{a_2\hat{x}}{(1+b_2\hat{x})(1+c_2\hat{z})^2} < 0,$$

$$b_{31} = \frac{\kappa_2\hat{z}^2}{\hat{x}^2} > 0, b_{22} = r_1 - \frac{m_1 r_2\hat{x}}{\kappa_2}, b_{33} = -r_2 < 0.$$

Following Routh Hurwitz criteria E_{13} will be locally asymptotically stable if and only if,

$$b_{22} < 0 \Rightarrow \Delta_2 > 0 \text{ and } b_{22} < 0 \Rightarrow \bar{x} > \frac{r_1\kappa_2}{m_1 r_2}.$$

Hence the proposition is proved.

Proposition 3.4. *If the equilibrium E^* exists following the conditions as specified in (4), then E^* is locally asymptotically stable if the following conditions hold,*

$$Z_1 > 0, Z_3 > 0, Z_1Z_2 - Z_3 > 0,$$

where Z_1, Z_2, Z_3 are given in equation (5).

Proof. The characteristic equation of the Jacobian matrix evaluated at E^* is calculated to be,

$$\lambda^3 + Z_1\lambda^2 + Z_2\lambda + Z_3 = 0,$$

where,

$$Z_1 = -(m_{11} + m_{22} + m_{33}),$$

$$Z_2 = m_{22}m_{33} - m_{23}m_{32} + m_{11}m_{33} - m_{13}m_{31} + m_{11}m_{22} - m_{12}m_{21},$$

$$Z_3 = m_{12}m_{23}m_{32} + m_{12}m_{21}m_{33} + m_{13}m_{22}m_{31} - m_{11}m_{22}m_{33}$$

$$-m_{12}m_{23}m_{31} - m_{13}m_{21}m_{32}$$

$$m_{11} = r - \frac{2rx^*}{K} - \frac{a_1y^*}{(1+b_1x^*)^2(1+c_1y^*)} - \frac{a_2z^*}{(1+b_2x^*)^2(1+c_2z^*)},$$

$m_{12} = -\frac{a_1 x^*}{(1+b_1 x^*)(1+c_1 y^*)^2}, m_{13} = -\frac{a_2 x^*}{(1+b_2 x^*)(1+c_2 z^*)^2},$

$m_{21} = \frac{\kappa_1 y^{*2}}{x^{*2}}, m_{22} = r_1 - \frac{2\kappa_1 y^*}{x^*} - m_1 z^*, m_{23} = -m_1 y^*,$

$m_{31} = \frac{\kappa_2 z^{*2}}{x^{*2}}, m_{32} = -m_2 z^*, m_{33} = r_2 - \frac{2\kappa_2 z^*}{x^*} - m_2 y^*.$

Hence, the theorem is proved following Routh-Hurwitz Criteria.

4 Numerical simulations

To perform numerical simulations, we have used the following parameter values:

Parameter Set – I: $r = 0.3, K = 40, a_1 = 0.4, b_1 = 0.07, c_1 = 0.25, a_2 = 0.4, b_2 = 0.07,$
$c_2 = 0.25, r_1 = 0.02, \kappa_1 = 0.04, r_2 = 0.02, \kappa_2 = 0.04.$

We wish to observe the effects of interspecific competition. For that we consider the following three cases.

Case – 4.1. *When both the predators are equal competitors, i.e., the rates of interspecific competitions are same, i.e.,* $m_1 = m_2$.

When $m_1 = m_2 = 0$, i.e., there is no competition, then E^*exists in a stable mode satisfying the stability conditions as specified in Proposition 3.4. In this case, $Z_1 = 0.23441 > 0, Z_3 = 13.8721 > 0, Z_1 Z_2 - Z_3 = 2.5641 > 0$. Therefore, all trajectories converge to E^*. This scenario shows that in absence of inter specific competitions, all the species coexist in a stable manner. This case has been depicted in Fig. 2.

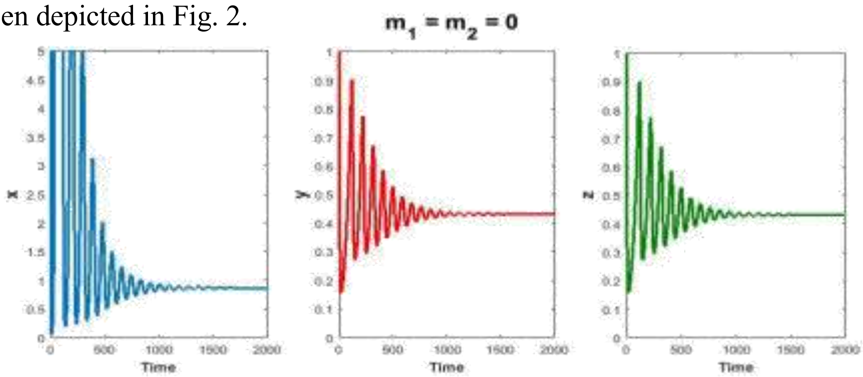

Fig. 2. *Time evolution of system (3) when* $m_1 = m_2 = 0$. *Other parameters are as in*

Parameter Set – I.

Next, we consider inter specific competition at a low rate, i.e., $m_1 = m_2 = 0.01$ keeping all other parameters fixed at Parameter Set – I. This case actually describes the scenario, when both the predators compete with each other, but in a low intensity. We observe that for this low intensity inter specific competition, all the populations coexist and system converges to a stable E^*. This scenario has been described in Fig. 3. For these parameter values we have calculated $Z_1 = 3.6723 > 0, Z_3 = 5.9822 > 0, Z_1Z_2 - Z_3 = 0.7754 > 0$.

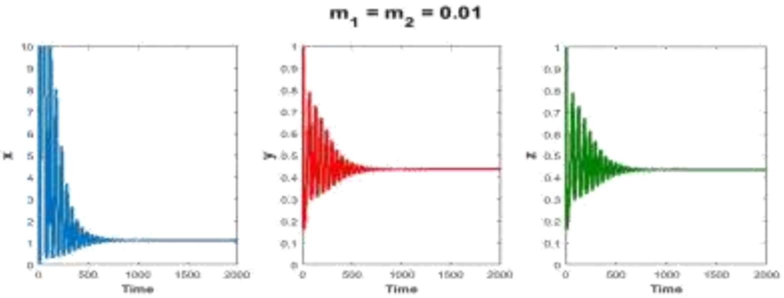

Fig. 3.*Time evolution of system (3) when* $m_1 = m_2 = 0.01$. *Other parameters are as in Parameter Set – I.*

As the rate of interspecific competition is increased further, we see that the system becomes unstable. For example, when we take $m_1 = m_2 = 0.02$, E^* exists but becomes unstable through a Hopf Bifurcation in backward direction. Therefore, with increasing competition, the coexistence of all the species becomes unstable. This situation has been shown in Fig. 4. Here we have calculated, $Z_1 = 0.6673 > 0, Z_3 = 2.7323 > 0, Z_1Z_2 - Z_3 = -3.1193 < 0$.

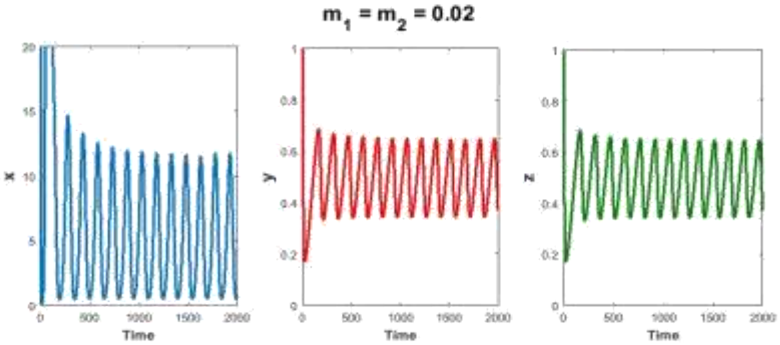

Fig. 4.*Time evolution of system (2) when* $m_1 = m_2 = 0.02$. *Other parameters are as in Parameter Set – I.*

Finally, when the intensity of the interspecific competition is increased further to $m_1 = m_2 = 0.05$, the system again becomes stable through a forward Hopf Bifurcation and all populations coexist in a stable state. In fact, in this case E^* exists and stable. This scenario has been depicted in Fig. 5.For these parameter values one can calculate, $Z_1 = 0.8845 > 0, Z_3 = 7.1826 > 0, Z_1 Z_2 - Z_3 = 1.3286 > 0$.Thus, we assert that, when both the predators are equal competitors, then all the populations will coexist. However, for low or high competition, the coexistence is stable. But for moderate level of interspecific competition, the populations will coexist in an unstable state.

Case – 4.2. ***When $z -$ predator is a better competitor than the $y -$ predator, i.e., when $m_1 > m_2$.***

To analyze this case we consider, $m_1 = 0.07$ and $m_2 = 0.05$. We see that, when the $z -$ predator is much more effective than the y −predator, then due to higher competition y −predator species dies out and system converges to unstable E_{13}, i.e., only the x −prey and z −predator exist in an unstable state. This case has been shown in Fig. 6.

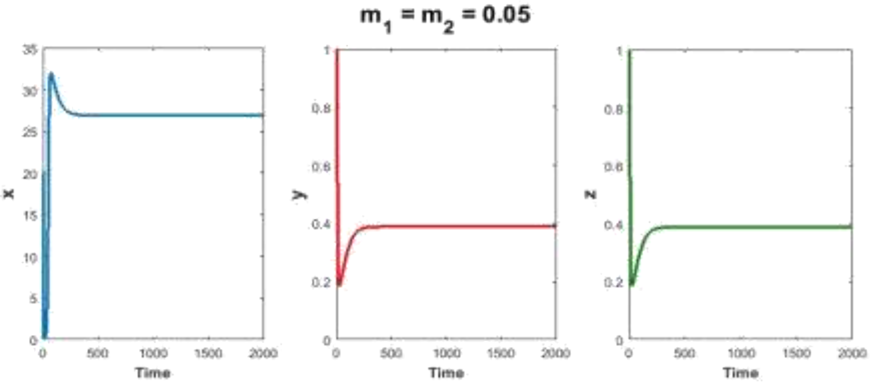

Fig. 5. *Time evolution of system (2) when $m_1 = m_2 = 0.05$. Other parameters are as in Parameter Set – I.*

Case – 4.3. ***When $y -$ predator is a better competitor than the $z -$ predator, i.e., when $m_2 > m_1$.***

To analyze this case we consider, $m_2 = 0.07$ and $m_1 = 0.05$. In this case the $y -$ predator is considered to be more effective competitor than the z −predator. Eventually, the $y -$ predator puts more pressure on $z -$ predator and after sufficiently large time, as a result of higher competition, thez −predator species goes to extinction. Hence the system converges to unstable E_{12}, i.e., only the x −prey and y −predator exist in an unstable state. This case has been shown in Fig. 7.

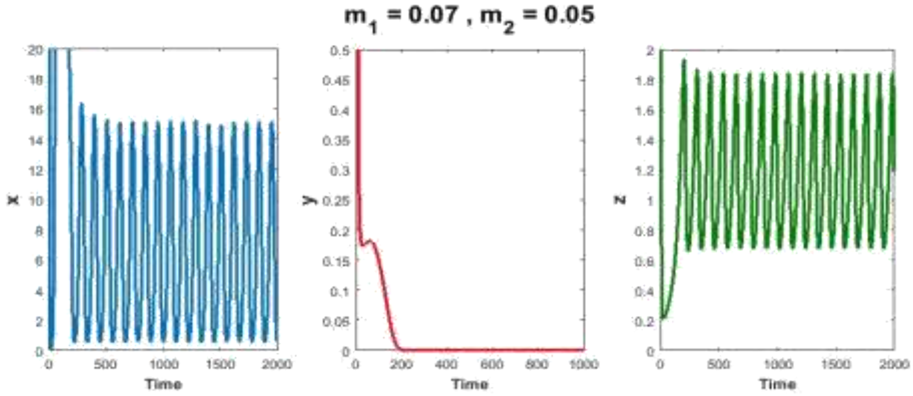

Fig. 6.*Time evolution of system (2) when* $m_1 = 0.07,\ m_2 = 0.05$*. Other parameters are as in Parameter Set – I.*

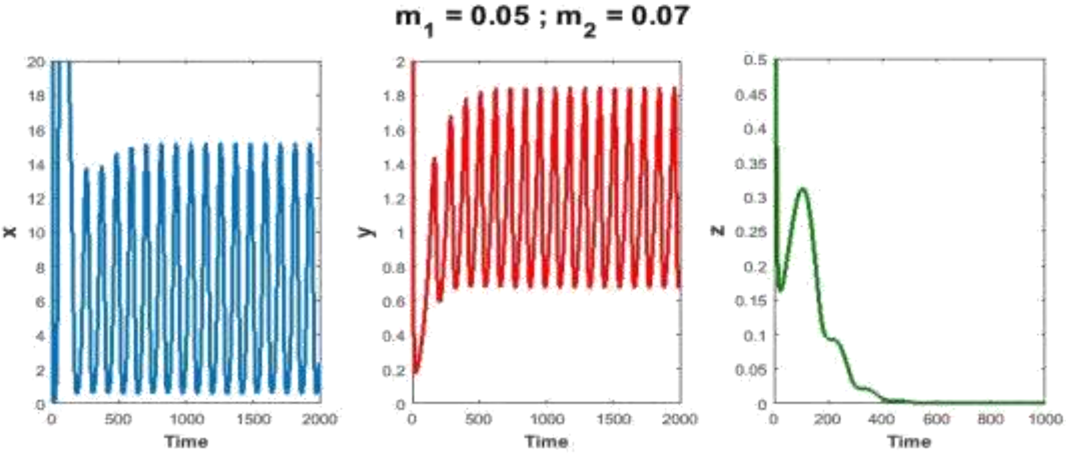

Fig. 7.*Time evolution of system (2) when* $m_2 = 0.07,\ m_1 = 0.05$*. Other parameters are as in Parameter Set – I.*

Hence, we obtain that unequal competition will lead to extinction of one of the competing predators. In long run, the better competitor will get advantage and will persist. Whereas, the weaker competitor will perish.

5. Conclusion

In this work a three dimensional deterministic model involving two competing predator species fighting for one prey species has been studied to analyze the impact of competition on the system dynamics. It has been assumed that prey population grows logistically. Whereas, the predator population grows following Leslie-Gower growth and follows Crowley-Martin functional response. It has been observed that in absence of any inter-specific competition between the predator species all population can coexist in a stable mode with stable E^*. In presence of low and equal rate of inter-specific competition between the predator species all population can coexist in a stable mode with stable E^*. But as the rate of competition increases in equal proportion the system undergoes a backward Hopf Bifurcation and becomes unstable. In this case all population coexist in an unstable mode with unstable E^*. However, with further increase of the rate of competition in equal proportion the system again undergoes a forward Hopf Bifurcation and becomes stable again. At this point all population coexist in a stable mode with stable E^*. Now, keeping the effect of competition of $z-$ predator species on y predator species (m_1) at the stable level if we increase the effect of competition of $y-$ predator species on $z-$ predator species (m_2), then due to higher competition $z-$ predator goes to extinction and system converges to unstable E_{12} equilibrium. Similarly, keeping the effect of competition of $y-$predator species on $z-$ predator species (m_2) at the stable level if we increase the effect of competition of $z-$ predator species on $y-$ predator species (m_1), then due to higher competition $y-$predator goes to extinction and system converges to unstable E_{13}equilibrium. Hence, we assert that equal impact of inter specific competition has both stabilizing and destabilizing effect. Variation of the equal rate of inter specific competition may be the reason for periodic fluctuations observed in prey-predator ecosystems in nature. However, unequal interspecific competition has a destabilizing effect on the system and may eliminate an otherwise stable coexistence equilibrium by causing extinction of the predator species with lower impact of competition on its counterpart. In this case system finally converges to an unstable E_{12} or E_{13} equilibrium.

6 References

[1] P. H. Leslie, S*ome further notes on the use of matrices in population mathematics*, Biometrika, 35 (1948) 213-245.

[2] P. Leslie and J. Gower, *The properties of a stochastic model for the predator-prey type of interaction between two species*, Biometrika ,47 (1960) 219-234.

[3] C. S. Holling, *The components of predation as revealed by a study of small mammal predation of the European pine sawfly*. Canadian Entomologist, 91 (1959) 293–320.

[4] C. S. Holling, *Some characteristics of simple types* of predation and parasitism. Canadian Entomologist, 91 (1959) 385–398.

[5] J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency,* Journal of Animal Ecology 51 (1975) 331–340.

[6] D. L. DeAngelis, R. A. Goldstein and R. V. O'Neill, *A model for trophic interaction,* Ecology 56 (1975) 881–892.

[7] P. H. Crowley and E. K. Martin, *Functional responses and interference within and between year classes of a dragonfly population*. Journal of the North American Benthological Society 8 (1989) 211–221.

[8] M. Hassell, (1971). *Mutual interference between searching insect parasites*. The Journal of Animal Ecology, 40 (1971) 473–486.

[9] X. Zhou, J. Cui. *Global stability of the viral dynamics with Crowley-Martin functional response*. Bull. Korean Math. Soc., 48(3) (2011) 555–574.

[10] S. Xu. *Global stability of the virus dynamics model with Crowley-Martin functional response*. Electronic Journal of Qualitative Theory of Differential Equations, 12(9) (2012) 1–10.

[11] C. Kang, H. Miao, X. Chen, J. Xu, D. Huang. *Global stability of a diffusive and delayed virus dynamics model with Crowley-Martin incidence function and CTL immune response*. Advances in Difference Equations, (324) (2017) 1–16.

[12] N. Kumari, N. Mohan. *Cross Diffusion Induced Turing Patterns in a Tritrophic Food Chain Model with Crowley Martin Functional Response*. Mathematics, 7(229) (2019) 1–25.

[13] H. Yin, X. Xiao, X. Wen, K. Liu. *Pattern analysis of a modified Leslie – Gower predator – prey model with Crowley – Martin functional response and diffusion*. Computers and Mathematics with Applications, 67(8) (2014) 1607–1621.

[14] H. Li,*Asymptotic behavior and multiplicity for a diffusive Leslie – Gower predator – prey system with Crowley – Martin functional response*. Computers and Mathematics with Applications, 68(7) (2014) 693–705.

[15] J. Zhou*, Positive solutions for a modified Leslie – Gower prey – predator model with Crowley – Martin functional responses*. Nonlinear Differential Equations and Applications, 21(1) (2014) 621–661.

[16] J. Zhou, *Qualitative analysis of a modified Leslie-Gower predator-prey model with Crowley-Martin functional responses*. Communications on Pure and Applied Analysis, 14(3) (2015) 1127–1145.

[17] R. Sivasamy, K. Sathiyanathan, K. Balachandran, *Dynamics of a Modified Leslie-Gower Model with Crowley-Martin Functional Response and Prey Harvesting*. Journal of Applied Nonlinear Dynamics, 8(4) (2019) 621–636.

[18] Y. Zhang, S. Chen, S. Gao, *Analysis of a nonautonomous stochastic predator-prey model with Crowley-Martin functional response*. Advances in Difference Equations, 2016(264) (2016) 1–28.

[19] C. Xu, G. Ren, Y. Yu*, Extinction Analysis of Stochastic Predator–Prey System with Stage Structure and Crowley– Martin Functional Response*. Entropy, 21(252) (2019) 1–13.

[20] S. Rana, *Bifurcation analysis and chaos control in a discrete-time predator-prey system with Crowley-Martin functional response*. Computational Ecology and Software, 9(2) (2019) 37–57.

[21] C. Liu, S. Li, Y. Yan, *Hopf Bifurcation Analysis of A Density Predator-Prey Model With Crowley-Martin Functional Response And Two Time Delays*. Journal of Applied Analysis and Computation, 9(4) (2019) 1589–1605, 2019.

[22] M. R. Heithaus, *Predator-prey and competitive interactions between sharks (order Selachii) and dolphins (suborder Odontoceti): a review*, J. Zool., Lond. 253 (2001) 53-68.

[23] S. J. Wright, *Competition Between Insectivorous Lizards and Birds in Central Panama,* Amer. Zool., 19 (1979) 1145-1156.

[24] N. Nagumo, *Uber die lage der integralkurvengewnlicherdi_erantialgleichungen*, Proc. Phys. Math. Soc. Jpn. 24 (1942) 551.

[25] F. Chen, *On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay*, J. Comput. Appl. Math. 180(1) (2005) 33–49.

Representation of a Triangle Through Algebraic Forms

Dhananjay Kumar and Prabhat Chandra

[a]Department of Mathematics, Sudha Madhav Degree College, Tajpur, Saran, Bihar, India
[b]Department of Mathematics, Patna University, Patna, Bihar, India

Email: [a]dhananjay.jpu@gmail.com and [b]prabhatchandra9450@gmail.com

Abstract

An attempt has been made to represent a triangle by a transformation which has been defined as a transformation of linear form. Let $(x, y) \in R^2$. Then a transformation of linear form denoted by E is defined by

$$E\,(x, y) = (ax + by + c_1, cx + dy + c_2)$$

where a, b, c, d, c_1, c_2 are real numbers.

Consider a triangle ABC with vertices $A(a_1, b_1)$, $B(a_2, b_2)$ and $C(a_3, b_3)$. It has been shown that by suitable values of a, b, c, d, c_1, c_2.

$E(a_1, b_1) = (a_2, b_2)$, $E(a_2, b_2) = (a_3, b_3)$ and $E\,(a_3, b_3) = (a_1, b_1)$.

Keywords: Triangle, transformation of linear form.

I. Introduction

In this paper, ideas of geometry and functional analysis have been combined. From books of functional analysis we are familiar with ideas of linear transformation given in references such as [1], [2], [3] and [4]. We have generalised idea of linear transformation and called it transformation of linear form.

II. Definition

Let $(x, y) \in R^2$. Then a transformation of linear form, denoted by E is a transformation on R^2 defined by

$$E\,(x, y) = (ax + by + c_1, cx + dy + c_2)$$

where a, b, c, d, c_1, c_2 are real numbers.

When $c_1 = c_2 = 0$, then E is a linear transformation.

III. Main Results

Theorem 1. For given numbers *m* and *n*,

$$E^2\,(x, y) = \Big((an-m)\,x + bny + (a+1)\,c_1 + bc_2, cnx + (n^2 - an - m)\,y + cc_1 + (d+1)c_2\Big)$$

and

$$\begin{aligned} E^3\ (x,\ y) = = &\Big(\big(a\,(an-m)+bcn\big)x+\big(abn+b\,(n^2-an-m)\big)\Big)\,y + a\,(a+1)\,c_1 + abc_2 \\ &+ bcc_1 + b\,(d+1)\,c_2 + c_1, \big(c(an-m)+dcn\big)\,x \\ &+ \big(cbn + d(n^2-an-m)\big)\,y + cc_1\,(a+1) + bcc_2 + cc_1d + d\,(d+1)\,c_2 + c_2\Big), \end{aligned}$$

where $n = a+d$ and $m = ad-bc$.

Proof: Let $a + d = n$ and $ad - bc = m.$

Then, $d = n - a.$

Hence, $ad - bc = m \Rightarrow a\,(n-a) - bc = m$

$$\Rightarrow a^2 + bc = an - m$$

$$\Rightarrow bc = an - m - a^2$$

Then, $bc + d^2 = an - m - a^2 + (n-a)^2 = n^2 - an - m.$

Let $A = a^2 + bc = an - m, B = b\,(a + d) = bn$

$C = c\,(a + d) = cn, D = bc + d^2 = n^2 - an - m.$

Then,

$$E^2\,(x, y) = E\big(E(x, y)\big) = E\,(ax + by + c_1, cx + dy + c_2)$$

$$= \big(a(ax + by + c_1) + b\,(cx + dy + c_2) + c_1, c(ax + by + c_1) + d(cx + dy + c_2) + c_2\big)$$

$$= \big((a^2 + bc)\,x + b\,(a + d)y + (a + 1)\,c_1 + bc_2, c\,(a + d)\,x + (bc + d^2)\,y + cc_1 + (d + 1)c_2\big)$$

$$= \big((an - m)\,x + bny + (a + 1)\,c_1 + bc_2, cnx + (n^2 - an - m)\,y + cc_1 + (d + 1)c_2\big)$$

Hence,

$$E^3\,(x, y) = E\big(E^2\,(x, y)\big)$$

$$= E\big((an - m)\,x + bny + (a + 1)c_1 + bc_2, cnx + (n^2 - an - m)\,y + cc_1 + (d + 1)c_2\big)$$

$$= \Big(\Big(a\big((an - m)\,x + bny + (a + 1)\,c_1 + bc_2\big) + b\,(cnx + (n^2 - an - m)\,y + cc_1$$

$$+ (d + 1)\,c_2\big) + c_1, c\,\big((an - m)\,x + bny + (a + 1)\,c_1 + bc_2\big)$$

$$+ d\,\big(cnx + (n^2 - an - m)\,y + cc_1 + (d + 1)\,c_2) + c_2\big)$$

$$= \Big(\big(a\,(an - m) + bcn\big)x + \big(abn + b\,(n^2 - an - m)\big)\Big)\,y + a\,(a + 1)\,c_1 + abc_2$$

$$+ bcc_1 + b\,(d + 1)\,c_2 + c_1, \big(c(an - m) + dcn\big)\,x$$

$$+ \big(cbn + d(n^2 - an - m)\big)\,y + cc_1\,(a + 1) + bcc_2 + cc_1d + d\,(d + 1)\,c_2 + c_2\big)$$

$$= \big(A'x + B'y + c_3, C'x + D'y + c_4\big), \text{say}$$

Then,

$$A' = a\,(an - m) + bcn = (a^2 + bc)\,n - am$$

$$= (an - m)\,n - am = an^2 - mn - am\,.$$

$$B' = abn + b\,(n^2 - an - m) = b\,(n^2 - m)\ .$$

$$C' = c\,(an - m) + cdn = cn\,(a + d) - cm = cn^2 - cm = c\,(n^2 - m)\,.$$

$$D' = cbn + d\,(n^2 - an - m) = dn^2 - n\,(ad - bc) - md$$

$$= (n - a)\,n^2 - mn - m\,(n - a)$$

$$= n^3 - an^2 - 2mn + am\,.$$

$$c_3 = a(a+1)\,c_1 + abc_2 + bcc_1 + b\,(d+1)\,c_2 + c_1$$

$$= c_1\,(a^2 + a + bc + 1) + c_2\,\big(ab + b\,(d+1)\big)$$

$$= c_1\,(a^2 + bc + a + 1) + c_2 b\,(a + d + 1)$$

$$= c_1\,(an - m + a + 1) + c_2 b\,(n+1)\,.$$

and

$$c_4 = cc_1(a+1) + bcc_2 + cc_1 d + d(d+1)\,c_2 + c_2$$

$$= cc_1(a + d + 1) + c_2\,(bc + d^2 + d + 1)$$

$$= cc_1(n+1) + c_2\,(n^2 - an - m + n - a + 1)\,.$$

So, we have calculated both $E^2\,(x, y)$ and $E^3\,(x, y)$.

Theorem 2. If $E(a_1, b_1) = (a_2, b_2)$, $E\,(a_2, b_2) = (a_3, b_3)$ and $E(a_3, b_3) = (a_1, b_1)$, then $E^3 = I$, identity transformation.

Proof: We have

$$E^2(a_1, b_1) = E\big(E(a_1, b_1)\big) = E(a_2, b_2) = (a_3, b_3).$$

Hence,

$$E^3\,(a_1, b_1) = E\,\big(E^2(a_1, b_1)\big) = E\,(a_3, b_3) = (a_1, b_1).$$

Thus,

$$E^3\,(a_1, b_1) = (a_1, b_1).$$

Also,

$$E^3\,(a_2, b_2) = E\left(E^2(a_2, b_2)\right) = E\left(E\left(E(a_2, b_2)\right)\right) = E\left(E(a_3, b_3)\right)$$

$$= E\,(a_1, b_1) = (a_2, b_2).$$

Thus,

$$E^3\,(a_2, b_2) = (a_2, b_2).$$

Also,

$$E^3\,(a_3, b_3) = E^2\left(E(a_3, b_3)\right) = E^2\,(a_1, b_1) = (a_3, b_3).$$

Hence, we investigate transformations E such that

$E^3 = I$, identity transformation.

Theorem 3. Show that $E^3 = I$ holds if $m = 1$ and $n = -1$.

Proof: Let $E^3 = I$. Then

$$E^3\,(x, y) = I\,(x, y) = (x, y) \text{ for } (x, y) \in R^2.$$

Now by theorem 1,

$$(A'x + B'y + c_3, C'x + D'y + c_4) = (x, y)$$

$$\Rightarrow A' = 1, B' = 0, c_3 = 0, C' = 0, D' = 1, c_4 = 0.$$

Now, $B' = b\,(n^2 - m) = 0$

If $b = 0$, it is clearly satisfied.

If $b \neq 0$, then let $n^2 - m = 0 \Rightarrow n^2 = m.$

Also, $C' = c\,(n^2 - m) = 0.$

If $c = 0$, it is clearly satisfied.

If $c \neq 0,$ then since $n^2 - m = 0,$ it is satisfied.

So, let $n^2 = m,$ so that $B' = C' = 0.$

$A' = 1 \Rightarrow a(n^2 - m) - mn = 1 \Rightarrow -mn = 1$

$\Rightarrow mn = -1.$

$\Rightarrow n^2 n = -1 \quad \Rightarrow n^3 = -1$

$\Rightarrow n = -1.$

Hence, $m = n^2 = 1.$

Then, $D' = n^3 - an^2 - 2mn + am$

$= -1 - a + 2 + a = 1.$

Also, $c_3 = c_1(-a - 1 + a + 1) + bc_2(-1 + 1) = 0.$

$c_4 = cc_1(-1 + 1) + c_2(1 + a - 1 - 1 - a + 1) = 0.$

So, if $m = 1, n \ -1,$ then $E^3 = I$ holds.

Theorem-4

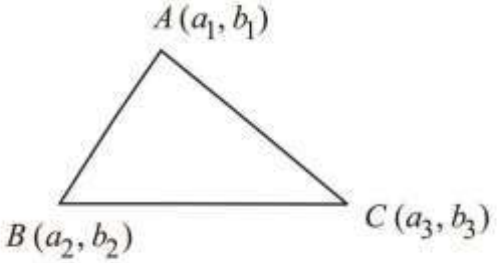

Figure 1

In triangle ABC as shown in Figure 1 and E as described earlier, if

$$a_1b_2 - a_2b_1 = r, a_2b_3 - a_3b_2 = s, a_3b_1 - a_1b_3 = t$$

and $r + s + t = k \neq 0,$ then

$$a = \frac{a_1(b_1 - b_2) + a_2(b_2 - b_3) + a_3(b_3 - b_1)}{k}$$

and $$b = -\left[\frac{a_1(a_1 - a_2) + a_2(a_2 - a_3) + a_3(a_3 - a_1)}{k}\right].$$

Proof: We have $E(a_1, b_1) = (a_2, b_2)$, $E(a_2, b_2) = (a_3, b_3)$ and $E(a_3, b_3) = (a_1, b_1)$.

Comparing 1^s coordinates of both sides, we have

$$a_2 = aa_1 + bb_1 + c_1 \qquad \text{... (1)}$$

$$a_3 = aa_2 + bb_2 + c_1 \qquad \text{... (2)}$$

$$a_1 = aa_3 + bb_3 + c_1 \qquad \text{... (3)}$$

Subtracting (1) from (2), (2) from (3), we have

$$a_3 - a_2 = a(a_2 - a_1) + b(b_2 - b_1)$$

$$a_1 - a_3 = a(a_3 - a_3) + b(b_3 - b_2).$$

or

$$a\,(a_1 - a_2) + b\,(b_1 - b_2) + a_3 - a_2 = 0$$

$$a\,(a_2 - a_3) + b\,(b_2 - b_3) + a_1 - a_3 = 0.$$

Let, $a_1 - a_2 = l_1,\ a_2 - a_3 = m_1, a_3 - a_1 = n_1$

and $b_1 - b_2 = l_1',\ b_2 - b_3 = m_1', b_3 - b_1 = n_1'.$

Then,

$$al_1 + bl_1' - m_1 = 0$$

$$am_1 + bm_1' - n_1 = 0.$$

Solving for *a* and *b*,

$$\frac{a}{m_1m_1' - l_1'n_1} = \frac{b}{l_1n_1 - m_1^2} = \frac{l}{l_1m_1' - l_1'm_1}$$

Now,

$$l_1m_1' - l_1'm_1 = (a_1 - a_2)(b_2 - b_3) - (b_1 - b_2)(a_2 - a_3)$$

$$= a_1b_2 - a_1b_3 - a_2b_2 + a_2b_3 - (a_2b_1 - a_3b_1 - a_2b_2 + a_3b_2)$$

$$= a_1b_2 - a_1b_3 + a_2b_3 - a_2b_1 + a_3b_1 - a_3b_2$$

$$= a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 + a_3b_1 - a_1b_3$$

$$= r + s + t = k \neq 0.$$

Also,

$$m_1m_1' - l_1'n_1 = (a_2 - a_3)(b_2 - b_3) - (b_1 - b_2)(a_3 - a_1)$$

$$= a_2b_2 - a_2b_3 + a_3b_3 - (a_3b_1 - a_1b_1 - a_3b_2 + a_1b_2)$$

$$= a_2b_2 - a_2b_3 - a_3b_2 + a_3b_3 - a_3b_1 + a_1b_1 - a_1b_2$$

$$= a_1(b_1 - b_2) + a_2(b_2 - b_3) + a_3(b_3 - b_1).$$

Hence,

$$a = \frac{a_1(b_1 - b_2) + a_2(b_2 - b_3) + a_3(b_3 - b_1)}{k}$$

Also,

$$l_1n_1 - m_1^2 = (a_1 - a_2)(a_3 - a_1) - (a_2 - a_3)^2$$

$$= a_1a_3 - a_1^2 - a_2a_3 + a_1a_2 - (a_2^2 + a_3^2 - 2a_2a_3)$$

$$= a_1a_3 - a_1^2 + a_2a_3 + a_1a_2 - a_2^2 - a_3^2$$

$$= -\left[a_1(a_1 - a_2) + a_2(a_2 - a_3) + a_3(a_3 - a_1)\right]$$

Hence,

$$b = -\left[\frac{a_1(a_1 - a_2) + a_2(a_2 - a_3) + a_3(a_3 - a_1)}{k}\right].$$

Theorem 5. In the triangle ABC as given in Figure 1, show that *c* and *d* are given by

$$c = \frac{b_1(b_1 - b_2) + b_2(b_2 - b_3) + b_3(b_3 - b_1)}{k}$$

and $$d = -\left[\frac{b_1(a_1 - a_2) + b_2(a_2 - a_3) + b_3(a_3 - a_1)}{k}\right].$$

Proof: We have,

$$E(a_1, b_1) = (a_2, b_2),\ E(a_2, b_2) = (a_3, b_3) \text{ and } E(a_3, b_3) = (a_1, b_1)$$

Comparing second coordinates on both sides,

$$ca_1 + db_1 + c_2 = b_2 \qquad \text{... (1)}$$

$$ca_2 + db_2 + c_2 = b_3 \qquad \text{... (2)}$$

$$ca_3 + db_3 + c_2 = b_1 \qquad \text{... (3)}$$

Subtracting (1) from (2) and (2) from (3), we have

$$b_3 - b_2 = c(a_2 - a_1) + d(b_2 - b_1)$$

$$b_1 - b_3 = c(a_3 - a_2) + d(b_3 - b_2).$$

This is equivalent to

$$c\,(a_1 - a_2) + d(b_1 - b_2) + b_3 - b_2 = 0$$

$$c(a_2 - a_3) + d(b_2 - b_3) + b_1 - b_3 = 0.$$

i.e.

$$cl_1 + dl_1' - m_1' = 0$$

$$cm_1 + dm_1' - n_1' = 0.$$

Solving for *c* and *d*,

$$\frac{c}{m_1'^2 - l_1'n_1'} = \frac{d}{l_1n_1' - m_1m_1'} = \frac{1}{l_1m_1' - l_1'm_1}$$

As in previous theorem,

$$l_1m_1' - l_1m_1' = k\ (\neq 0).$$

Now

$$\begin{aligned} m_1'^2 - l_1'n_1' &= (b_2-b_3)^2 - (b_1-b_2)(b_3-b_1) \\ &= b_2^2 + b_3^2 - 2b_2b_3 - (b_1b_3 - b_1^2 - b_2b_3 + b_1b_2) \\ &= b_2^2 + b_3^2 - b_2b_3 - b_1b_3 + b_1^2 - b_1b_2 \\ &= b_1(b_1-b_2) + b_2(b_2-b_3) + b_3(b_3-b_1). \end{aligned}$$

Hence,

$$c = \frac{b_1(b_1-b_2)+b_2(b_2-b_3)+b_3(b_3-b_1)}{k}.$$

Also,

$$\begin{aligned} l_1n_1' - m_1m_1' &= (a_1-a_2)(b_3-b_1) - (a_2-a_3)(b_2-b_3) \\ &= a_1b_3 - a_1b_1 - a_2b_3 + a_2b_1 - (a_2b_2 - a_2b_3 - a_3b_2 + a_3b_3) \\ &= a_1b_3 - a_1b_1 + a_2b_1 - a_2b_2 + a_3b_2 - a_3b_3 \\ &= -\left[b_1(a_1-a_2) + b_2(a_2-a_3) + b_3(a_3-a_1)\right]. \end{aligned}$$

Hence,

$$d = -\left[\frac{b_1(a_1-a_2)+b_2(a_2-a_3)+b_3(a_3-a_1)}{k}\right].$$

Theorem 6. Let $r+s+t = k \neq 0$, then the values of c_1, c_2 are given by

$$c_1 = \frac{a_1r + a_2s + a_3t}{k},\ c_2 = \frac{b_1r + b_2s + b_3t}{k}$$

Proof: From relation (1) in Theorem 4

$$c_1 = a_2 - aa_1 - bb_1$$

$$= a_2 - a_1\frac{[a_1(b_1-b_2)+a_2(b_2-b_3)+a_3(b_3-b_1)]}{k} + b_1\frac{[a_1(a_1-a_2)+a_2(a_2-a_3)+a_3(a_3-a_1)]}{k}$$

$$=\frac{ka_2-a_1^2(b_1-b_2)-a_1a_2(b_2-b_3)-a_1a_3(b_3-b_1)+a_1b_1(a_1-a_2)+a_2b_1(a_2-a_3)+a_3b_1(a_3-a_1)}{k}$$

The numerator is equal to

$$=a_2(a_1b_2-a_2b_1+a_2b_3-a_3b_2+a_3b_1-a_1b_3)-a_1^2(b_1-b_2)-a_1a_2(b_2-b_3)-a_1a_3(b_3-b_1)$$

$$+a_1b_1(a_1-a_2)+a_2b_1(a_2-a_3)+a_3b_1(a_3-a_1)$$

$$=a_1a_2b_2-a_2^2b_1+a_2^2b_3-a_2a_3b_2+a_2a_3b_1-a_1a_2b_3-a_1^2b_1+a_1^2b_2-a_1a_2b_2$$

$$+a_1a_2b_3-a_1a_3b_3+a_1a_3b_1+a_1^2b_1-a_1a_2b_1+a_2^2b_1-a_2a_3b_1+a_3^2b_1-a_1a_3b_1$$

$$=a_2^2b_3-a_2a_3b_2+a_1^2b_2-a_1a_3b_3-a_1a_2b_1+a_3^2b_1$$

$$=a_1^2b_2-a_1a_2b_1+a_2^2b_3-a_2a_3b_2+a_3^2b_1-a_1a_3b_3$$

$$=a_1(a_1b_2-a_2b_1)+a_2(a_2b_3-a_3b_2)+a_3(a_3b_1-a_1b_3)$$

$$=a_1r+a_2s+a_3t\,.$$

Hence,

$$c_1=\frac{a_1r+a_2s+a_3t}{k}$$

From relation (1) in theorem 5,

$$c_2=b_2-ca_1-db_1$$

$$=b_2-a_1\frac{[b_1(b_1-b_2)+b_2(b_2-b_3)+b_3(b_3-b_1)]}{k}+b_1\frac{[b_1(a_1-a_2)+b_2(a_2-a_3)+b_3(a_3-a_1)]}{k}$$

$$=\frac{\begin{array}{c}b_2k-a_1b_1(b_1-b_2)-a_1b_2(b_2-b_3)-a_1b_3(b_3-b_1)\\+b_1^2(a_1-a_2)+b_1b_2(a_2-a_3)+b_1b_3(a_3-a_1)\end{array}}{k}$$

Now Numerator is equal to

$$=b_2(a_1b_2-a_2b_1+a_2b_3-a_3b_2+a_3b_1-a_1b_3)-a_1b_1(b_1-b_2)-a_1b_2(b_2-b_3)$$

$$
\begin{aligned}
&- a_1b_3(b_3 - b_1) + b_1^2(a_1 - a_2) + b_1b_2(a_2 - a_3) + b_1b_3(a_3 - a_1)\\
&= a_1b_2^2 - a_2b_1b_2 + a_2b_2b_3 - a_3b_2^2 + a_3b_1b_2 - a_1b_2b_3 - a_1b_1^2 + a_1b_1b_2 - a_1b_2^2\\
&+ a_1b_2b_3 - a_1b_3^2 + a_1b_1b_3 + a_1b_1^2 - a_2b_1^2 + a_2b_1b_2 - a_3b_1b_2 + a_3b_1b_3 - a_1b_1b_3\\
&= a_2b_2b_3 - a_3b_2^2 + a_1b_1b_2 - a_1b_3^2 - a_2b_1^2 + a_3b_1b_3\\
&= a_1b_1b_2 - a_2b_1^2 + a_2b_2b_3 - a_3b_2^2 + a_3b_1b_3 - a_1b_3^2\\
&= b_1(a_1b_2 - a_2b_1) + b_2(a_2b_3 - a_3b_2) + b_3(a_3b_1 - a_1b_3)\\
&= b_1r + b_2s + b_3t\,.
\end{aligned}
$$

Hence,

$$c_2 = \frac{b_1r + b_2s + b_3t}{k}.$$

Theorem 7. Let $r+s+t = k \neq 0$ in a triangle *ABC* as shown in Figure 1, the show that

$$a + d = -1 \text{ and } ad - bc = 1.$$

Proof:

From theorems (4) and (5),

$$a + d = \frac{a_1(b_1 - b_2) + a_2(b_2 - b_3) + a_3(b_3 - b_1) - b_1(a_1 - a_2) - b_2(a_2 - a_3) - b_3(a_3 - a_1)}{k}$$

$$= \frac{a_1b_1 - a_1b_2 + a_2b_2 - a_2b_3 + a_3b_3 - a_3b_1 - a_1b_1 + a_2b_1 - a_2b_2 + a_3b_2 - a_3b_3 + a_1b_3}{k}$$

$$= \frac{- a_1b_2 - a_2b_3 - a_3b_1 + a_2b_1 + a_3b_2 + a_1b_3}{k}$$

$$= \frac{-(a_1b_2 - a_2b_1) - (a_2b_3 - a_3b_2) - (a_3b_1 - a_1b_3)}{k} = \frac{-r - s - t}{k} = \frac{-k}{k} = -1.$$

Also,

$$ad - bc = -\frac{[a_1(b_1 - b_2) + a_2(b_2 - b_3) + a_3(b_3 - b_1)][b_1(a_1 - a_2) + b_2(a_2 - a_3) + b_3(a_3 - a_1)]}{k^2}$$

$$+\frac{[a_1(a_1-a_2)+a_2(a_2-a_3)+a_3(a_3-a_1)]\,[b_1(b_1-b_2)+b_2(b_2-b_3)+b_3(b_3-b_1)]}{k^2}$$

Hence numerator, after putting $a_1 - a_2 = l_1, b_1 - b_2 = l_1'$ etc. and noting that

$$l_1 m_1' - l_1' m_1 = m_1 n_1' - m_1' n_1 = n_1 l_1' - n_1' l_1 = k$$

Numerator is equal to

$$= (a_1 l_1 + a_2 m_1 + a_3 n_1)\,(b_1 l_1' + b_2 m_1' + b_3 n_1') - (a_1 l_1' + a_2 m_1' + a_3 n_1')\,(b_1 l_1 + b_2 m_1 + b_3 n_1)$$

which is after simplification,

$$= (a_1 b_2 - a_2 b_1)\, l_1 m_1' + (a_2 b_1 - a_1 b_2)\, l_1' m_1 + (a_2 b_3 - a_3 b_2)\, m_1 n_1'$$

$$+ (a_3 b_2 - a_2 b_3)\, m_1' n_1 + (a_3 b_1 - a_1 b_3)\, l_1' n_1 + (a_1 b_3 - a_3 b_1)\, l_1 n_1'$$

$$= r\,(l_1 m_1' - l_1' m_1) + s\,(m_1 n_1' - m_1' n_1) + t\,(n_1 l_1' - l_1 n_1')$$

$$= rk + sk + tk = (r + s + t)\,k = k^2.$$

Hence $ad - bc = \frac{k^2}{k^2} = 1.$

References:

1. Dunford, N. and Schwartz. J. : Linear operators, Part I; Interscience Publishers, Inc., New York, 1967.
2. Edwards, R.E. : Functional Analysis; Holt, Reinhart and Winston, Inc., New York, 1965.
3. Lusternik, L.A. and Sobolev, V.J. : Elements of Functional Analysis : (Hindustan Publishing Corporation), India, 1961.
4. Taylor, A.E. : Introduction to Functional Analysis : Wiley, New York, 1958.

A Note on Two Person Zero-sum Game in Neutrosophic Environment

Abhishek Chakraborty[1], Sumanta Chakrabarti[2] and Runu Dhar[3*]
[1]Department of Mathematics, Maharaja Bir Bikram University, Agartala-799004, Tripura, India.
[2]Registrar, Maharaja Bir Bikram University, Agartala-799004, Tripura, India.
[3]Department of Mathematics, Maharaja Bir Bikram University, Agartala-799004, Tripura, India.
E-mail: [1]abcy3008@gmail.com,[2]imsumanta@gmail.com, [3]runu.dhar@gmail.com
∗Corresponding author: Runu Dhar (runu.dhar@gmail.com).

Abstract: In this paper, we discuss two-person zero-sum game in neutrosophic environment. We wish to define it in neutrosophic environment. Some basic properties and results of the newly defined game are to be investigated in neutrosophic environment. Our purpose is also to establish some theorems. Some numerical examples of game problems in neutrosophic environment are to be discussed.

Key words: Game, zero-sum, payoff matrix, optimal strategy, neutrosophic environment.

AMS 2020 Subject Classification No.: 03E72; 54A40; 91A86, 91A10, 91A80.

1. Introduction:

Game theory is a mathematical modeling technique used for decision making problems due to conflict or cooperation of two or more decision makers with each other. Each decision maker performs in order to outsmart the others. The wide application of game theory [14] is found in social policy and international and national policies, management etc. Krishnaveni and Ganesan [12] investigated the solution of two person zero sum fuzzy games and established effective and efficient tools for solving such game problems. Peski [15] referred necessary and sufficient conditions for zero-sum games. Players can exactly know all data in the traditional game problems. But in real-life situations, some games are found where players are not able to evaluate exactly some data due to inaccuracy of information and vague comprehension of situations by players. Berg and Engel [6] and Takahashi [22] introduced some techniques for finding the equilibrium strategies of these games.

To deal with real-life problems due to uncertainty, Zadeh [25] introduced the notion of fuzzy sets associating with membership values. It provides a natural way of dealing with game theory and other real-life problems in which the source of imprecision and vagueness occur. Atanassov [1] incorporated the concept of intuitionistic fuzzy sets concerning non-membership values of members including membership values and it plays an important role to deal with game theory as well as other real-life problems due to uncertainty. Thereafter, Smarandache [20, 21] recommended for membership, non-membership and indeterministic values and invented neutrosophic set which can be applied successfully to tackle decision making and game theory problems. Dubois and Prade [10] referred the use of algebraic operations on real numbers to fuzzy numbers. Bellman and Zadeh [5] focused on understanding the decision-making processes of participants in these scenarios. Selvakumari and Lavanya [17] and Thirucheran et al. [24] accelerated fuzzy game. Campos [7] studied fuzzy linear programming models to solve fuzzy matrix games. Sakawa and Nishizaki [16] used max-min principle of game theory under fuzzy environment. Bector et al. [3, 4] further studied on two-person zero-sum matrix game with the concept of linear programming models. Li and Hong [13] proposed an approach for solving constrained matrix games with triangular fuzzy numbers. Chen and Larboni [8] referred matrix game with triangular membership function. Sharma et al. [19] studied fuzzy linear programming models for the solution of two-person zero-sum game with triangular fuzzy number. Bandyopadhyay et al. [2] studied matrix game with triangular intuitionistic fuzzy numbers. Seikh et al. [18] talked on matrix games in intuitionistic fuzzy environment. Das and Chakraborty [9] talked about trapezoidal neutrosophic linear programming problems.

The outlay of the paper is organized as follows: In section 2, basic concepts and results which are relevant for the investigation are procured. In section 3, two-person zero-sum game in neutrosophic environment is defined. Section 4 focuses on two-person zero-sum game in crisp environment. We formulate and comment about reasonable solutions of the same concept for neutrosophic environment in section 5 and section 6 respectively. We discuss two numerical examples of two-person zero-sum game in neutrosophic environment in section 7. Finally, some concluding remarks are reported in section 8.

2. Preliminaries:

In this section, some known results and definitions would be procured for ready reference.

Definition 2.1. [14] Matrix game theory gives a scope to a player to determine the opponents' possible courses along his own courses of action in order to win the game. If a game involves only two players, namely, players I and II, then it is called two-person game. The game involving more than two players is referred to as n-person game. We consider player I is maximizing player who always aims to make maximum gain and player II is minimizing player who always tries to minimize his loss. A game of two players is termed a two-person zero-sum game, if losses of one is equivalent to the gains of other so that total sum is zero.

Definition 2.2. [14] The strategy for a player is a set of rules that will specify which of the available courses of action will be chosen at each play. There are two types of strategy- Pure strategy and Mixed strategy.

(i) Pure strategy is a decision-making rule in which a player will select a particular course of action.

(ii) Mixed strategy refers a rule in which a player decides, in advance to choose his course of action with some definite probabilities. In mixed strategies, a player is always kept guessing about the opponents choice. More specially, mixed strategy is a selection of pure strategy with some probability.

Definition 2.3. [14] The payoffs in terms of gains or losses, when players select their courses of action, can be represented in the form of a matrix, called the payoff matrix. Let the player I has m pure strategies $A_1, A_2,\ldots\ldots,A_m$ and the player II has n pure strategies $B_1, B_2,\ldots\ldots,B_n$. Let a_{ij} denotes the payoff to the player I, when player I chooses a pure strategy A_i and player II chooses a pure strategy B_j. Then the payoff matrix to the player I can be represented as an m $\times\ n$ matrix as

$$\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n} \\ .. & \cdots & \ldots \\ a_{m1}a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The payoff matrix for the player II can simply be constructed by replacing a_{ij} by $-a_{ij}$. In this report, we have considered twoperson matrix games in which elements of payoff are imprecise in nature rather than exact numbers.

Definition 2.4. [14] The game problem is viewed in such a manner that each player is interested in determining his optimal strategies which guarantees a payoff that can never be worsened by the choice of his opponent. It is referred to as maximin-minimax principle.

Definition 2.5. [14] For maximizing player, minimum value in each row represents the least pay-off to him if he chooses this particular strategy. He then selects the strategy which provides largest gain among row minimum values. Such choice is called maximin principle and corresponding gain is called the maximin value of the game.

Definition 2.6. [14] For minimizing player, maximum value in each column represents the maximum loss to him if he chooses this particular strategy. He then selects the strategy that gives minimum loss among the column maximum values. Such choice is called minimax principle and corresponding loss is called the minmax value of the game.

Definition 2.7. [14] A saddle point of a payoff matrix is that position in the payoff matrix where maximin value coincides with minimax value. Thus, in pay-off matrix $(a_{ij})_{m\times n}$the (p, q)-th position will be called a saddle point if and only if a_{pq} = max{mina_{ij}} = min{maxa_{ij}}.
The payoff at the saddle point is known as value of the game and the corresponding strategies are called optimal strategies.

Definition 2.8. [25] Let X be a universal set. A fuzzy set A in X is characterized by a membership function μ_A which associates each point x in X with a real number in [0, 1], where$\mu_A(x)$ at x representing the "grade of membership" of X in A.
Let X be a universal set. A Fuzzy set A in X is a set of order pair as defined below:

A = {(x, $\mu_A(x)$): $x \in X$}

where

μ_A: $X \rightarrow [0,1]$ and $\mu_A(x)$ is the degree of belongingness of x in A.

Definition 2.9. [1] An intuitionistic fuzzy set A in a non-empty set X is $A = \{(x, \mu_A(x), \nu_A(x) : x \in X\}$, where $\mu_A(x), \nu_A(x) : X \longrightarrow [0, 1]$ provide respectively, the degree of membership and non - membership of x to the set A, a subset of X and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each x. Further, we have $\pi_A(\mathrm{x}) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(\mathrm{x})$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(\mathrm{x}) \in [0, 1]$ i.e., $\pi : X \longrightarrow [0, 1]$ and $0 \leq \pi_A(\mathrm{x}) \leq 1$ for every $x \in X$. $\pi_A(\mathrm{x})$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Remark 2.10. If $\pi_A(x) = 0\ \forall\ x \in A$, then the intuitionistic fuzzy set reduces to fuzzy set i.e. every fuzzy set is an intuitionisitic fuzzy set but converse may not be true.

Example 2.11. A real-life example of Intuitionistic Fuzzy Sets (IFS) is evaluating "Has Disease" of a patient. Let us consider “Patients with symptoms” be the universe of discourse. The doctor has found that particular patient ‘P’ having IFS elements be (0.7, 0.2, 0.1) for a disease, meaning 70% chance of having it, 20% chance of not and 10% uncertainty due to unclear signs.

Definition 2.12. [20] A neutrosophic set N in an universal set U is a set containing the triplet, namely, truthness, falseness and indeterminacy membership values taken from [0, 1]. These can be characterized independently and denoted by T_N, F_N, I_N respectively. We denote it as below:

$N = \{(x, T_N(x), F_N(x), I_N(x)) : x \in U$ and $T_N(x), F_N(x), I_N(x) \in [0,1]\}$ where$0 \leq T_N(x) + F_N(x) + I_N(x) \leq 3$.

Example 2.13. A soccer club wants to hire the best new player from a set of candidates. The manager evaluates each candidate based on several criteria (e.g., speed, age, height, experience). In traditional set theory, a player is either "fast" (1) or "not fast" (0). In fuzzy logic, a player might be fast with membership value 0.8 (a degree of truthness).

A neutrosophic set, however, accounts for incomplete, inconsistent, or indeterminate information by assigning three independent membership values to each criterion for a given player as given below:

- **Truth membership (T):** The degree to which the player meets the criterion (e.g., is a good player).
- **Indeterminacy membership (I):** The degree to which the information is vague, unknown, or uncertain (e.g., the manager is unsure about the goodness of a new player).
- **Falsity membership (F):** The degree to which the player does not meet the criterion.

For a specific player A, the manager might assign a neutrosophic triplet like this:

“Player A is a good player = (0.7, 0.1, 0.4)”. It indicates the following:

- The coach is 70% sure the player is good (T = 0.7).
- The coach has a 10% doubt or indeterminacy (I = 0.1).

- The coach has a 40% belief that the player is not good (F = 0.4).

3. Two-person zero-sum neutrosophic game:

Definition 3.1. A two-person zero-sum fuzzy game extends classic game theory by incorporating fuzzy logic, where payoffs aren't precise numbers but fuzzy sets (like triangular fuzzy numbers) to handle real-world uncertainty, while still maintaining the zero-sum rule (Player I's gain equals Player II's loss). It models situations where outcomes are vague, using fuzzy numbers for payoffs and solving techniques that handle uncertainty, unlike crisp games with exact values.

Example 3.2. Let us consider that there are two companies. We referrer these two companies as Player A and Player B. Suppose that these two companies have decided to launch a new product in the market for business. It is assumed that success of the business depends on 'market condition'. Here the 'market condition' is a fuzzy concept. Let us consider that there are two strategies, namely, (a) Aggressive Marketing and (b) Conservative Marketing, are available for both the players. Instead of fixed profits, payoffs are fuzzy as described below:

(i) A (Aggressive) vs. B (Aggressive) : "Moderate Profit"
(ii) A (Aggressive) vs. B (Conservative) : "High Profit".
(iii) A (Conservative) vs. B (Aggressive): "Low Profit".
(iv) A (Conservative) vs. B (Conservative): "Medium Profit".

Example 3.3. A two-person zero-sum intuitionistic fuzzy game extends classic zero-sum games (where one player's gain is the other's loss) by allowing payoff values to be intuitionistic fuzzy numbers, representing vagueness with membership (belief) and non-membership (disbelief) degrees, capturing uncertainty better than crisp numbers or standard fuzzy numbers.

Example 3.4. Real-life examples of intuitionistic fuzzy (IF) game problems involve complex decision-making where outcomes aren't just 'yes/no' but include 'may be' (hesitation). like market share battles between companies (satisfaction vs. rejection vs. abstention), Let us consider that there are two real estate companies competing for market share. We referrer these two companies as Player A and Player B. Suppose that these two companies have decided to launch a new product in the market for market share. Payoffs aren't exact numbers but IF numbers. We consider Player A's payoff (Satisfaction: 0.7, Rejection: 0.1, Hesitation: 0.2) reflects 70% satisfied customers, 10% opposed and 20% abstaining/undecided. We also consider Player B's payoff (Satisfaction: 0.6, Rejection: 0.2, Hesitation: 0.2) reflects 60% satisfied customers, 20% opposed and 20% abstaining/undecided. This captures market uncertainty better than crisp numbers.

Definition 3.5. A two-person zero-sum game in neutrosophic environment extends the classic game by representing payoffs as neutrosophic members, where players express not just the degree of membership (likelihood of a payoff) but also the degree of non-membership as well as degree of hesitation, keeping view in mind that the membership function, non-membership function and hesitation function are independent to each other and their sum is less than or equal to 3. It allows for better modeling of real-world vagueness where payoffs aren't precise, leading to solution methods using ranking functions and possibility/credibility measures for neutrosophic sets.

Example 3.6. Real-life examples of neutrosophic fuzzy game problems involve complex decision-making with uncertainty. International Conflict Resolution is a real-life example of neutrosophic game problem. Political negotiations between two conflicting countries, such as the Israel-Palestine issue, involve significant ambiguity in intentions, promises and potential outcomes. Neutrosophic game theory can model the degrees of cooperation, non-cooperation, and the inherent ambiguity (indeterminacy) in the actions and speeches made by these two countries.

4. Two-person zero-sum game in crisp environment:

We consider a payoff matrix A = (a_{ij}), (i = 1, 2, . . . ,p; j = 1, 2, . . . , q) of real numbers for such games. If player I chooses to play row i and player II chooses to play column j, then the payoff to player I is a_{ij} and that of player II is − a_{ij}. They wish to select strategies which benefit them individually.

Take S_1(= {α_1, α_2, . . . , α_p}) and S_2(= {β_1, β_2, . . . , β_q}) as pure strategies. We consider mixed strategies for two players I and II as: Y = {y = (y_1, y_2, . . . , y_p)^T :$\sum_{i=1}^{p} y_i$ is1, $y_i \geq 0$, i = 1, 2, . . . , p} and Z = {z = (z_1, z_2, . . . , z_q)^T: $\sum_{j=1}^{q} z_j$is1, $z_j \geq 0$, j = 1, 2, . . . , q} respectively. If player I chooses $\alpha_i \in S_1$with probability (i = 1, 2, . . . , p) and Player II chooses $\beta_j \in S_2$with probability z_j (j = 1, 2, . . . , q), then the game is defined as G ≡ (Y, Z, A).

5. Two-person zero-sum game in triangular neutrosophic environment:

In real World situation, there are many game problems where pay off elements are in imprecise nature. We consider the payoff matrix$\widetilde{M}$where $\widetilde{M}$= ($\widetilde{m_{ij}}$) with triangular neutrosophic numbers (in short, TNNs) and the corresponding game is G ≡ (Y,Z, $\widetilde{M}$). We consider two players, player I (maximizing player) and player II (minimizing player) of the game. Player I and player II wish to use the best strategies for maximizing the gain and minimizing the loss respectively. The maximin and minimax principle provides the best strategies for both the players. The game possesses a saddle point when maximin for player I and minimax of player II are equal.

Let (p, q)-th position of $\widetilde{M}$= $\widetilde{(m_{ij})_{r\times s}}$, the payoff matrix, is a saddle point, then,

$$\bar{a} =\{(m_{pq}^1, m_{pq}^2, m_{pq}^3, m_{pq}^4); \alpha_{\overline{m_{pq}}}, \beta_{\overline{m_{pq}}}, \gamma_{\overline{m_{pq}}}\}$$

$$= \underbrace{max}_{i}\{\underbrace{min}_{j}\{(m_{ij}^1, m_{ij}^2, m_{ij}^3, m_{ij}^4); \alpha_{\overline{m_{ij}}}, \beta_{\overline{m_{ij}}}, \gamma_{\overline{m_{ij}}}\}$$

$$= \underbrace{min}_{j}\{\underbrace{max}_{i}\{(m_{ij}^{1}, m_{ij}^{2}, m_{ij}^{3}, m_{ij}^{4}); \alpha_{\overline{m_{ij}}}, \beta_{\overline{m_{ij}}}, \gamma_{\overline{m_{ij}}}\}.$$

This implies that the saddle point entry is, (p, q)-th position which is the value of the game. Here$\alpha_{\overline{m_{ij}}}$, $\beta_{\overline{m_{ij}}}$, *and* $\gamma_{\overline{m_{ij}}}$ respectively denote the degree of acceptance, degree of indeterminacy and degree of non-acceptance of$(m_{ij}^{1}, m_{ij}^{2}, m_{ij}^{3}, m_{ij}^{4})$.If $\exists\widetilde{v^*}\in$ V and $\widetilde{w^*}\in$ W and$\nexists$ $\tilde{v}$ and $\widetilde{w}$such that $\widetilde{v^*} \lesseqqgtr \tilde{v}$ and $\widetilde{w^*} \gtreqqless \widetilde{w}$, where, V and W indicate all reasonable game values $\tilde{v}$ and $\widetilde{w}$ for players I and II respectively, then $(y^*, z^*, \widetilde{v^*}, \widetilde{w^*})$ is called a solution of the triangular neutrosophic matrix game (in short, TNMG). And y∗ and z∗ are called maximin and minimax strategies for players I and II respectively and $\widetilde{v^*}$ and $\widetilde{w^*}$are called respectively player I's gain-floor and player II's loss-ceiling. Let $\widetilde{\pi^*} = \widetilde{v^*} \wedge \widetilde{w^*}$ with the membership function $\alpha_{\widetilde{\pi^*}}$(x) = min$\{\alpha_{\widetilde{v^*}}$(x),$\alpha_{\widetilde{w^*}}(x)\}$. Then $\widetilde{\pi^*}$ is called a fuzzy value of TNMG. Also, y∗ and z∗, maxi-min and mini-max strategies for players I and II respectively are obtained by solving fuzzy mathematical programming problems, given below:

maximize $\tilde{v}$

$$\text{subject}\begin{cases} y^T\widetilde{M}z \gtreqqless \breve{v}, \ z\epsilon Z \\ y\epsilon Y, \breve{v}\ \epsilon TNN(R) \end{cases}$$

and minimize $\widetilde{w}$

$$\text{subject}\begin{cases} y^T\widetilde{M}z \lesseqqgtr \breve{w}, \ y\epsilon Y \\ z\epsilon Z, \breve{w}\ \epsilon TNN(R). \end{cases}$$

Here'$\gtreqqless$ ' and '$\lesseqqgtr$ 'denote neutrosophic inequalities and 'TNN(R)' indicates triangular neutrosophicreal numbers.

E($\widetilde{M}$) = $y^T\widetilde{M}z$

$$= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{s}\widetilde{a_{ij}}\, y_i z_j$$

$$= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{s}\{(m_{pq}^{1}, m_{pq}^{2}, m_{pq}^{3}, m_{pq}^{4}); \ \alpha_{\overline{m_{pq}}}, \beta_{\overline{m_{pq}}}, \gamma_{\overline{m_{pq}}}\}\, y_i z_j$$

$$= \{(\textstyle\sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^{1}\, y_i z_j, \sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^{2}\, y_i z_j, \sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^{3}\, y_i z_j, \sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^{4}\, y_i z_j \alpha_{\overline{m_{pq}}}, \beta_{\overline{m_{pq}}}, \gamma_{\overline{m_{pq}}}\}$$

and

E$(\widetilde{-M}) = y^T(\widetilde{-M})$z

$= \sum_{i=1}^{r}\sum_{j=1}^{s}(\widetilde{-a_{ij}})y_i z_j$

$= \sum_{i=1}^{r}\sum_{j=1}^{s}\{-(m_{pq}^1, m_{pq}^2, m_{pq}^3, m_{pq}^4);\ \alpha_{\overline{m_{pq}}}, \beta_{\overline{m_{pq}}}, \gamma_{\overline{m_{pq}}}\} y_i z_j$

$=$

$\{-(\sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^1\, y_i z_j, \sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^2\, y_i z_j, \sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^3\, y_i z_j, \sum_{i=1}^{r}\sum_{j=1}^{s} m_{pq}^4\, y_i z_j \alpha_{\overline{m_{pq}}}, \beta_{\overline{m_{pq}}}, \gamma_{\overline{m_{pq}}}\}$.

Both E$(\widetilde{M})$ and E$(\widetilde{-M})$ are TNNs.

6. Reasonable solutions and strategies:

Assume $\tilde{v} = \{(v^1, v^2, v^3, v^4);\ \alpha_{\tilde{v}}, \beta_{\tilde{v}}, \gamma_{\tilde{v}}\} \in TNN(\mathbb{R})$ and $\tilde{w} = \{(w^1, w^2, w^3, w^4);\ \alpha_{\tilde{w}}, \beta_{\tilde{w}}, \gamma_{\tilde{w}}\} \in TNN(\mathbb{R})$ be two triangular neutrosophic number. Suppose that $\exists y^* \in$ Y, $z^* \in$ Z. Then $(y^*, z^*, \tilde{v}, \tilde{w})$ refers a reasonable (highly acceptable) solution of the matrix game and for any$y^* \in$ Y and $z^* \in$ Z, we have $y^{*^T}\widetilde{M}$z$\gtrapprox \tilde{v}$ and $y^T\widetilde{M}z^* \lessapprox \tilde{w}$. Then $\tilde{v}\ and \tilde{w}$indicate reasonable values for players I and II respectively for a reasonable solution $(y^*, z^*, \tilde{v}, \tilde{w})$ of the NMG. Similarly$y^*\ and\ z^*$refer reasonable strategies for players I and II respectively.

7. Numerical Examples:

Thamaraiselvi and Santhi [23] defined single valued trapezoidal neutrosophic number. H. A. Khalifa [11] explained the procedure for the solution two person zero-sum matrix game with trapezoidal neutrosophic numbers. Here we discuss the solutions of two numerical examples.

Example 7.1. Game Description:

Consider a game between two players, Player I and Player II. Player I has two strategies: A$_1$ and A$_2$. Player II has two strategies: B$_1$ and B$_2$. The payoffs are represented by neutrosophic numbers.

Payoff matrix The payoff matrix for Player I is:

Strategies	B$_1$	B$_2$
A$_1$	(0.6, 0.2, 0.2)	(0.3,0.4,0.3)
A$_2$	(0.4, 0.3, 0.3)	(0.8,0.1,0.1)

The payoff matrix for Player II is the negative of the above matrix.

Solution: To solve this game, we can use the score function method. Let us define a score function S for a neutrosophic number A = (T, I, F) as:

S(A) = T − F

Score Matrix:

Table: The score matrix for Player I

Strategies	B_1	B_2
A_1	0.4	0.0
A_2	0.1	0.7

Optimal strategies: The optimal strategies can be found using the minimax theorem. Let us find the minimum score for each row:
For A_1: min(0.4, 0) = 0
For A_2: min(0.1, 0.7) = 0.1
The maximum of these minimum scores is 0.1 which corresponds to strategy A_2.
Optimal strategy for Player I: The optimal strategy for Player I is A_2.
Optimal Strategy for Player II:
To find the optimal strategy for Player II, we need to find the maximum score for each column:
For B_1: max(0.4, 0.1) = 0.4
For B_2: max(0, 0.7) = 0.7
The minimum of these maximum scores is 0.4, which corresponds to strategy B_1.
Optimal Strategy for Player II: The optimal strategy for Player II is B_1.

Value of the game: The value of the game is the score corresponding to the optimal strategies:
$V = (0.4, 0.3, 0.3)$
The score of the value is:
$S(V) = 0.4 - 0.3 = 0.1$.

Interpretation and conclusion:
Optimal strategies: Player I should choose strategy A_2 to maximize their payoff.
Player II should choose strategy B_1 to minimize Player I's payoff.
Value of the game: The value of the game is (0.4, 0.3, 0.3) which represents the neutrosophic payoff. The score of the value is 0.1, which indicates that Player I has a slight advantage.

Interpretation: Player I's advantage: Player I has a slight advantage in the game, as the score of the value is positive (0.1).
Uncertainty: The neutrosophic payoff (0.4, 0.3, 0.3) indicates that there is uncertainty and indeterminacy in the game which can be due to various factors such as lack of information or randomness.

Example 7.2. Game Description:

Let us consider the following two-person zero-sum matrix game with trapezoidal neutrosophic numbers:

$$\begin{bmatrix} (0,1,3,6);7,5,3 & (5,8,10,14);3,6,6 & (5,8,10,14);0 \\ (9,11,14,16);5,4,7 & (12,15,19,22);6,4,5 & (3,5,6,8);6,5,4 \\ (5,8,10,14);3,6,6 & (3,5,6,8);6,5,4 & (15,17,19,22);4,8,4 \end{bmatrix}$$

Solution: The corresponding crisp payoff:

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & 7 & 2 \\ 3 & 5 & 6 \end{bmatrix}$$

max (y_1+ y_2+ y_3)

Subject to:

$y_1+3y_2+0y_3\leq 1$

$4y_1+7y_2+2y_3\leq 1$

$3y_1+5y_2+6y_3\leq 1$

y_1, y_2, $y_3\geq 0$.

Table1:The optimal strategy of player II

Variables	Optimal strategies	Game Value
y_1	0	9.769
y_2	0.231	
y_3	0.769	

max (x_1+ x_2+x_3)

Subject to:

$4x_1+11x_2+x_3\geq 1$

$3x_1+12x_2+2x_3\geq 1$

$5x_1+9x_2+7x_3\geq 1$

$6x_1+10x_2+7x_3\geq 1$

$x_1, x_2, x_3 \geq 0$.

Table2: The optimal strategies of Player I

Optimal strategies	Game Value
$x_1 = 0$ $x_2 = 0.923$ $x_3 = 0.077$	9.769

It is clear from the duality theorem that the optimal strategy of player I is: (0, 0.923, 0.077) which are the coefficients slack variables in the final table of the simplex method.

Player I	Player II	Game value
$x_1 = 0$	$y_1 = 0$ $y_2 = 0$ $y_3 = 0.231$, $y_4 = 0.769$	(13.5625, 16.3488, 21.0241, 27.4141)
$x_2 = 0.923$		
$x_3 = 0.077$		

8. Conclusion:

NS deals with decision making problems occurring indeterminacy due to uncertainty. In this paper, we have introduced two-person zero-sum game in neutrosophic environment. We have explained two-person zero-sum games in crisp as well as triangular neutrosophic environment. We have defined the reasonable solutions and strategies of such games. Lastly, two numerical examples have been illustrated. We hope the article will be helpful for future research work of games in neutrosophic environment.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986) 87 - 96.

[2] S. Bandyopadhyay, P. K. Nayakand M. Pal, Solution of matrix game with triangular intuitionistic fuzzy payoff using score function, *Open Journal of Optimization*, 2 (2013)9 -15.

[3] C. R. Bector, S. Chandra and V. Vijay, Duality in linear programming with fuzzy parameters and matrix games with fuzzy payoffs, *Fuzzy Sets and Systems*,146 (2004), 253- 269.

[4] C. R. Bector, S. Chandra and V. Vijay, Matrix games with fuzzy goals and fuzzy linear programming duality. *Fuzzy Optimization and Decision Making*, 3 (2004), 255- 269.

[5] R. E. Bellmanand L. A. Zadeh, Decision making in a fuzzy environment, *Management Science*,17 (1970), 141-164.

[6] J. Berg and A. Engel, Matrix games, mixed strategies and statistical mechanics,*Physical Review Letters,*81(1998) 4999- 5002.

[7] L. Campos, Fuzzy linear programming models to solve fuzzy matrix games, *Fuzzy Sets and Systems*, 32 (1989) 275- 289.

[8] Y. W. Chen and M. Larboni, Two person zero-sum game approach for fuzzy multiple attribute decision-making problem, Fuzzy Sets and Systems, 157(2006) 34- 51.

[9] S. K. Das and A. Chakraborty, A new approach to evaluate linear programming problem in pentagonal neutrosophic environment, *Complex Intell. Syst.*, 7(1) (2021) 101 - 110.

[10] D. Dubois and H. Prade, Possibility Theory: An Approach to Computerized Processing of Uncertainty, Plenum, New York, (1980).

[11] H. A. Khalifa, An approach for solving two-person zero-sum matrix games in neutrosophic environment, *Journal of Industrial and Systems Engineering,* 12(2) (2019) 186 - 198.

[12] G. Krishnaveni and K. Ganesan, New approach for the solution of two-person zero-sum fuzzy games, *International Journal of Pure and Applied Mathematics,* 119(9)(2018) 405- 414.

[13] F. D. Li and X. F. Hong, Solving constrained matrix games with payoffs of triangular fuzzy numbers, *Computers & Mathematics with Applications,* (64) (2012) 432- 448.

[14] J. V. Neumann and D. Morgenstern, The Theory of Games in Economic Behavior, Wiley, New York, (1944).

[15] M. Peski, Comparison of information structures in zero-sum games,*Games and Economic Behavior,* 62(2)(2008) 732- 735.

[16] M. Sakawa and I. Nishizaki, Max-min solutions for fuzzy multiobjective matrix games,*Fuzzy Sets and Systems,* (67) (1994) 53- 69.

[17] K. Selvakumari and S. Lavanya, An approach for solving fuzzy game problem,*Indian Journal of Science and Technology,* 8(15) (2015) 1-6.

[18] M. R. Seikh, K. P. Nayakand M. Pal, Matrix games in intuitionistic fuzzy environment, *International Journal of Mathematics in Operational Research,*5(6) (2013) 693 - 708.

[19] G. Sharma, S. K. Das and G. Kumar, Solving zero-sum two-person game with triangular fuzzy number payoffs using new fully fuzzy linear programming models,*OPSEARCH,*60 (2023) 1456 - 1487.

[20] F. Smarandache, Neutrosophy. Neutrosophic Probability, Set and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 (1998).

[21] F. Smarandache, A Unifying Fieldinlogics, Neutrosophy: Neutrosophic Probability, Set and Logic, American Research Press (1999).

[22] S. Takahashi, The number of pure Nash equilibria in a random game with nondecreasing best responses, *Games and Economic Behavior,* 63(1 (2008) 328- 340.

[23] A. Thamaraiselvi and R. Santhi, A new approach for optimization of real-life transportation problem in neutrosophic environment, *Mathematical Problem in Engineering,* Article ID 5950747, doi: 10.1155/ 20l6/ 5950747, (2016), 1 - 9.

[24] M. Thirucheran, R. E. Meena and S. Lavanya, A new approach for solving fuzzy game problem, *International Journal of Pure and Applied Mathematics,* 114(6) (2017) 67- 75.

[25] L. A. Zadeh, Fuzzy sets, *Information and Control,* 8 (1965) 338 - 353.

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Effect of cannibalism in both prey and predator with seasonal variation

Debadatta Adak*[a], Shekhar Das[b]
Department of Mathematics, Maharaja Bir Bikram University, Agartala, Tripura, India
Email: [a]dev.adak.math95325@gmail.com, and [b]shekhar2018agt@gmail.com
* Corresponding author. Email Id: dev.adak.math95325@gmail.com

Abstract: Cannibalism, also known as intraspecific predation, is a significant and widely observed phenomenon in various animal species ranging from protozoa to mammals. Numerous researches suggest that cannibalism can significantly alter the dynamics of the interacting species, especially the prey-predator systems. Most of the researchers found so far emphasize on predator cannibalism where adult predators predate on juvenile ones. However, in nature there are also evidence of cannibalism in prey population. In this study we consider a prey-predator system where both of prey and predator practices cannibalism. We have proposed and analyzed a simple three-dimensional model with the help of nonlinear ordinary differential equations. Predator population has been divided age structurally in two groups, namely adult and juvenile. The existence and stability conditions of the equilibrium points have been determined analytically and numerically. Moreover, we have incorporated seasonal environmental variations on the growth rates of prey and juvenile predator populations. This effect of seasonality is considered as the sinusoidal perturbation on the growth rates of prey & juvenile predator and significantly affects the system dynamics by causing high periodic fluctuations that is often observed in nature. Therefore, the obtained results are helpful in understanding the impact of cannibalism on prey – predator systems as well as on the survival of the interacting species.

Keywords: Prey-predator system, age-structure, cannibalism, mathematical model, seasonality, stability analysis, Hopf bifurcation, chaos.
Mathematics Subject Classification: 34D, 34H, 90A, 92B

1 Introduction

Cannibalism is the act of intraspecific predation, that includes both killing and whole or partial consumption of conspecific individuals. The evidence of cannibalism has been documented in different human cultures and populations all over the world. Different indigenous South American and New Guinean tribes practice the act of cannibalism as a social norm **[1]**. The cases of human cannibalism have also reported in various native tribes of Africa **[2]**. Though cannibalism is considered an inhuman and barbaric practice among the human civilizations, it is quite a common trait in ecology, especially in prey – predator systems **[3]**. Several important factors such as population density, age structure, stage structure, population size, temperature etc. can influence the occurrence of cannibalism **[4]**. Active cannibalistic behavior can be observed in more than 1300 animal species in nature **[5]** for example, insects **[6]**, primates **[7]**, frogs **[8]**, fish **[9]**, carnivore mammals **[10]**, spiders **[11]**, etc. are to name a few.

In nature it is often observed that, cannibalistic practice can alter population dynamics significantly. It has been documented that a population, which is at the verge of extinction, can persist through cannibalism **[12]**. It can also stabilize an otherwise unstable population **[13]**. Various researchers have studied the effect of cannibalism on intra- and inter-specific systems. A single species cannibalism model has been studied by Kang et al. **[14]**. In this study they have age structurally divided the population in two classes. One class is eggs and other is the adult class. A three-dimensional stage structured prey-predator model with predator cannibalism has been analyzed by Zhang et al. **[15]**. They divided the predator population into two subclasses such as juvenile predator and adult predator, considering adults prey on juveniles following Holling Type – I functional response. They have performed local and global stability analysis of the equilibrium points incorporating bifurcation analysis. Few researchers have explored the destabilizing effect of cannibalism in two species models **[12, 16–18]**. Basheer et al. **[19]** studied a three species prey-predator system and showed that cannibalism can destabilize the population dynamics. The effect of maturation delay on a diffusive cannibalistic predator–prey model is studied by Li et al. **[20]**. Sun et al. **[21]** proposed and analyzed a spatiotemporal prey-predator model that incorporates predator cannibalism. The stabilizing effect of cannibalism has been studied by **[13, 22]**. Buonomoa et al. **[23]** established the stabilizing effect of cannibalism in a stage structured population. The stabilizing consequences of adult-on-juvenile cannibalism has been investigated by Cushing et al. **[24]** who proposed some matrix models and applied bifurcation theory.

It is interesting to observe that majority of the prey-predator systems that incorporates cannibalism, considers only predator cannibalism. But there are strong evidences in ecology that cannibalism also occurs in prey population **[25, 26]**. To analyze the impact of pry cannibalism, Rudolf **[27]** studied a prey predator system where the dragonfly-larvae, both large and small, acts as the prey for other predator species such as tadpoles, frogs etc. In this prey-predator system the larger dragonfly larvae cannibalize on the smaller larvae and it has been established that such prey cannibalism can alter the system dynamics significantly. If fact due to this prey cannibalism the predator's functional response alters and decreases the predator induced prey mortality almost by 47% **[27]**. The effect of prey cannibalism in a multi component structured model has been studied by Solis et al. **[28]** with the help of integer differential equations. Chow and Jang **[29]** studied a discrete dynamical model considering prey cannibalism and found that the prey cannibalism can either stabilize or destabilize the system dynamics, depending on the rate of cannibalism rates and other sensitive system parameters. A three species structured deterministic model have been proposed and analyzed by Zhang et al. **[30]**. By considering prey cannibalism they established that the prey cannibalism can stabilize the system, as well as can emphasize the persistence of the coexistence of the interacting species.

In this study we propose a simple stage structured deterministic prey-predator model with the help of nonlinear differential equations to explore the simultaneous effect of prey and predator cannibalism on a prey-predator system.

2 Mathematical model

Let, $X(t), U(t), V(t)$ respectively be the prey, juvenile predator and adult predator density at any time t. To formulate our model, we consider the following assumptions:

Assumption 1: The adult predators $V(t)$ predate the prey population $X(t)$ as well as the juvenile predators represented by $U(t)$. The rate of predation by adult predator on prey is represented by the parameter γ and the same on juvenile predator is denoted by the parameter λ.

Assumption 2: Prey population grows at a rate r_1 and experience intra-specific competition at a rate κ.

Assumption 3: The birth rate of juvenile predators is r_2. The natural death rate of juvenile and adult predators is respectively, d_1 and d_2.

Assumption 4: Juvenile predators mature to adulthood at a rate b.

Assumption 5: The rate of cannibalism of prey population is denoted by c. It is evident that the prey population gains an extra energy due to consuming its conspecifics as a result of cannibalism. This gain increases the rate of reproduction of the prey species and eventually contributes to an extra growth of prey population and denoted by the parameter c_1.

Assumption 6: The conversion efficiency of adult prey due to consuming juvenile predator is represented by the parameter m. Whereas, n denotes conversion efficiency of adult prey due to consuming prey.

Assumption 7: All parameters are positive.

Based on these assumptions, we have drawn the following schema diagram in **Fig. 1** depicting the intra and inter specific interactions between prey and predator.

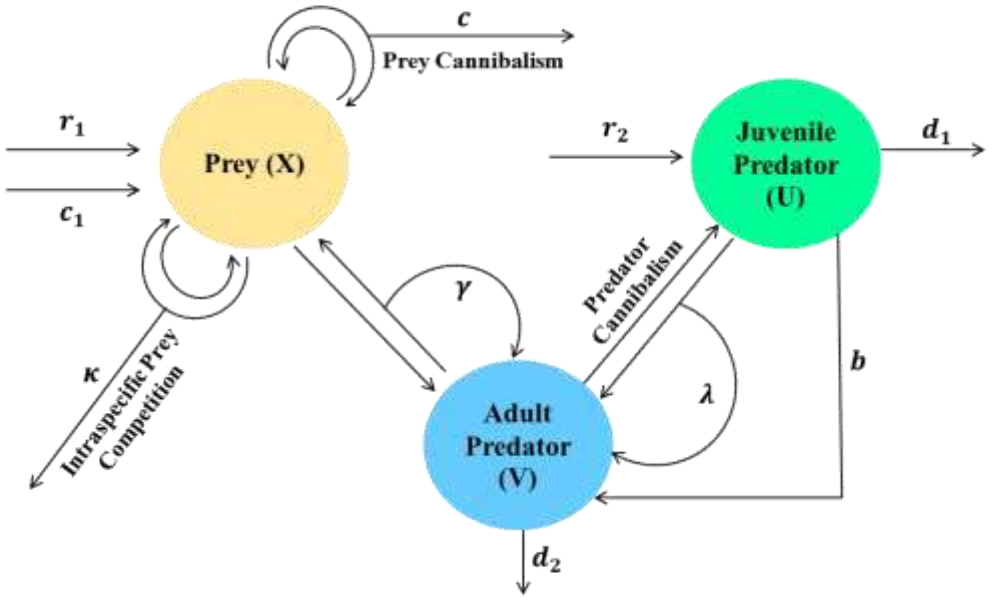


Fig. 1. *Schema diagram based on the above assumptions.*

According to the above stated assumptions and the schema diagram we propose the following deterministic prey-predator model incorporating prey as well as predator cannibalism.

$$\left.\begin{aligned}\frac{dX}{dt} &= r_1X + c_1X - \kappa X^2 - cX^2 - \gamma XV,\\ \frac{dU}{dt} &= r_2V - \lambda UV - bU - d_1U,\\ \frac{dV}{dt} &= bU + n\gamma XV + m\lambda UV - d_2V.\end{aligned}\right] \quad \ldots(1)$$

We study model system (1) with respect to the following initial conditions.

I.C:$X(0) > 0, U(0) \geq 0, V(0) \geq 0.$ ...(2)

Due to seasonal variations of resources the growth of prey population may vary and as a result the growth of predator population will vary too **[31 – 34]**. We consider the seasonal perturbations to be of sinusoidal type and periodic function of time. Hence the seasonally perturbed growth rate of prey population is given by:

$$r_1 \rightarrow r_1[\,1 + \epsilon_1 \sin\theta t] \quad \ldots(2a)$$

and the same for the juvenile predator's birth rate is considered as:

$$r_2 \rightarrow r_2[\,1 + \sin(\theta t + \phi)]. \quad \ldots(2b)$$

Here ϵ_1, ϵ_2 are the degrees of seasonality; $\epsilon_1 r_1, \epsilon_2 r_2$ are the magnitudes of perturbation; θ is the angular frequency of the fluctuations caused by seasonality. ϕ is the phase angle and $0 \leq \phi \leq 2\pi$. Clearly $0 \leq \epsilon_1, \epsilon_2 \leq 1$. When $\epsilon_1, \epsilon_2 = 0$ there is no effect of seasonality. Whereas, $\epsilon_1, \epsilon_2 = 1$ implies maximum level of seasonality. Finally, we modify system (1) by the seasonally perturbed parameters as defined in (2a) & (2b) and obtain the following seasonally perturbed system with $Z(t) = \theta t$:

$$\left.\begin{aligned}
\frac{dX}{dt} &= r_1X + c_1X - \kappa X^2 - cX^2 - \gamma XV + r_1\epsilon_1 X\sin Z,\\
\frac{dU}{dt} &= r_2V - \lambda UV - bU - d_1U + r_2\epsilon_2\, V\sin(Z+\phi),\\
\frac{dV}{dt} &= bU + n\gamma XV + m\lambda UV - d_2V,\\
\frac{dz}{dt} &= \theta\ ,\ \ z(0) = 0.
\end{aligned}\right\} \quad \dots(3)$$

3 Existence and stability of equilibrium points

Model system (1) has four equilibrium points. Namely,

1. The trivial equilibrium point $E_0(0,0,0)$, which always exists.
2. The predator-free equilibrium point$E_1(\bar{X},0,0)$ where $\bar{X} = \frac{r_1+c_1}{\kappa+c}$. This equilibrium always exists.
3. The prey-free equilibrium point $E_{23}(0,\widehat{U},\widehat{V})$ where, $\widehat{V} = \frac{b\widehat{U}}{d_2-\lambda m\widehat{U}}$ and $\widehat{U} = \frac{br_2-d_2(b+d_1)}{\lambda[b-m(b+d_1)]}$. This equilibrium exists if the following parametric conditions are satisfied:

   $$(i) r_2 > \frac{d_2(b+d_1)}{b}, (ii)\ m < \frac{b}{b+d_1}\ , (iii) \frac{d_2[b+m(b+d_1)]}{m[br_2+d_2(b+d_1)]} > 1. \quad \dots(4)$$

4. The coexistence equilibrium point $E^*(X^*,U^*,V^*)$ where, $X^* = \frac{(r_1+c_1)-\gamma V^*}{k+c}$, $U^* = \frac{r_2V^*}{\lambda V^*+b+d_1}$ and V^* is given by the positive root of the following quadratic equation: $A_1V^{*2} + A_2V^* + A_3 = 0$, where $A_1 = n\gamma^2\lambda > 0, A_2 = n\gamma^2(b+d_1) - n\gamma\lambda(r_1 + c_1) - (k+c)\lambda(mr_2 - d_2)$,$A_3 = (b+d_1)[d_2(k+c) - n\gamma(r_1+c_1)] - br_2(k+c)$. The equilibrium E^* exists if the system parameters satisfies the following conditions:

$$(I) r_2 < \frac{d_2(b+d_1)}{b},$$

$$(II)\ \gamma > \max\left\{\frac{d_2\kappa}{n(r_1+c_1)}, \frac{d_2\kappa(b+d_1)-\kappa b r_2}{n(b+d_1)(r_1+c_1)}\right\}$$

$$(II)\ c > \max\left\{\frac{n\gamma(r_1+c_1)}{d_2} - \kappa, \frac{(b+d_1)n\gamma(r_1+c_1)}{(b+d_1)d_2 - br_2} - \kappa\right\}. \tag{5}$$

We have proved the following propositions regarding the local stability conditions of the equilibrium points.

Proposition 3.1. *The trivial equilibrium point* E_0 *is always unstable.*

Proof. The characteristic equation of the Jacobian matrix of system (1) evaluated at E_0 is given by,

$$(\xi - m_{11})(\xi^2 + B_1\xi + B_2) = 0, \qquad \dots(6)$$

where, $B_1 = -(m_{22} + m_{33}) > 0$, $B_2 = m_{22}m_{33} - m_{23}m_{32}, m_{11} = r_1 + c_1 > 0$, $m_{22} = -(b + d_1) < 0, m_{23} = r_2 > 0, m_{32} = b > 0, m_{33} = -d_2 < 0$. Clearly, one of the roots of (6) is $\xi_1 = m_{11} = r_1 + c_1 > 0$. Hence, E_0 is always unstable. This proves the proposition.

Proposition 3.2. *The predator-free equilibrium* E_1 *is locally asymptotically stable if and only if* $P_1 > 0$ *and* $P_2 > 0$, *where* P_1 *and* P_2 *are given by the equation (8).*

Proof. The characteristic equation of the Jacobian matrix of system (1) evaluated at E_1 is given by,

$$(\xi - n_{11})(\xi^2 + P_1\xi + P_2) = 0, \qquad \dots(7)$$

where,

$$\left.\begin{aligned} &P_1 = -(n_{22} + n_{33}) > 0,\ P_2 = n_{22}n_{33} - n_{23}n_{32}, n_{11} = -(r_1 + c_1) < 0,\\ &n_{13} = \gamma\left(\frac{r_1+c_1}{\kappa+c}\right) > 0, n_{22} = -(b + d_1) < 0, n_{23} = r_2 > 0, n_{32} = b > 0,\\ &n_{33} = n\gamma\left(\frac{r_1+c_1}{\kappa+c}\right) - d_2. \end{aligned}\right\} \dots(8)$$

One can easily see that one of the roots of (7) is $\xi_1 = n_{11} < 0$. Hence following Routh Hurwitz criteria, E_1 will be locally asymptotically stable if and only if $P_1 > 0$ and $P_2 > 0$. Hence the proposition.

Proposition 3.3. *If the prey-free equilibrium E_{23}exists satisfying the conditions as specified in (4), then it is locally asymptotically stable if and only if $a_{11} < 0,\ Q_1 > 0$ and $Q_2 > 0$, where a_{11}, Q_1 and Q_2 are given by the equation (9).*

Proof. The characteristic equation of the Jacobian matrix of system (1) evaluated at E_{23} is given by,

$$(\xi - a_{11})(\xi^2 + Q_1\xi + Q_2) = 0,$$

where,

$$\left.\begin{aligned}&Q_1 = -(a_{22} + a_{33}) > 0,\ Q_2 = a_{22}a_{33} - a_{23}a_{32},\ a_{11} = (r_1 + c_1) - \gamma\hat{V},\\ &a_{22} = -(\lambda + b + d_1),\ a_{23} = r_2 > 0,\ a_{31} = n\gamma\hat{V} > 0,\ a_{32} = b + m\lambda\hat{V} > 0,\\ &a_{33} = m\lambda\hat{U} - d_2.\end{aligned}\right\} \quad \dots(9)$$

One can easily see that one of the roots of (4) is $\xi_1 = a_{11}$. Other two roots are given by the quadratic equation $\xi^2 + Q_1\xi + Q_2 = 0$ and following Routh Hurwitz criteria both the roots will have negative real part if and only if $Q_1 > 0$ and $Q_2 > 0$. Therefore, E_{23} will be locally asymptotically stable if and only if $a_{11} < 0, Q_1 > 0, Q_2 > 0$. Thus, the proposition is proved.

Proposition 3.4. *If the coexistence equilibrium E^*exists satisfying the conditions as specified in (5), then it is locally asymptotically stable if and only if $R_1 > 0, R_3 > 0$ and $R_1R_2 - R_3 > 0$, where R_1, R_2, R_3 are given by the equation (11).*

Proof. The characteristic equation of the Jacobian matrix of system (1) evaluated at E^* is given by,

$$\xi^3 + R_1\xi^2 + R_2\xi + R_3 = 0, \qquad \dots(10)$$

where,

$$\left.\begin{aligned}&R_1 = -(b_{11} + b_{22} + b_{33}),\\ &R_2 = b_{22}b_{33} + b_{11}b_{33} + b_{11}b_{22} - b_{23}b_{32} - b_{13}b_{31},\\ &R_3 = b_{13}b_{31}b_{22} + b_{11}b_{23}b_{32} - b_{11}b_{22}b_{33},\\ &b_{11} = r_1 + c_1 - 2\kappa X^* - 2cX^* - \gamma V^*, b_{13} = \gamma X^* > 0,\\ &b_{22} = -(\lambda V^* + b + d_1), b_{23} = r_2 > 0, b_{31} = n\gamma V^*, b_{32} = b + m\lambda V^*,\\ &\qquad\qquad b_{33} = n\gamma X^* + m\lambda U^* - d_2\end{aligned}\right\} \quad \dots(11)$$

Therefore, following Routh Hurwitz criteria E^* will be locally asymptotically stable if and only if $R_1 > 0, R_3 > 0, R_1R_2 - R_3 > 0$. Hence the proposition.

4. Numerical simulations of the model system (1)

In this section we perform numerical simulations of the model system (1) using MATLAB R2022a software. We consider the following parameter values given in Parameter Set – I:

Parameter Set I: $r_1 = 0.5;\ c_1 = 0.04;\ \kappa = 0.001; \gamma = 0.03;\ r_2 = 0.08; \lambda = 0.04;$
$d_1 = 0.2; b = 0.01;\ m = 0.1;\ n = 1.5;\ d_2 = 0.025.$

Our objective is to analyze the effect of prey cannibalism and predator cannibalism on the system dynamics. Hence, we will vary the parameters, c and λ, which respectively represent the rate of prey and predator cannibalism. First, we consider $c = 1.5$ and $\lambda = 0.04$. For these chose parameter values we calculate, $P_1 = 0.2188 > 0$ and $P_2 = 0.0011 > 0$ using the expressions of P_1 and P_2 as derived in equation (8). Following Proposition 3.2, one can easily conclude that in this case the predator-free equilibrium $E_1(0.3598,0,0)$ is locally asymptotically stable. Therefore, if the rate of prey cannibalism is high then, predators will face shortage of food due to nonavailability of sufficient number of prey and finally will die out in long run. This case has been depicted in Fig. 2.

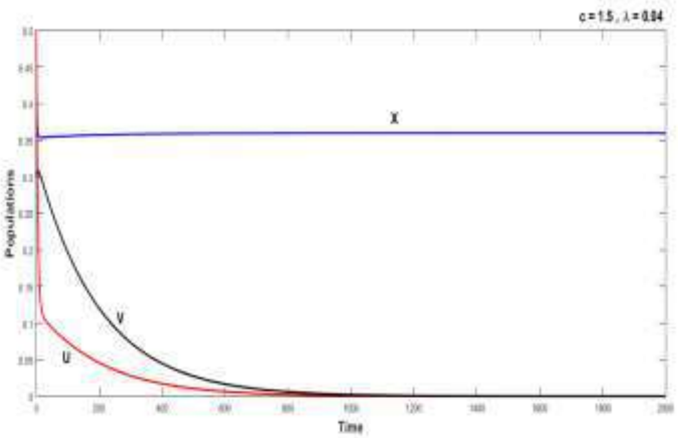


Fig. 2. *Time evolution of model system (1) for parameter values as specified in Parameter Set – I with* $c = 1.5$ *and* $\lambda = 0.04$.

To determine the existence of the prey-free equilibrium E_{23} we consider the existence conditions as specified in (4). For the parameter values as specified in Parameter Set – I, with

$c = 1.5$ and $\lambda = 0.04$ one can calculate $(i)\frac{d_2(b+d_1)}{b} = 0.525,\ (ii)\frac{b}{b+d_1} = 0.0476$.Therefore, we consider $r_2 = 0.8$ and $m = 0.03$ to satisfy the existence conditions of E_{23} as specified in (4). Moreover, for these parameter values, we found that $\frac{d_2[b+m(b+d_1)]}{m[br_2+d_2(b+d_1)]} = 1.0252 > 1$. Hence according to the existence condition (4), the prey-free equilibrium $E_{23}(0, \widehat{U}, \widehat{V})$ exists, where, $\widehat{U} = 18.5811$ and $\widehat{V} = 68.75$. Next, we discuss the stability of E_{23}. For the chosen parameter values, we evaluate $a_{11} = -1.5225 < 0, Q_1 = 0.2527 > 0, Q_2 = 0.0733 > 0$. Following Proposition 3.3 we conclude the prey-free equilibrium E_{23} is locally asymptotically stable in this case. The time evolution diagram of model system (1) for the above chosen parameter values has been drawn in Fig. 3, that depicts the stability of E_{23} implying the extinction of the prey population in long run.

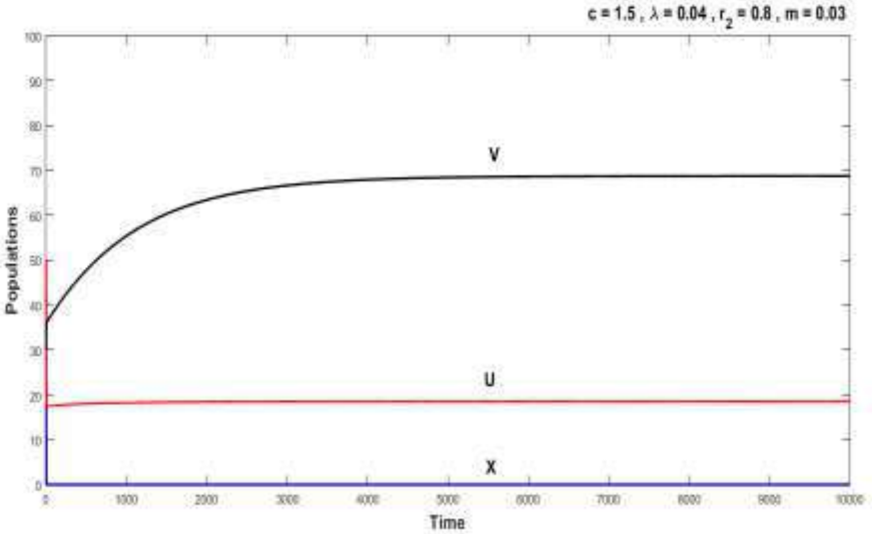


Fig. 3. *Time evolution of model system (1) with* $r = 0.8$ *and* $m = 0.03$. *Other parameters are as in Fig 2.*

The coexistence equilibrium $E^*(X^*, U^*, V^*)$will exist if the parameter values satisfy the conditions deduced in (5). For the parameter values as specified in Parameter Set – I we calculate $\frac{d_2(b+d_1)}{b} = 0.525 > r_2 = 0.08$. Thus, the condition (I) of (5) is readily satisfied.

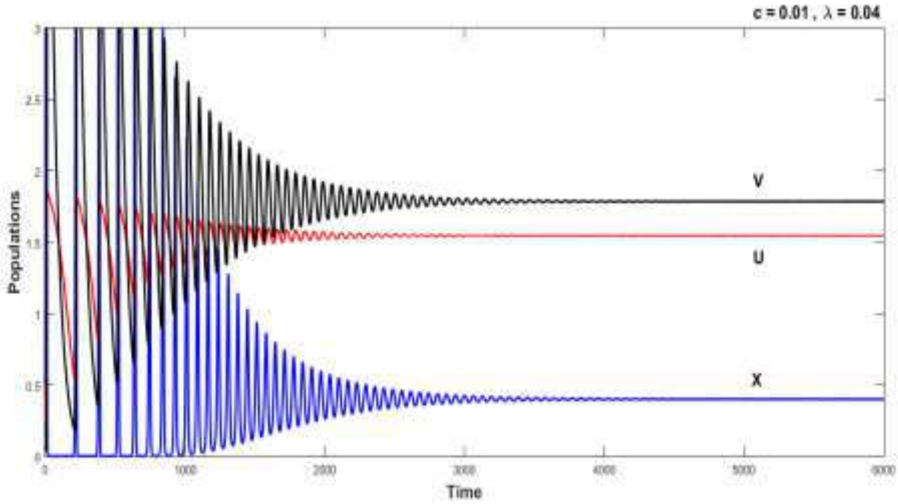


Fig. 4. *Time evolution of model system (1) with* $c = 0.01$ *and* $\lambda = 0.04$. *Other parameters are as in Fig 2.*

We have evaluated $\frac{d_2\kappa}{n(r_1+c_1)} = 0.00031$ and $\frac{d_2\kappa(b+d_1)-\kappa b r_2}{n(b+d_1)(r_1+c_1)} = 0.00026$ using Parameter Set – I. Clearly the chosen value of $\gamma = 0.03$ is greater than $\max\{0.00031, 0.00026\}$. Hence, the condition (II) of (5) holds true. Again, the maximum value of the expressions $\frac{n\gamma(r_1+c_1)}{d_2} - \kappa$ and $\frac{(b+d_1)n\gamma(r_1+c_1)}{(b+d_1)d_2-br_2} - \kappa$ is found to be 0.000097. Therefore, to satisfy the condition (III) of (5) we set $c = 0.01$ along with $\lambda = 0.04$. Hence, for these parameter values E^* exists with $X^* = 0.3989, U^* = 1.5455, V^* = 1.7857$. Proposition 3.4 specifies the stability conditions of E^*. We therefore, calculate $R_1 = 0.9294 > 0, R_3 = 0.0089 > 0, R_1R_2 - R_3 = 0.0016 > 0$. Hence, following Proposition 3.4, for the parameter values given in Parameter Set – I with $c = 0.01$ and $\lambda = 0.04$, we obtain that the coexistence equilibrium E^* is locally asymptotically stable. This case has been shown in Fig. 4.

Next, we analyze the impact of prey cannibalism and predator cannibalism on the dynamics of system (1). First, we observe the effect of variation of prey cannibalism, represented by the parameter c, when predator cannibalism, represented by λ remains fixed. In this regard we consider three different values of predator cannibalism, i.e., low, moderate and high and vary the rate of prey cannibalism.

Case – 4.1: Effect of prey cannibalism when predator cannibalism is low

In this case we consider predator cannibalism $\lambda = 0.001$. Keeping all other parameters as in Parameter Set – I, we vary the rate of prey cannibalism, i.e., c. We have previously calculated that maximum value of the expressions $\frac{n\gamma(r_1+c_1)}{d_2} - \kappa$ and $\frac{(b+d_1)n\gamma(r_1+c_1)}{(b+d_1)d_2-br_2} - \kappa$ as 0.000097. Moreover, as these two expressions does not depend on λ, they will not change for different values of λ. So we vary c for $c \geq 0.001$, so that the condition (III) of (5) is satisfied. Moreover, we have already seen that, for the parameter values as specified in Parameter Set – I, condition (I) and (II) are readily satisfied. In this way, existence of E^* is guaranteed for $\lambda = 0.001$ and $c \geq 0.001$, provided other parameters are as in Parameter Set – I.

We have drawn the phase portraits of system (1) for $c = 0.001$ (Fig. 5a), for $c = 0.01$ (Fig. 5b) and for $c = 0.1$ (Fig. 5c). In all the cases we observe that system trajectories converge to the stable coexistence equilibrium E^* implying the stable coexistence of all the system populations. Therefore, if the rate of prey cannibalism is low, then all the population will coexist, as long as the rate of predator cannibalism remains below a critical level. But if the rate of predator cannibalism exceeds that critical level and reaches as high as $c = 1.2$ or above, then predator population goes to extinction in long run and system converges to stable E_1. This scenario has been depicted in Fig. 5d.

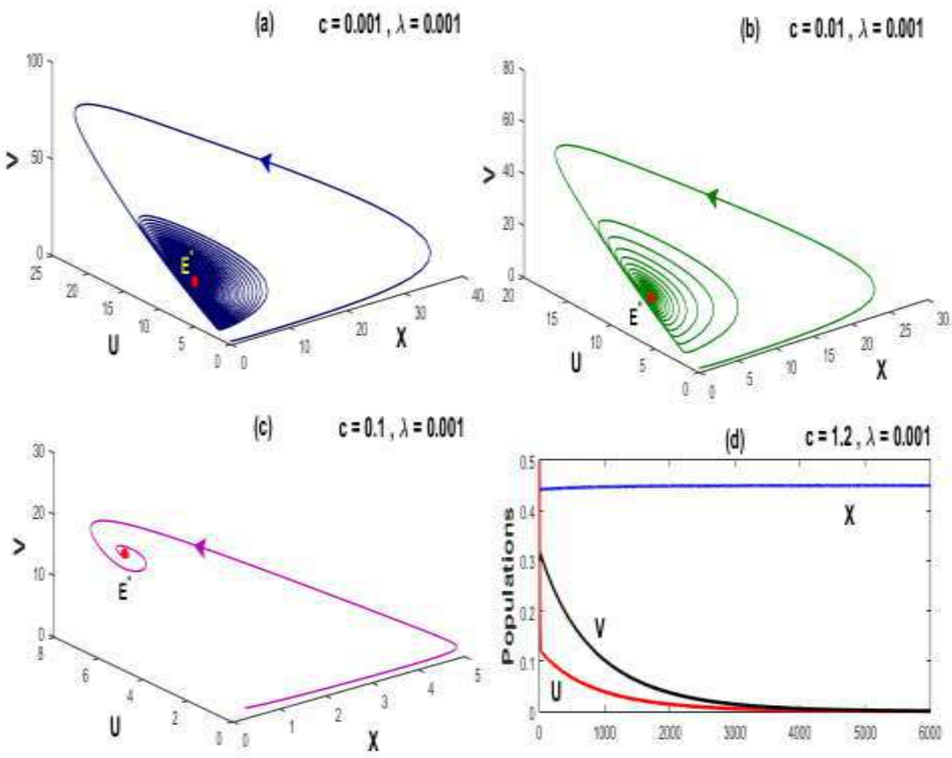


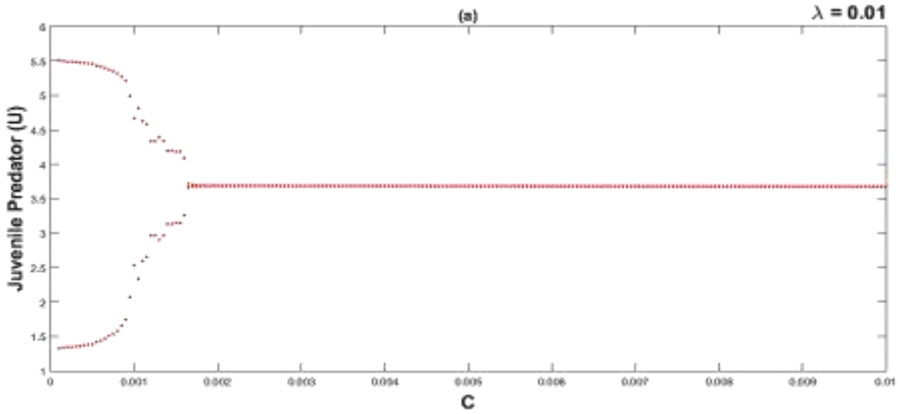
Fig. 5. *Phase portrait of system (1) for* $\lambda = 0.001$ *and different values of* c. *(a)* $c = 0.001$, *(b)* $c = 0.01$, *(c)* $c = 0.1$ *and (d)* $c = 1.2$. *Other parameters are as in Parameter Set – I.*

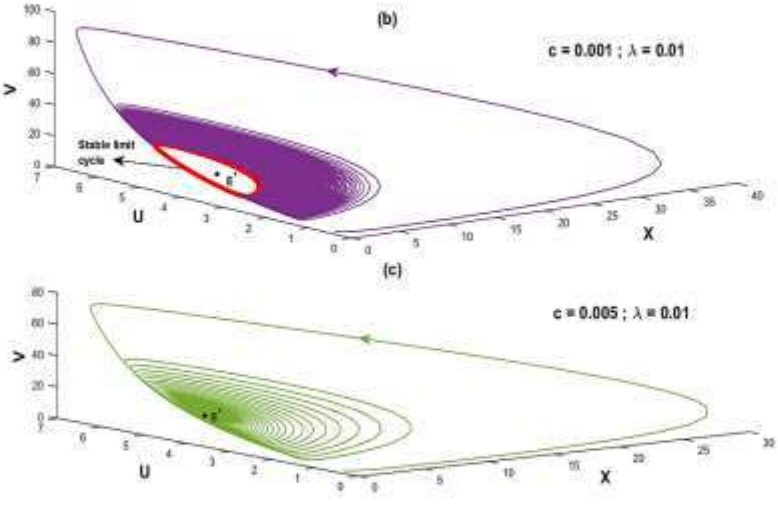
Case – 4.2: Effect of prey cannibalism when predator cannibalism is moderate

To analyze this scenario, we fix $\lambda = 0.01$ and vary c for $c \geq 0.0001$, at the same time keeping it less than a critical, which guarantees the existence of E^* (see Case – 4.1). Here we keep other parameters same as specified in Parameter Set – I. The bifurcation diagram of the system (1) with respect to the juvenile predator population has been drawn in Fig. 6a. It shows that for lower values of c , say $c = 0.001$, the coexistence equilibrium E^* becomes unstable and system trajectories converge to a stable limit cycle (Fig. 6b). However, as c increases, the mode of stability of E^* switches through a Hopf Bifurcation and finally E^* becomes stable implying stable coexistence of all the populations (Fig. 6c). Therefore, we assert that, prey cannibalism plays a stabilizing role when predator cannibalism is moderate.

Case – 4.3: Effect of prey cannibalism when predator cannibalism is high

Next we increase the rate of predator cannibalism to a higher level and fix $\lambda = 0.1$. As previous, we increase c from $c = 0.0001$ so that E^* exists for such values, keeping other parameters as in Parameter Set – I. Moreover, we do not increase c to a very high level where E^* ceases to exist (see Case – 4.1). For this case we have drwan the bifurcation diagram of system (1) with respect to the juvenile prey population, in Fig. 7. The nature of stability of E^* for increasing c, is found to be similar to the Case – 4.2. For lower values of c, E^* is unstable. However, as c is increased, E^* attains its stability through a Hopf bifurcation much faster, when λ is high. Therefore, we obtain that, when predator cannibalism is high, the coexistence equilibrium E^* is unstable for low prey cannibalism and system trajectories converge to a stable limit cycle around E^*. But as prey cannibalism increases, a Hopf bifurcation occurs switching the stablity of unstable E^* and then all trajectories converge to a stable E^* signifying the coexistence all system populations. Moreover, prey cannibalism also plays a stabilizing role in this case when predator cannibalsim is high.





ulation (U)
005. *Other*

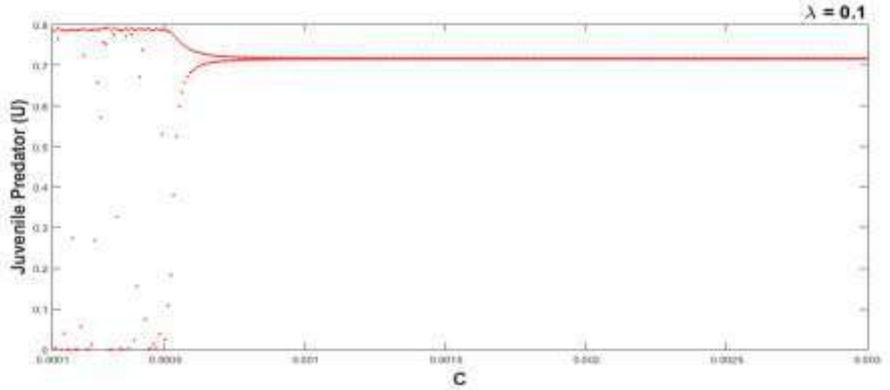


Fig. 7. *Bifurcation diagram of system (1) with respect to the juvenile predator population (U) for varying* c. *Other parameters are as in Parameter Set – I with* $\lambda = 0.1$.

In the next part we discuss the effect of predator cannibalism, when the prey cannibalism is fixed. For that purpose, keeping all other parameters fixed at Parameter Set – I, we consider three different values of prey cannibalism, i.e., $c = 0.001, c = 0.01$ and $c = 0.1$ signifying low, moderate and high value of prey cannibalism. And we vary predator cannibalism represented by λ. As discussed in Case – 4.1, if we consider $c > max\left\{\frac{n\gamma(r_1+c_1)}{d_2} - \kappa, \frac{(b+d_1)n\gamma(r_1+c_1)}{(b+d_1)d_2-br_2} - \kappa\right\} = 0.000097$, then E^* will exist satisfying the existence conditions evaluated in (5) for varying λ, as long as c does not exceed a critical value as well as all other parameters are kept same as in Parameter Set – I.

Case – 4.4: Effect of predator cannibalism when prey cannibalism is low

For this case we consider $c = 0.001$ and draw the bifurcation diagram with respect to juvenile predator population for varying λ in Fig. 8a.

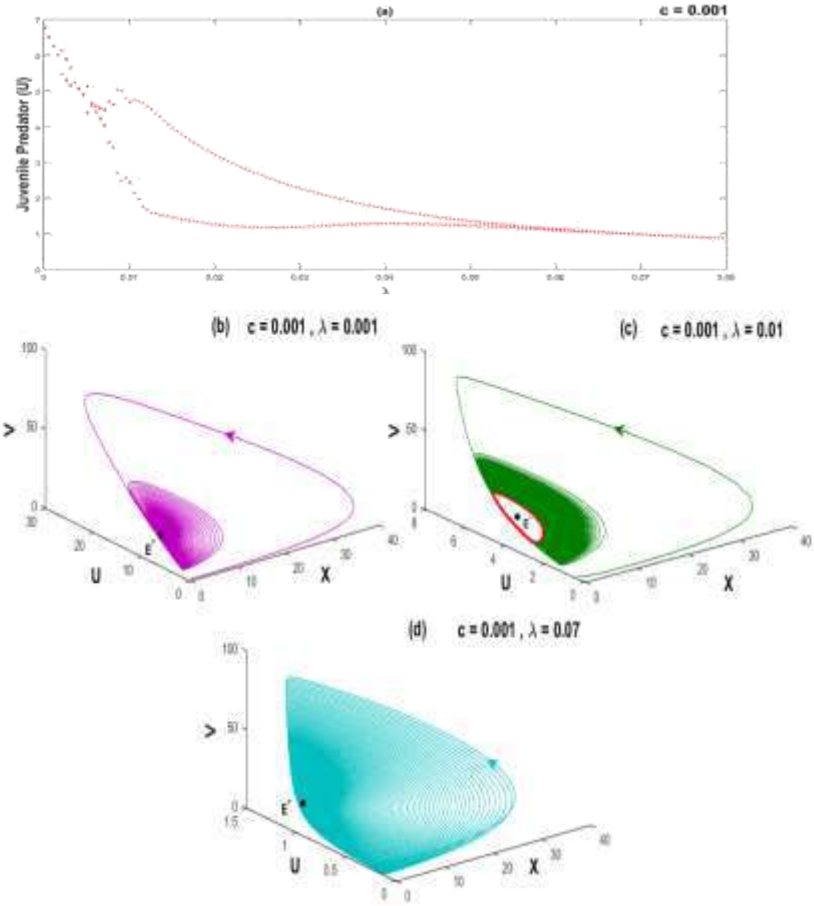


Fig. 8. *Bifurcation diagram of system (1) with respect to the juvenile predator population (U) for varying* λ *(a) and phase portrait of system (1) for (b)* $\lambda = 0.001$, *(c)* $\lambda = 0.01$, *(d)* $\lambda = 0.07$. *Other parameters are as in Parameter Set – I with* $c = 0.001$.

Form this diagram it can be seen that, for lower values of λ, E^* is stable. Now as λ increases, E^* switches its stability through a Hopf bifurcation and becomes unstable. In this case all trajectories converge to a stable limit cycle around E^*. Moreover, E^* switches its stability again and finally becomes stable through another Hopf bifurcation as λ is increased further. Therefore, we assert that, when prey cannibalism is low, then low predator cannibalism destabilizes the system. Whereas, high predator cannibalism stabilizes the system. To corroborate these results, we have drawn the phase portraits of system (1) for different values of λ in Fig. 8b ($\lambda = 0.001$), Fig. 8c ($\lambda = 0.01$) and Fig. 8d ($\lambda = 0.07$). Clearly, for $\lambda = 0.001\&0.07$, we find E^* is stable and all populations coexist in a stable mode. But for $\lambda = 0.001$, all populations coexist in an unstable mode as E^* is unstable in this case.

Case – 4.5: Effect of predator cannibalism when prey cannibalism is moderate and high

Here we consider $c = 0.01$ as moderate prey cannibalism and $c = 0.1$ as high. For these chosen values of c, the bifurcation diagrams of system (1) for varying λ have been drawn in Fig 9a and Fig. 9b respectively. In both the cases it is observed that, E^* remains stable and therefore, system trajectories converge to stable E^* implying stable coexistence of all the populations.

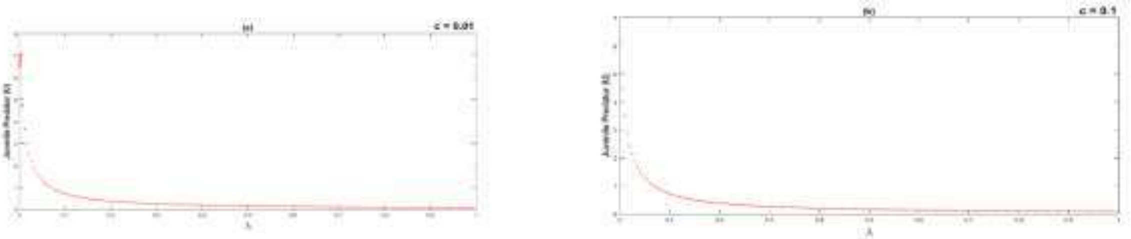


Fig. 9. *Bifurcation diagram of system (1) with respect to the juvenile predator population (U) for varying* λ *with (a)* $c = 0.01$ *and (b)* $c = 0.1$*. Other parameters are as in Parameter Set – I.*

5 Numerical simulations of the model system (2)

In this section we analyze the effect of seasonal variations considering the model system (2). We set the values of the degrees of seasonality $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.4$. We discuss the effect of θ, the angular frequency of the fluctuations caused by seasonality by considering three different values of the phase angle ϕ i.e., $\phi = 0, \pi$ and 2π.Moreover, we consider all the other parameters as in Parameter Set – I, with $c = 0.01$ and $\lambda = 0.04$. From Case – 4.5 and bifurcation diagram Fig. 9a one can easily see that, the model system (1) converges to stable E^* for these parameter values. Our objective is therefore, to deduce the impact of seasonal variations on an otherwise stable system.

Case – 5.1: $\phi = 0$

First, we draw the bifurcation diagram of system (2) with respect to the juvenile predator (U) population, for varying θ and $\phi = 0$ in Fig. 10. From this diagram one can clearly see that for increasing θ the system (2) exhibits critical aperiodic behavior. The dynamics of system (2) for some specific values of θ have been drawn in Fig. 11.

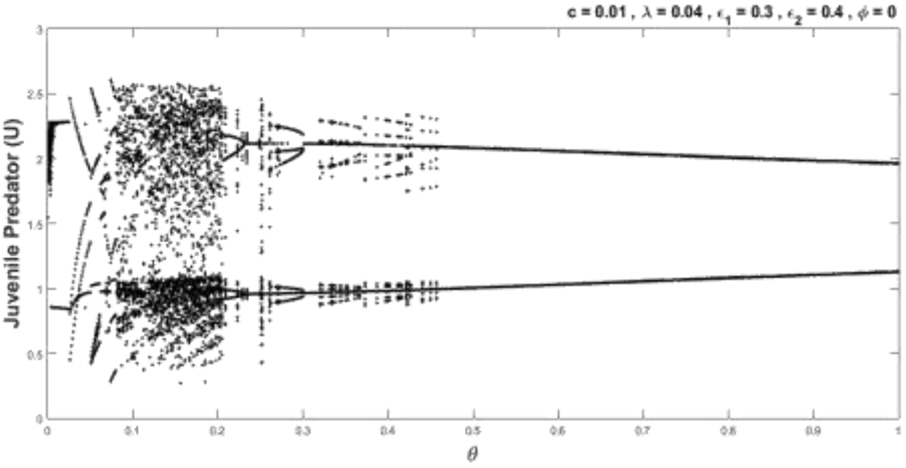


Fig. 10. *Bifurcation diagram of system (2) with respect to the juvenile predator population (U) for varying* θ. *In this diagram* $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.4$, $\phi = 0, c = 0.01$ and $\lambda = 0.04$. *Other parameters are as in Parameter Set – I*

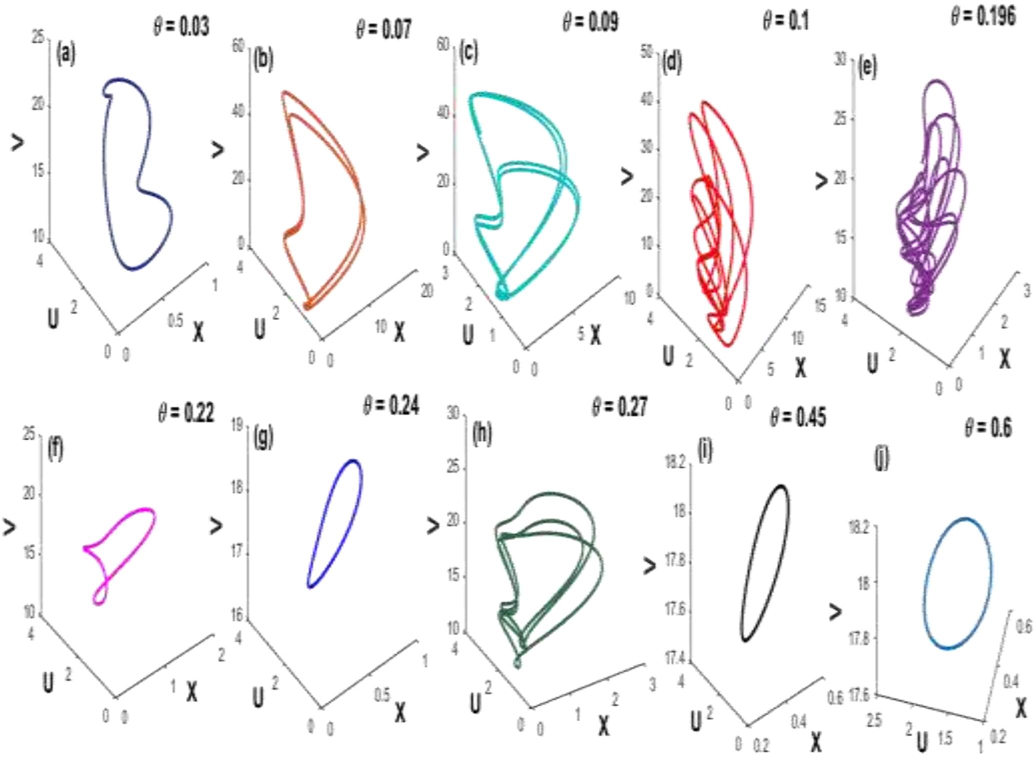


Fig. 11. *Phase portraits of system (2) for different values of* θ. *(a)* $\theta = 0.03$*, (b)* $\theta = 0.07$*, (a)* $\theta = 0.09$*,, (a)* $\theta = 0.1$*,, (a)* $\theta = 0.196$*,, (a)* $\theta = 0.22$*,, (a)* $\theta = 0.24$*,, (a)* $\theta = 0.27$*,, (a)* $\theta = 0.45$*,, (a)* $\theta = 0.6$. *In this diagram* $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.4$, $\phi = 0, c = 0.01$ and $\lambda = 0.04$. *Other parameters are as in Parameter Set – I.*

From this diagram we observe that, for $\theta = 0.03$ we obtain unstable E^* (Fig. 11a). As θ increases to $\theta = 0.07$, period doubling occurs and we obtain two periodic solutions (Fig. 11b). Increasing θ further results in another period doubling and 4-periodic solutions are observed when $\theta = 0.09$(Fig. 11c). After that for $\theta = 0.1$, chaotic oscillations can be seen (Fig. 11d). It is evident that the route to chaos in this case is period doubling bifurcations. This chaotic behavior of system (2) continues and even for $\theta = 0.196$ system shows chaotic oscillations (Fig. 11e). However, if we increase θ to $\theta = 0.22$, chaotic oscillations disappear and trajectories exhibit 2 periodic oscillations (Fig. 11f). Then as θ increases to $\theta = 0.24$, we obtain single periodic solutions (Fig. 11g). However, for $\theta = 0.27$, the solution trajectories exhibit 3 – periodic solutions (Fig. 11h). But as we increase θ more, to $\theta = 0.45$ (Fig. 11i) and $\theta = 0.6$ (Fig. 11j) system trajectories converge to a stable limit cycle around the unstable E^*.

Case – 5.2: $\phi = \pi$ and $\phi = 2\pi$

The bifurcation diagrams of system (2) with respect to the juvenile predator (U) population, for varying θ and $\phi = \pi \& \phi = 2\pi$ has respectively been drawn in Fig. 12 and 13. In both the cases, we can see that for increasing θ the system (2) exhibits critical aperiodic behavior and the dynamics of system (2) in both the case for varying θ is similar to the Case – 5.1.

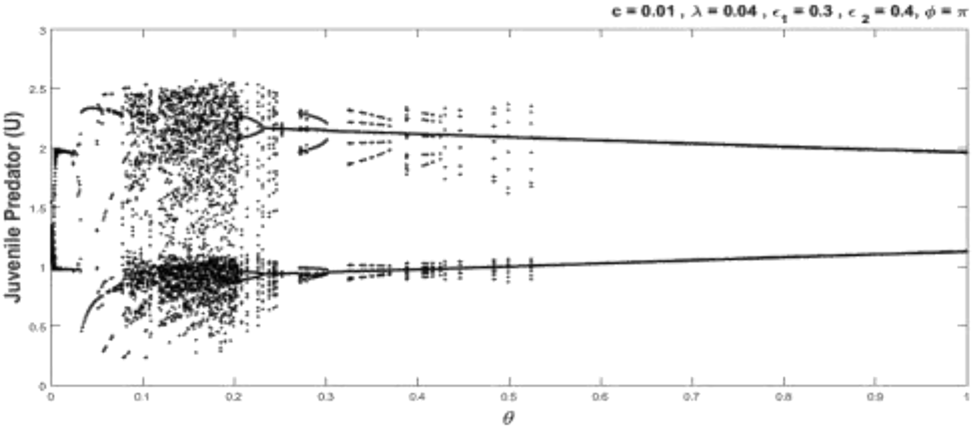
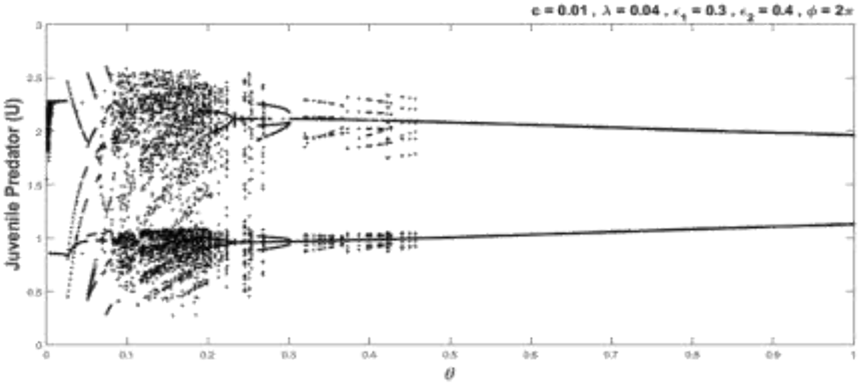


Fig. 12. *Bifurcation diagram of system (2) with respect to the juvenile predator population (U) for varying* θ*. In this diagram* $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.4$, $\phi = \pi, c = 0.01$ and $\lambda = 0.04$. *Other parameters are as in Parameter Set – I.*



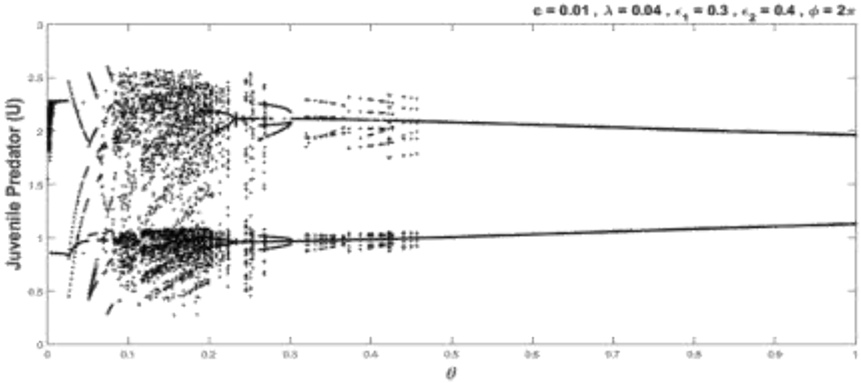


Fig. 13. *Bifurcation diagram of system (2) with respect to the juvenile predator population (U) for varying* θ. *In this diagram* $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.4$, $\phi = 2\pi, c = 0.01$ and $\lambda = 0.04$. *Other parameters are as in Parameter Set – I.*

5. Conclusion

In this study we have proposed a three dimensional stage structured prey-predator model incorporating cannibalism in both of prey and predator species. We have then extended our model considering periodic sinusoidal type time dependent seasonal variationsin the growth rate of prey and predator population. In the previous studies, it has been shown that cannibalism has either stabilizing or destabilizing effect on system dynamics may it be prey or predator cannibalism. However, through our study we observed that, in a prey-predator system in presence of both prey and predator cannibalism, if the predator cannibalism is low then as long as the prey cannibalism remains below a critical value, all populations coexist in a stable mode. But if in any scenario, the prey cannibalism exceed a critical value, the predator populations die in long run. Now if the rate of predator cannibalism is moderate or high, then for lower values of prey cannibalism, system becomes unstable. But as prey cannibalism increases, system stabilizes through a Hopf bifurcation. However, for high rate of predator cannibalism, system stabilizes faster at a lower prey cannibalism rate. Whereas, when predator cannibalism is moderate, it takes higher rate of prey cannibalism to stabilize the otherwise unstable system. Therefore, in general, if predator cannibalism is fixed, then prey cannibalism plays a stabilizing role. On the other hand, when the prey cannibalism is low, then predator cannibalism shows stabilizing as well as destabilizing effect. For low and high predator cannibalism the system becomes stable. But for moderate values of predator cannibalism, the system becomes unstable. However, if the prey cannibalism is moderate or high, then system always remain stable for varying predator cannibalism.

Finally, we have studied the effect of seasonal variations on system dynamics by varying the angular frequency θ of the fluctuations caused by seasonalityand for three different values of the phase angle ϕ, namely, $\phi = 0, \pi, 2\pi.$ We have found that, for different values of θ and ϕ the system exhibits critical periodic and chaotic nature. And the route to chaos is always the period doubling bifurcations.

Thus, we conclude, in absence of seasonal variations, lower rate of prey cannibalism plays a destabilizing role, provided the predator cannibalism is moderate or high. Whereas, lower and higher predator cannibalism can stabilize the system, but moderate predator cannibalism destabilizes the system, when prey cannibalism is low. Moreover, in presence of seasonal variations the system shows critical periodic and chaotic behavior. These results are significant from the ecological point of view, as it signifies the impact of prey and predator cannibalism in a combined ecological scenario and can explain the stabilizing & destabilizing system dynamics of a cannibalistic species as well as fluctuations of population densities observed in nature.

Reference

[1] D. Kelly. *The culture that still practices cannibalism*, October **2013**.
[2] C. R. Pennell. *Cannibalism in early modern North Africa. British Journal of Middle Eastern Studies*, **18**(**2**):169-185, **1991**.
[3] D. Claessen, A. M. De Roos and Lennart Persson. *Population dynamic theory of size - dependent cannibalism*. Proceedings of the Royal Society of London B: Biological Sciences, **271**(**1537**):333-340, **2004**.
[4] J. Ye, J. Li. *Factors affecting cannibalism by Mallada basalis*. Biocontrol Sci. Technol., **30**:442–450, **2020**.
[5] G. Polis. *The evolution and dynamics of intraspecific predation*. Annual Review of Ecology and Systematics, 225-251, **1981**.
[6] M. L. Richardson, R. F. Mitchell, P. F. Reagel, L. M. Hanks. *Causes and consequences of cannibalism in non-carnivorous insects*. Annu. Rev. Entomol. **55:**39–53, **2010**.

[7] P. Fedurek, et. al. *Maternal cannibalism in two populations of wild chimpanzees*. Primates **61**; 181–187, **2020**.

[8] A. Muñoz-Saravia et. al. *Cannibalism in the high Andean Titicaca Water Frog, Telmatobiusculeus Garman, 1875*. Amphib. Reptile Conserve. **14**:156–161, **2020**.

[9] M. Frye et. al., *Cannibalism and protective behavior of eggs in Arctic charr (Salvelinus alpinus)*. Ecol. Evol. **11**:14383–14391, **2021**.

[10] M. Gonzalvez et. al., *Smart carnivores think twice: Red fox delays scavenging on conspecific carcasses to reduce parasite risk*. Appl. Anim. Behav. Sci. **243**: 105462, **2021**.

[11] A. M. Koltz and J. P. Wright. *Impacts of female body size on cannibalism and juvenile abundance in a dominant arctic spider*. J. Anim. Ecol. **89:**1788–1798, **2020.**

[12] P. Getto, O. Diekmann, and Andre M De Roos. *On the (dis) advantages of cannibalism*. Journal of Mathematical Biology, **51**(**6**):695-712, **2005**.

[13] C. Kohlmeier and W. Ebenhoh. *The stabilizing role of cannibalism in a predator-prey system*. Bulletin of Mathematical Biology, 57(**3**):401-411, **1995**.

[14] Y. Kang et al. *Ecological and evolutionary dynamics of two-stage models of social insects with egg cannibalism*. J. Math. Anal. Appl. **430**, 324–353, **2015**.

[15] F. Zhang, Y. Chen, J. Li, *Dynamical analysis of a stage-structured predator-prey model with cannibalism*. Math. Biosci. **307**, 33–41, **2019**.

[16] A. A. Basheer, R. D. Parshad, E. Quansah, S. Yu, R. K. Upadhyay. *Exploring the dynamics of a Holling–Tanner model with cannibalism in both predator and prey population*, Int. J. Biomath. **11**:1850010, **2018**.

[17] S. Fasani, S. Rinaldi, *Remarks on cannibalism and pattern formation in spatially extended prey-predator systems*, Nonlinear Dyn. **67**:2543–2548, **2012**.

[18] C. Kaewmanee, I. M. Tang, *Cannibalism in an age structured predator–prey system*, Ecol. Model. **167**:213–220, **2003**.

[19] A. A. Basheer, J. Lyu, A. Giffin, R. D. Parshad, *The (destabilizing) effect of cannibalism in a spatially explicit three-species age structured predator–prey model*, Complexity **2017**.

[20] Y. Li, H. Liu, R. Yang, *A delayed diffusive predator–prey system with predator cannibalism*, Comput. Math. App. **75**:1355–1367, **2018**.

[21] G.Q. Sun, G. Zhang, Z. Jin, L. Li, *Predator cannibalism can give rise to regular spatial pattern in a predator–prey system*, Nonlinear Dyn. **58**:75–84, **2009**.

[22] A. Veprauskas, J. M. Cushing, *A juvenile-adult population model: climate change, cannibalism, reproductive synchrony, and strong allee effects*, J. Bio. Dyn. **11**: 1–24, **2016**.

[23] B. Buonomoa, D. Lacitignola, S. Rionero, *Effect of prey growth and predator cannibalism rate on the stability of a structured population model*, Non- linear Ana. RWA. **11**:1170–1181, **2010**.

[24] J. M. Cushing, S. M. Henson, J. L. Hayward, *An evolutionary game-theoretical model of cannibalism*, Nat. Res. Model. **28 (4)**:497–521, **2015**.

[25] V. HW Rudolf. *Consequences of stage-structured predators: cannibalism, behavioral effects, and trophic cascades*. Ecology **88(12)**:2991-3003, **2007**.

[26] V. HW Rudolf. *The interaction of cannibalism and omnivory: consequences for community dynamics*. Ecology, **88(11)**:2697-2705, **2007**.

[27] Volker HW Rudolf, *The impact of cannibalism in the prey on predator–prey systems*, Ecology, **89**:3116–3127, **2008**.

[28] F. J Solis and R. A Ku-Carrillo. *Birth rate effects on an age-structured predator prey model with cannibalism in the prey*, Abstract and Applied Analysis, **501**:241312, **2014**.

[29] Y. Chow and S. R J Jang. *Cannibalism in discrete-time predator-prey systems*. Journal of Biological Dynamics, **6(1)**:38-62, **2012**.

[30] L. Zhang and C. Zhang. *Rich dynamic of a stage-structured prey-predator model with cannibalism and periodic attacking rate*. Communications in Nonlinear Science and Numerical Simulation, **15(12)**:4029-4040, **2010**.

[31] S. Gakkhar, B. Singh. *Dynamics of modified Leslie–Gower-type prey–predator model with seasonally varying parameters*. Chaos, Solitons Fractals, **27**:1239–55, **2006**.

[32] S. Gakkhar, K. Negi, S. K. Sahani. *Effects of seasonal growth on ratio dependent delayed prey–predator system*. Commun Nonlinear Sci Numer. Simul, **14**:850–62, **2009**.

[33] S. Gakkhar, K. Sahani SK, K. Negi. *Effects of seasonal growth on delayed prey–predator model*. Chaos, Solitons Fractals,**39**:230–9, **2009**.

[34] M. Gao, H. H. Shi, Z. Z. Li. *Chaos in a seasonally and periodically forced phytoplankton–zooplankton system*. Nonlinear Anal RWA, **10**:1643–50, **2009**.

Inverse of a Neutrosophic Fuzzy Matrix

Deepraj Das[1], Debadatta Adak[2] and Runu Dhar[3]
[1, 2, 3]Department of Mathematics, Maharaja Bir Bikram University, Agartala-799004, Tripura, India.
E-mail:[1]deeprajd43@gmail.com, [2]dev.adak.math95325@gmail.com and [3]runu.dhar@gmail.com
∗Corresponding author: Runu Dhar (runu.dhar@gmail.com).

Abstract: The article reveals the introduction of the notion of inverse of a nutrosophic fuzzy matrix. We introduce neutrosophic fuzzy vectors and investigate some of their basic properties and results. We incorporate some suitable examples of neutrosophic fuzzy vectors. We further investigate the notion of generalized inverse of a neutrosophic fuzzy matrix. We formulate some basic properties and investigate some characterization theoremsof generalized inverse of a neutrosophic fuzzy matrix.

Keywords: Neutrosophic set, neutrosophic fuzzy matrix, neutrosophic vectors, generalized inverse of a neutrosophic fuzzy matrix.

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1. Introduction:

Many mathematical tools have been developed to model and solve real life problems. Zadeh [20] introduced the notion of fuzzy set as an appropriate mathematical instrument for description of uncertainty observed in nature. Intensive acceptability in various fields of fuzzy sets has been found since its inception. Traditionally, fuzzy sets is characterized by the membership value or the grade of membership value. Sometimes we may face to assign the membership value for fuzzy sets. Thereafter, the concept of interval valued fuzzy sets was introduced by Zadeh [19] to capture the uncertainty of grade of membership value. Sometimes, real life problems in belief system, expert system, information fusion and so on, we must consider the truth membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate to handle such an ambiguous situation. In order to resolve such a situation, Atanassov [1] introduced the notion of intuitionistic fuzzy sets by associating both the truth membership (or simply membership) and falsity-membership (or non-membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system.

Smarandache [16] introduced and studied the concept of neutrosophic set as a mathematical tool for handling real life problems involving imprecise, indeterminacy and inconsistent data. Each element of neutrosophic set is associated with three independent functions, namely, the membership function (T), the non-membership function (F) and the indeterminacy function (I) defined on the universe of discourse X. The indeterminacy function of the set is the non-deterministic part of the situation. In the recent years, the concept of neutrosophic set has been applied successfully by Pal et al. [13, 14], Broumi et al. [2, 3] and others.

Matrices have important role in science and technology. It is fact that matrix formulation of a mathematical formula gives extra advantages to solve the problem. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. Such type of problems are solved by using fuzzy matrix. Thomas [17] introduced the notion of fuzzy matrices to represent fuzzy relation in a system based on fuzzy set theory and discussed about the convergence of powers of fuzzy matrix. Fuzzy relational maps and neutrosophic relational maps have been introduced by Kandasamy and Smarandache [8]. In classical algebra, matrices are playing an important role in the theory of vector spaces. They [9] further generalized neutrosophic matrices. Khaled et al. [10] investigated on the rectangle neutrosophic fuzzy matrices. Topal et al. [18] studied on bipolar neutrosophic matrices. Dhar et al. [6] defined neutrosophic fuzzy matrices and studied about square neutrosophic fuzzy matrices. Das et al. [4] investigated on algebraic operations on neutrosophic fuzzy matrices. Das et al. [5] investigated the properties of multiplication operation of neutrosophic fuzzy matrices. Kim and Baartmans [12] investigated on determinant theory for fuzzy matrices. Khan and Pal [11] and Pradhan and Pal [15] investigated on generalised inverse of intuitionistic fuzzy matrices.These works motivate us to investigate on inverse of a neutrosophic fuzzy matrix. Thearticle is subdivided as follows. The next section briefly focuses some definitions related to neutrosophic set, fuzzy matrices and neutrosophic matrices which are relevant to the article. In section 3, we investigate the notion of neutrosophic fuzzy vectors. In section 4, we study generalized inverse of a neutrosophic fuzzy matrix. Lastly, section 5 gives conclusion.

2. Preliminaries:

In this section, we recall some basic concepts and results which are relevant for this article.

Definition 2.1. [16] Let X be an universal set. A neutrosophic set A in X is a set containing the triplettruthness, falseness and indeterminacy membership values that can be characterized independently, denoted by T_A, F_A, I_A in [0,1]. The neutrosophic set is defined as follows:

$A = \{(x, T_A(x), F_A(x), I_A(x)) \colon x \in X, \text{ and} T_A(x), F_A(x), I_A(x) \in [0,1]\}$, where

$0 \leq T_A(x) + F_A(x) + I_A(x) \leq 3$.

The null and full neautrosophic sets on a nonempty set X are denoted by 0_N and 1_N respectively.

Example 2.2. Assume that the universe of discourse X = $\{x_1, x_2, x_3\}$, where x_1, $x_2$and $x_3$characterises the quality, reliability and the price of the objects. It may be further assumed that the values of $\{x_1, x_2, x_3\}$are in [0, 1] and they are obtained from some investigations of some experts. The experts may impose their opinion in threecomponents viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a neutrosophic set (NS) of X, such that

A = $\{(x_1, 0.4, 0.5, 0.3), (x_2, 0.7, 0.2, 0.4), (x_3, 0.8, 0.3, 0.4)\}$, where for x_1 the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc.

Definition 2.3.[7] By a fuzzy matrix, we mean a matrix over the fuzzy algebra F = [0, 1] under the fuzzy operations (+, .) defined as a + b = max{a, b} and a.b = min{a, b} ∀ a, b∈F. A fuzzy matrix can be interpreted as a binary fuzzy relation, which is defined as below:
A fuzzy matrix (in short, FM) of order m × n is defined as A = $(a_{ij}, a_{ij\mu})$ where $a_{ij\mu}$ is the membership value of the ij-th element a_{ij} *in A*. We denote the set of all fuzzy matrices of order m × n by $F_{m\times n}$. If m = n, in short we write F_n, the set of all square fuzzy matrices of order n.
For simplicity, we write A = $[a_{ij}]_{mxn}$ where $a_{ij} = (a_{ij\mu})$.
A fuzzy square matrix of order two is expressed in the following way

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where the entries a,b,c, d all belong to the interval [0,1].

Example 2.4. Let A = $\begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.3 \end{bmatrix}$ be a fuzzy matrix of order 2.

Definition 2.5. (One may refer tmo [15]) Fuzzy matrices deal with only membership values, not non-membership values. To handle the non-membership values of the elements also we consider the intuitionistic fuzzy matrix (in short, IFM), which is defined as below:
An IFM of order m × n is defined as A = $[a_{ij}, (a_{ij\mu}, a_{ij\vartheta})]_{m\times n}$ where $a_{ij\mu}$ and $a_{ij\vartheta}$ are called membership and non-membership values of the ij-th element x_{ij} *in A.,* maintaining the condition $0 \le a_{ij\mu} + a_{ij\vartheta} \le 1$.
For simplicity, we write A = $[a_{ij}]_{m\times n}$ where $a_{ij} = (a_{ij\mu}, a_{ij\vartheta})$.

Definition 2.6. Intuitionistic fuzzy matrices deal with membership and non-membership values but cann't deal with indeterminacy values . To handle the indeterminacy values of the elements also, we consider the neutrosophic fuzzy matrix (in short, NFM), which is defined as below:
An NFM of order m × n is defined as A = $[a_{ij}, (\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{F}})]_{m\times n}$ where $\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}$is called membership value,$\mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}$is called indeterminacy value and $\mathrm{a}_{\mathrm{ij}}^{\mathrm{F}}$is called non - membership value of the ij-th element x_{ij} *in A,* maintaining the condition $0 \le \mathrm{a}_{\mathrm{ij}}^{\mathrm{T}} + \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}} + \mathrm{a}_{\mathrm{ij}}^{\mathrm{F}} \le 3$. Here each of $\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}$ *and* $\mathrm{a}_{\mathrm{ij}}^{\mathrm{F}}$takes from the interval [0, 1].
For simplicity, we write A = $[a_{ij}]_{m\times n}$ where $a_{ij} = (\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{F}})$.
A neutrosophic fuzzy matrix for which m = n (i.e., the number of rows is equal to the number of columns) and whose each of degree of membership value of the three independent functions takes from the interval [0, 1] is called a square neutrosophic fuzzy matrix of order n.
In arithmetic operations, only the values of $\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}$ *and* $\mathrm{a}_{\mathrm{ij}}^{\mathrm{F}}$ are needed. Thus, from here we only consider the values of $a_{ij} = (\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{F}})$.

Example 2.7. A = $\begin{bmatrix} (0.4, 0.2, 0.3) & (0.6,\ 0.7,\ 0.4) \\ (0.5, 0.3,\ 0.2) & (0.4, 0.3,\ 0.2) \end{bmatrix}$ is a neutrosophic fuzzy matrix of order 2 × 2.

Definition 2.8 Let A = $((a^T_{ij}, a^I_{ij}, a^F_{ij})) = (a_{ij})$ and B = $((b^T_{ij}, b^I_{ij}, b^F_{ij})) = (b_{ij})$ be two neutrosophic fuzzy matrices of order m× n and n × p respectively. Then the product of A and B is defined as

$$AB = (\vee_{k=1}^{n}(a^T_{ik} \wedge a^T_{kj}), \vee_{k=1}^{n}(a^I_{ik} \wedge a^I_{kj}), \wedge_{k=1}^{n}(a^F_{ik} \vee a^F_{kj})).$$

The order of AB will be m × p. The product AB is defined if and only if the number of columns of A is same as the number of rows of B. A and B are said to be comfortable for multiplication.

3. Neutrosophic fuzzy vectors:

In this section, we introduce neutrosophic fuzzy vectors and investigate several properties related to it.

Definition 3.1. Let $V_n$be the set of all n-tuples $((x_{1\alpha},\ x_{1\beta},\ x_{1\gamma}), (x_{2\alpha},\ x_{2\beta},\ x_{2\gamma}), \dots\dots\dots, (x_{n\alpha},\ x_{n\beta},\ x_{n\gamma}))$ over the field F. An element of V_n is called a neutrosophic fuzzy vector (in short, NFV) of dimension n, where $x_{i\alpha},\ x_{i\beta}$ and $x_{i\gamma}$ are the membership, indeterminacy and non-membership values of the component x_i.

The operations addition (+) and multiplication (.) are defined on $V_n$as follows:

Let $x = ((x_{1\alpha},\ x_{1\beta},\ x_{1\gamma}), (x_{2\alpha},\ x_{2\beta},\ x_{2\gamma}), \dots\dots\dots, (x_{n\alpha},\ x_{n\beta},\ x_{n\gamma}))$ and

$y = ((y_{1\alpha},\ y_{1\beta},\ y_{1\gamma}), (y_{2\alpha},\ y_{2\beta},\ y_{2\gamma}), \dots\dots\dots, (y_{n\alpha},\ y_{n\beta},\ y_{n\gamma}))$ be two NFVs in V_n. Then

$x + y$ = ((max($x_{1\alpha},\ y_{1\alpha}$), min($x_{1\beta}, y_{1\beta}$), min($x_{1\gamma}, y_{1\gamma}$)), (max($x_{2\alpha},\ y_{2\alpha}$), min($x_{2\beta}, y_{2\beta}$), min($x_{2\gamma}, y_{2\gamma}$)),………, (min($x_{n\alpha},\ y_{n\alpha}$), min($x_{n\beta}, y_{n\beta}$), min($x_{n\gamma}, y_{n\gamma}$)))

and

ax = ((min(a, $x_{1\alpha}$), max($1 - a$, $x_{1\beta}$), max($1 - a$, $x_{1\gamma}$)), (min(a, $x_{2\alpha}$), max($1 - a$, $x_{2\beta}$), max($1 - a$, $x_{2\gamma}$)),………….., (min(a, $x_{n\alpha}$), max($1 - a$, $x_{n\beta}$), max($1 - a$, $x_{n\gamma}$)))for$a \in [0, 1]$.

The system V_n together with these operations of componentwise addition and multiplication forms neutrosophic fuzzy vectors space (in short, NFVS).

Definition 3.2. Let A = $((a^T_{ij}, a^I_{ij}, a^F_{ij})) \in F_{m\times n}$ be a NFM. Then the element $(a^T_{ij}, a^I_{ij}, a^F_{ij})$ is the ij-the entry of A. Let A_{i*} (respectively A_{*j}) be denote the i-th row (respectively j-th column) of A. The row space R(A) of A is the subspace of $V_n$generated by the rows $\{A_{i*}\}$ of A. The column space C(A) of A is the subspace of $V_n$generated by the column $\{A_{*j}\}$ of A.

Definition 3.3. The row rank $\rho_r(A)$of a NFM is the number of independent rows which generate the row space R(A) of A. The column rank $\rho_c(A)$of a NFM is the number of independent columns which generate the column space C(A) of A.

Then the element $(\mathrm{a}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{I}}, \mathrm{a}_{\mathrm{ij}}^{\mathrm{F}})$ is the ij-the entry of A. Let A_{i*} (respectively A_{*j}) be denote the i-th row (respectively j-th column) of A. The row space R(A) of A is the subspace of $V_n$generated by the rows $\{A_{i*}\}$ of A. The column space C(A) of A is the subspace of $V_n$generated by the column $\{A_{*j}\}$ of A.

Example 3.4. Let us consider the NFM

$$\mathrm{A} = \begin{bmatrix} (0.5, 0.5, 0.5) & (04.,0.5, 0.5) \\ (0.8, 0.2, 0.3) & (0.4, 0.2, 0.4) \end{bmatrix}$$

The row vectors of A are respectively R_1 = ((0.5, 0.5, 0.5), (0.4, 0.5, 0.5)) and R_2 = ((0.8, 0.2, 0.3), (0.4, 0.2, 0.4)) which are linearly dependent as R_1 = cR_2 for c = 0.5∈F.
Therefore, the row rank of A is $\rho_r(A)$ =1.

The column vectors of A are respectively C_1 = ((0.5, 0.5, 0.5), (0.8, 0.2, 0.3)) and R_2 = ((0.4, 0.5, 0.5), (0.4, 0.2, 0.4)) which are linearly independent as $C_1 \neq$ cR_2 for any c ∈F.
So, the column rank of A is $\rho_c(A)$ = 2.

That is, $\rho_r(A) \neq \rho_c(A)$.

Now we consider the NFM

$$\mathrm{B} = \begin{bmatrix} (0.8, 0.2, 0.5) & (0.6,0.4, 0.5) \\ (0.5, 0.4, 0.3) & (0.7, 0.3, 0.4) \end{bmatrix}$$

The set of row and column vectors of B are both linearly. So the row and column ranks of B are equal and is equal to 2.

Definition 3.5. Let S = $\{a_1, a_2$,………,$a_\mathrm{n}\}$ be a set of NFVS of dimension n. The linear combination of elements of the set S is a finite sum $\sum_{i=}^{p} c_i a_i$ where a_i ∈S and c_i ∈ [0, 1]. The set of all linear combinations of the set elements of S is called the span of S, denoted $\langle S \rangle$.

An example of V_3 and its spanning set is given below:

Example 3.6. Let S = $\{a_1, a_2, a_3\}$ be a subset of V_3, where,

a_1 = $(\langle 0.6,\ 0.3, 0.5\rangle, \langle 0.4,\ 0.5, 0.3\rangle,\ \langle 0.7,\ 0.1, 0.3\rangle)$,

a_2 = $(\langle 0.5,\ 0.4, 0.6\rangle, \langle 0.8,\ 0.4, 0.1\rangle,\ \langle 0.6,\ 0.2, 0.5\rangle)$, and $a_3 = (\langle 0.7,\ 0.1, 0.4\rangle, \langle 0.6,\ 0.2, 0.4\rangle,\ \langle 0.5,\ 0.5, 0.4\rangle)$.

Then,

$\langle S\rangle = \{c_1(\langle 0.6,\ 0.3, 0.5\rangle, \langle 0.4,\ 0.5, 0.3\rangle,\ \langle 0.7,\ 0.1, 0.3\rangle) + c_2(\langle 0.5,\ 0.4, 0.6\rangle, \langle 0.8,\ 0.4, 0.1\rangle,\ \langle 0.6,\ 0.2, 0.5\rangle), c_3(\langle 0.7,\ 0.1, 0.4\rangle, \langle 0.6,\ 0.2, 0.4\rangle,\ \langle 0.5,\ 0.5, 0.4\rangle)\}$, where $c_1, c_2, c_3 \in [0, 1]$.

Definition 3.7. Let S be a set of NFVS. Then S is called independent if and only if each element of Scan not be expressed as a linear combination of elements of S, that is, no element $s\in S$ is a linear combination of $S\setminus \{s\}$.

A vector α may be expressed by some other vectors. If it is possible then the vector α is called dependent otherwise it is called independent.

Example 3.8. Let $S = \{a_1, a_2, a_3\}$ be a subset of V_3, where,

$a_1 = (\langle 0.6,\ 0.3, 0.5\rangle, \langle 0.4,\ 0.5, 0.3\rangle,\ \langle 0.7,\ 0.1, 0.3\rangle)$,

$a_2 = (\langle 0.5,\ 0.4, 0.6\rangle, \langle 0.8,\ 0.4, 0.1\rangle,\ \langle 0.6,\ 0.2, 0.5\rangle)$,

and $a_3 = (\langle 0.7,\ 0.1, 0.4\rangle, \langle 0.6,\ 0.2, 0.4\rangle,\ \langle 0.5,\ 0.5, 0.4\rangle)$.

Here the set S is an independent set. If not, then

$a_1 = \alpha(\langle 0.5,\ 0.4, 0.6\rangle, \langle 0.8,\ 0.4, 0.1\rangle,\ \langle 0.6,\ 0.2, 0.5\rangle) + \beta(\langle 0.7,\ 0.1, 0.4\rangle, \langle 0.6,\ 0.2, 0.4\rangle,\ \langle 0.5,\ 0.5, 0.4\rangle)$.

It is not possible to find any $\alpha, \beta \in F$such that the corresponding coefficients on both sides will be equal. That is, $a_1 \neq \alpha a_2 + \beta a_3$.

Similarly, $a_2 \neq \alpha a_1 + \beta a_3$ and $a_3 \neq \alpha a_2 + \beta a_1$.

So the set S is independent.

Example 3.9. Let $S = \{a_1, a_2\}$ be a subset of V_3, where,

$a_1 = (\langle 0.7,\ 0.3, 0.3\rangle, \langle 0.5,\ 0.3, 0.4\rangle,\ \langle 0.6,\ 0.4, 0.3\rangle)$,

$a_2 = (\langle 0.8, 0.3, 0.1\rangle, \langle 0.5,\ 0.1, 0.4\rangle,\ \langle 0.6,\ 0.4, 0.3\rangle)$.

Here $a_1 = \mathrm{c}a_2$ for c = 0.7. So S is a dependent set.

Definition 3.10. Let W be a neutrosophic fuzzy subspace of V_n and S be a subset of W such that the elements of S are independent. If every element of W can be expressed uniquely as a linear combination of the elements of S, then S is called a basis of W.

Definition 3.11. A basis B of NFVS W is a standard basis if and only if whenever $b_i = \sum_{j=1}^{n} a_{ij} b_j$ for b_i, $b_j \in B$ and $a_{ij} \in [0, 1]$, then $a_{ii} = b_i$.

Example 3.12. Let $S = \{a_1, a_2, a_3\}$ be a subset of V_3, where,

$a_1 = (\langle 0.5,\ 0.2, 0.3\rangle, \langle 0.8,\ 0.4, 0.2\rangle,\ \langle 0.7,\ 0.3, 0.2\rangle)$,

$a_2 = (\langle 0.5,\ 0.3, 0.2\rangle, \langle 0.7,\ 0.3,\ 0.2\rangle,\ \langle 0.6,\ 0.5, 0.2\rangle)$,

and $a_3 = (\langle 0.4,\ 0.2, 0.3\rangle, \langle 0.6,\ 0.4, 0.2\rangle,\ \langle 0.6,\ 0.3, 0.2\rangle)$.

Here the set S is an independent set, since $a_1 \neq c_1 a_2 + c_2 a_3$, $a_2 \neq c_3 a_1 + c_4 a_3$ and $a_3 \neq c_5 a_1 + c_6 a_2$. So $\{a_1, a_2, a_3\}$ is a basis for ⟨S⟩. Now this is a standard basis also. For $a_1 = c_{11} a_1 + c_{12} a_2 + c_{13} a_3$ holds if $c_{11} = 0.8$, $c_{12} = 0.5$ and $c_{13} = 0.8$. Also $a_1 = c_{11} a_1$ for $c_{11} = 0.8$. Similarly, for a_2 and a_3.

4. Generalized inverse of a neutrosophic fuzzy matrix:

In this section, we investigate the generalized inverse of a neutrosophic fuzzy matrix.

Definition 4.1. A neutrosophic fuzzy matrix of order $m \times n$ is said to be regular if there exists another neutrosophic fuzzy matrix X of order $m \times n$ such that $AXA = A$. In this case, X is called a generalized (g - inverse) of A and it is denoted by A^{-}.

The g- inverse of a neutrosophic fuzzy matrix is not unique, that is, a neutrosophic fuzzy matrix has many g- inverses. The set of all such g- inverses of A are denoted by $A\{1\}$.

For a regular neutrosophic fuzzy matrix, the row rank and column rank are equal. Then this value is called the rank (ρ) of the neutrosophic fuzzy matrix. The regular neutrosophic fuzzy matrices are the generalization of the invertible matrices.

Theorem 4.2. Let A be a neutrosophic fuzzy matrix whose non - zero rows form a standard basis. If for some neutrosophic fuzzy permutation matrix P, A satisfy the matrix equation $APA = A$ under the max - min operation, then A is regular.

Proof. Here the non-zero rows of a neutrosophic fuzzy matrix A form a standard basis. Let $PA = X$, the rows of X are rearrangement of rows of A. Then X is an idempotent neutrosophic fuzzy matrix, that is $X^2 = X$, having same row space as A with the non-zero rows of X form a standard basis also. Since standard basis are unique, therefore $A = PX$, for some neutrosophic fuzzy permutation matrix P. Then,

$AP^T A = PXP^T PX = PXX = PX = A$, that is, $APA = A$.

Therefore, A is regular.

Definition 4.3. For aneutrosophic fuzzy matrix A of order $m\times n$, a neutrosophic fuzzy matrix of the same order is sad to be outer inverse of A if $GAG = G$ and is denoted by $A\{2\}$.
G is said to be $\{1, 2\}$ inverse or semi inverse of A, if $AGA = A$ and $GAG = G$ and is denoted by $A\{1, 2\}$.
The neutrosophc fuzzy matrix G is said to be $\{1, 3\}$ inverse or least square g- inverse of A if, $AGA = A$ and $(AG)^T = AG$ and is denoted by $A\{1, 3\}$.
Again G is said to be $\{1, 4\}$ inverse or minimum norm g - inverse of A if $AGA = A$ and $(GA)^T = GA$ and is denoted by $A\{1, 4\}$.

Definition 4.4. A square neutrosophic fuzzy matrix is called neutrosophic fuzzy permutation matrix if every row and column contain exactly one (1, 0, 0) and all other entries are (0, 1, 1).

Definition 4.5. The NFMs A$\in F_m$ and B$\in F_n$ are said to be similar, denoted by A≂B if ∃ an idempotent NFM X$\in F_{m\times n}$ and the NFM Y$\in F_{n\times m}$ such that A = XBY, B = YAX and X = XYX.

Definition 4.6. The NFMs A$\in F_m$ and B$\in F_n$ are said to be semi-similar, denoted by A≈B if ∃ NFMs X$\in F_{m\times n}$ and Y$\in F_{n\times m}$ such that A = XBY and B = YAX.

Remark 4.7. Every pair of neutrosophic fuzzy similar matrices are semi-similar.

Theorem 4.8. Let the NFMs A$\in F_m$ and B$\in F_n$. Then the following are equivalent:

(i) A≂B.

(ii) ∃ an idempotent NFM X$\in F_{m\times n}$ and another NFM Y$\in F_{n\times m}$ such that A = XBY, B = YAX and XY$\in F_m$ is idempotent.

(iii) ∃ an idempotent NFM X$\in F_{m\times n}$ and Y$\in F_{n\times m}$ such that A = XBY, B = YAX and YX$\in F_n$ is idempotent.

Proof. (i) ⇒ (ii).
From definition of similarity, A≂B implies A = XBY, B = YAX and X = XYX. From the third relation,

X = XYX

or, XY = XYXY
or, XY = $(XY)^2$
That is, XY is idempotent.
(i) ⇒ (iii).
Let
X = XYX

or, YX = YXYX
or, XY = $(YX)^2$

That is, YX is idempotent.

(ii) ⇒ (i).
We have, A = XBY = XYAXY = (XYX)B(YXY).
Similarly, B = YAX = YXBYX = (YXY)A(XYX).
Putting C = XYX and D = YXY, we get, A = CBD and B = DAC.
Now, CD = (XYX)(YXY) = (XY)(XY)(XY) = XY, as XY is idempotent.
Again, (CD)(CD) = (XY)(XY) = XY.
Thus, CD is also idempotent.
Let us consider E = CDC and F = DCD.
Then, A = CBD = CDACD = (CDC)B(DCD) = EBF.
Similarly, B = DAC = DCBDC = DCDACDC = FAE.
Again, EFE = (CDC)(DCD)(CDC) = (CD)(CD)(CD)C =CDC = E as CD is idempotent.
Therefore, A≂B.
Similarly, we can show that (iii) ⇒ (i).

Theorem 4.9. Let A∈ F_m and B∈ F_n be two NFMs such that A≂B. Then A is idempotent if and only if B is idempotent.

Proof: Since A≂B, then from the definition, ∃ an idempotent NFM X∈ $F_{m\times n}$ and the NFM Y∈ $F_{n\times m}$ such that A = XBY, B = YAX and X = XYX.
Then XYA = XYXBY = XBY = A.
Suppose A is idempotent. Then A^2 = A.
Now, B^2 = (YAX)(YAX) = YA(XYA)X = YA^2X = YAX = B.
That is B is idempotent.
Similarly, we can prove the converse part.

Theorem 4.10. Let A∈ F_m and B∈ F_n be two NFMs such that A≂B. Then for idempotent NFM X∈ $F_{m\times n}$ and Y∈ $F_{n\times m}$ A is regular if and only if B is regular.

Proof: Since A≂B, then from the definition, ∃ an idempotent NFM X∈ $F_{m\times n}$ and the NFM Y∈ $F_{n\times m}$ such that A = XBY, B = YAX and X = XYX.
Now Y is idempotent, that is Y^2 = Y and A is regular, then ∃ G∈ F_m such that AGA = A.
Let us define U = YX.
Then clearly, U∈ F_n. Then
BUB = (YAX)YX(YAX)
= YA(XYX)YAX
= YA(XYA)X
= Y(AGA)(XYA)X (as AGA = X)
= YAGAAX (as XYA = A)

= YAGAX
= YAX
= B.

Hence B is regular.
Similarly, we can prove the converse part.

Theorem 4.11. Let A and B be two neutrosophic fuzzy matrices having each of order $m\times n$. If A is regular, then

(a) $R(B)\subseteq R(A)$ if and only if $B = BA^{-}A$ for each $A^{-} \in A\{1\}$.
(b) $C(B) \subseteq C(A)$ if and only if $B = AA^{-}B$ for each $A^{-} \in A\{1\}$.

Proof. (a) Let $R(B)\subseteq R(A)$. Then each row of B is a linear combination of the rows of A. Hence

$B_{j^*} = \sum x_{ij}A_{j^*}$, where $x_{ij} \in \langle F\rangle$.
That is, $B = XA$ (for some neutrosophic fuzzy matrix X of order m)
or, $B = XAA^{-}A$(since $A = AA^{-}A$)
or, $B = BA^{-}A$.
Conversely, if $B = BA^{-}A$, then $B = XAA^{-}A$ (for some neutrosophic fuzzy matrix X of order m)
or, $B = XA$ (since $A = AA^{-}A$).
This implies that $R(B)\subseteq R(A)$.
(b) Let $C(B) \subseteq C(A)$. Then B = AY (for some neutrosophic fuzzy matrix Y of order n)
or, $B = AA^{-}AY$,(as$A = AA^{-}A$).
That is $C(B) \subseteq C(A)$.

5. Conclusion:

Neutrosophic fuzzy matrix provides extra advantages to solve real-life problems involving uncertainties, occurring in an imprecise environment. In this paper, we have introduced neutrosophic fuzzy vectors. We have investigated several properties of it. We further incorporated some suitable numerical examples of neutrosophic fuzzy vectors. We have studied the generalized inverse of a neutrosophic fuzzy matrix. We have established some of its basic properties. We have defined the condition for similarity of two neutrosophic fuzzy matrices and established some of its basic properties. We hope the article will be helpful for future research work of games in neutrosophic environment.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986) 87 - 96.
[2] S. Broumi,L. H. Son, A. Bakali, M.Talea, F. Smarandache and G. Selvachandran, Computing Operational Matrices in Neutrosophic Environments: A Matlab Toolbox, *Neutrosophic Sets and Systems*, 18 (2017) 58-66.

[3] S. Broumi, A. Bakali, M. Talea and F. Smarandache, (2017). A Matlab Toolbox for interval valued neutrosophic matrices for computer applications, *UluslararasıYönetim Bilişim Sistemlerive Bilgisayar Bilimleri Dergisi*, 1(1) (2017) 1-21.
[4] R. Das, F. Smarandache and B. C. Tripathy, Neutrosophic Fuzzy Matrices and Some Algebraic Operations, *Neutrosophic Sets and Systems*, 32 (2020) 401-409.
[5] D. Das, R. Dhar and B. C. Tripathy, Properties of multiplication operation of neutrosophic fuzzy matrices.*Neutrosophc Sets and Systems*, 61 (2023) 134 - 148.
[6] M. Dhar, S. Broumi and F. Smarandache, A Note on Square Neutrosophic Fuzzy Matrices, *Neutrosophic Sets and Systems*, 3 (2014) 37-41.
[7] M. Dhar, A Note on Determinant and Adjoint of Fuzzy Square Matrix, *I. J. Intelligent Systems and Applications,* 5 (2013) 58-67.
[8] W. B. V. Kandasamy and F. Smarandache, Fuzzy Relational Maps and Neutrosophic Relational Maps, *HEXIS Church Rock*, 302 (2004).
[9] W. B. V. Kandasamy and F. Smarandache, Neutrosophic Rings, Hexis, Phoenix, Arizona: Infinite Study, (2006).
[10] H. Khaled, A. Younus and A. Mohammad, The Rectangle Neutrosophic Fuzzy Matrices, *Faculty of Education Journal*, 15 (2019).
[11] S. K. Khan and A. Pal, The Generalised Inverse of Intuitionistic Fuzzy Matrices, *Journal of Physical Sciences*, 11 (2017) 62-67.
[12] J. B. Kim and A. Baartmans, Determinant Theory for Fuzzy Matrices. *Fuzzy Sets and Systems,* 29 (1989) 349-356.
[13] G. Pal, R. Dhar and B. C. Tripathy, Minimal structures and grill in neutrosophic topological Spaces, *Neutrosophic Sets and Systems*, 51 (2022) 134-145.
[14] G. Pal, B. C. Tripathy and R. Dhar, On continuity in minimal structure neutrosophic topological space, *Neutrosophic Sets and Systems*, 51 (2022) 360-370.
[15] R. Pradhan and M. Pal, Some Results on Generalised Inverse of Intuitionistic Fuzzy Matrices, *Fuzzy Inf. Eng*., 6 (2014) 133-145.
[16] F. Smarandache, Neutrosophy. Neutrosophic Probability, Set and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 (1998).
[17] M. G. Thomas, Convergence of powers of a fuzzy matrix, *J.Math. Annal. Appl.*, 57 (1977) 476-480.
[18] S. Topal, S. Broumi, A. Bakali, M. Talea and F. Smarandache, (2019). A Python Tool for Implementations on Bipolar Neutrosophic Matrices, *Neutrosophic Sets and Systems*, 28 (2019) 138-161.
[19] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Information Science,* 8(3) (1975) 199-249.
[20] L. A. Zadeh, Fuzzy sets, *Information and Control,* 8 (1965) 338 - 353.

Equalities on Intuitionistic Fuzzy Operators and Operations

Jaydip Bhattacharya

Department of Mathematics, Bir Bikram Memorial College
Agartala, West Tripura, Pin-799004, India
e-mail: `jay73bhatta@gmail.com`

Abstract: It is well known that intuitionistic fuzzy sets can be transformed into fuzzy sets with the help of modal operators. Such operators are playing significant role in intuitionistic fuzzy set theory. There are various operators and operations in intuitionistic fuzzy set theory. With the help of first type modal operators and some special operators we establish some new equalities in intuitionistic fuzzy sets.

Keywords: Fuzzy sets, Intuitionistic fuzzy sets, Modal operators, Operations.
AMS Classification: 54A40, 03E72.

1 Introduction

L.A. Zadeh [14] invented and developed the concept of a fuzzy set in 1965. As an extension of fuzzy sets, Atanassov [1] presented the idea of intuitionistic fuzzy sets in 1983, which was eighteen years later. The basic difference of these two concepts is that in fuzzy set theory only membership function has been taken into account while in intuitionistic fuzzy set theory membership function and non-membership function both are considered along with hesitation margin. Researchers [5, 7, 8, 9, 10, 11, 12, 13] are working hard to develop and improve this subject. There are some operators D_α, $F_{\alpha,\beta}$, $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, and $J^*_{\alpha,\beta}$ as well as some operations like $*$, $\odot$, $\bowtie$, ∞, $\lhd$ and $\rhd$ in intuitionistic fuzzy set theory which are playing important role. We attempt to focus on the relationship between these operators and operations in the main section of this paper, as well as to demonstrate some important equalities on intuitionistic fuzzy sets.

2 Preliminaries

Definition 2.1[2] Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form A=$\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in X\}$,where the functions $\mu_A,\nu_A : x\rightarrow[0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x\in X$ to the set A, which is a subset of X, and for every element $x\in X$,$0\leq\mu_A(x)+\nu_A(x)\leq 1$. Furthermore, we have $\pi_A(x)$= 1-$\mu_A(x)$-$\nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of $x\in X$ to the IFS A and $\pi_A(x)\in[0,1]$ that is $\pi_A : x\rightarrow[0,1]$ and $0\leq\pi_A(x)\leq 1$ for every $x\in X$.
$\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.
Definition 2.2 [4] Let X be a nonempty set. If A is an IFS drawn from X, then,

1. $\Box A$= $\{\langle x,\mu_A(x),1-\mu_A(x)\rangle : x\in X\}$

2. $\Diamond A$= $\{\langle x,1-\nu_A(x),\nu_A(x)\rangle : x\in X\}$

For a proper IFS, $\Box A\subset A\subset\Diamond A$ and $\Box A\neq A\neq\Diamond A$.
Definition 2.3 [4] Let X be a nonempty set. If A is an IFS drawn from X, then,

1. $\boxplus A$= $\{\langle x,\frac{\mu_A(x)}{2},\frac{\nu_A(x)+1}{2}\rangle : x\in X\}$

2. $\boxtimes A$= $\{\langle x,\frac{\mu_A(x)+1}{2},\frac{\nu_A(x)}{2}\rangle : x\in X\}$

For a proper IFS, $\boxplus A\subset A\subset\boxtimes A$ and $\boxplus A\neq A\neq\boxtimes A$.
Definition 2.4 [4]Let $\alpha,\beta\in[0,1] and A\in$ IFS X. Then the following operators can be defined as

1. $D_\alpha(A)=\{\langle x,\mu_A(x)+\alpha\pi_A(x),\nu_A(x)+(1-\alpha)\pi_A(x)\rangle : x\in X\}$

2. $F_{\alpha,\beta}(A)=\{\langle x,\mu_A(x)+\alpha\pi_A(x),\nu_A(x)+\beta\pi_A(x)\rangle : x\in X\}$, **where** $\alpha+\beta\leq 1$.

3. $G_{\alpha,\beta}(A)=\{\langle x,\alpha\mu_A(x),\beta\nu_A(x)\rangle : x\in X\}$, **where** $\alpha+\beta\leq 1$.

4. $H_{\alpha,\beta}(A)=\{\langle x,\alpha\mu_A(x),\nu_A(x)+\beta\pi_A(x)\rangle : x\in X\}$, **where** $\alpha+\beta\leq 1$.

5. $H^*_{\alpha,\beta}(A)=\{\langle x,\alpha\mu_A(x),\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))\rangle : x\in X\}$, **where** $\alpha+\beta\leq 1$.
6. $J_{\alpha,\beta}(A)=\{\langle x,\mu_A(x)+\alpha\pi_A(x),\beta\nu_A(x)\rangle : x\in X\}$, **where** $\alpha+\beta\leq 1$.

7. $J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x)\rangle : x \in X\}$, **where** $\alpha + \beta \leq 1$.

Definition 2.5 [3, 6] Let X be a nonempty set. If A and B be two IFSs drawn from X, then,

1. $A * B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2(\mu_A(x)+\mu_B(x)+1)}, \frac{\nu_A(x)+\nu_B(x)}{2(\nu_A(x)+\nu_B(x)+1)}\rangle : x \in X\}$
2. $A \odot B = \{\langle x, \frac{\mu_A(x)\mu_B(x)}{2(\mu_A(x)\mu_B(x)+1)}, \frac{\nu_A(x)\nu_B(x)}{2(\nu_A(x)\nu_B(x)+1)}\rangle : x \in X\}$
3. $A \bowtie B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2(\mu_A(x)+\mu_B(x))+1}, \frac{\nu_A(x)+\nu_B(x)}{2(\nu_A(x)+\nu_B(x))+1}\rangle : x \in X\}$
4. $A \infty B = \{\langle x, \frac{\mu_A(x)\mu_B(x)}{2(\mu_A(x)\mu_B(x))+1}, \frac{\nu_A(x)\nu_B(x)}{2(\nu_A(x)\nu_B(x))+1}\rangle : x \in X\}$
5. $A \rhd B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{\mu_A(x)+\mu_B(x)+1}, \frac{\nu_A(x)+\nu_B(x)}{\nu_A(x)+\nu_B(x)+1}\rangle : x \in X\}$
6. $A \lhd B = \{\langle x, \frac{\mu_A(x)\mu_B(x)}{\mu_A(x)\mu_B(x)+1}, \frac{\nu_A(x)\nu_B(x)}{\nu_A(x)\nu_B(x)+1}\rangle : x \in X\}$

3 Main results

Throughout this paper, intuitionistic fuzzy set and fuzzy set are denoted by IFS and FS respectively.

Theorem 3.1 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, **where $\alpha + \beta \leq 1$, then**

1. $[\boxplus(D_\alpha(A * B))]^C = \boxtimes[D_\alpha(A * B)]^C$
2. $[\boxtimes(D_\alpha(A * B))]^C = \boxplus[D_\alpha(A * B)]^C$
3. $[\boxplus(F_{\alpha,\beta}(A * B))]^C = \boxtimes[F_{\alpha,\beta}(A * B)]^C$
4. $[\boxtimes(F_{\alpha,\beta}(A * B))]^C = \boxplus[F_{\alpha,\beta}(A * B)]^C$
5. $[\boxplus(G_{\alpha,\beta}(A * B))]^C = \boxtimes[G_{\alpha,\beta}(A * B)]^C$
6. $[\boxtimes(G_{\alpha,\beta}(A * B))]^C = \boxplus[G_{\alpha,\beta}(A * B)]^C$

7. $[\boxplus(H_{\alpha,\beta}(A*B))]^C = \boxtimes[H_{\alpha,\beta}(A*B)]^C$

8. $[\boxtimes(H_{\alpha,\beta}(A*B))]^C = \boxplus[H_{\alpha,\beta}(A*B)]^C$

9. $[\boxplus(H^*_{\alpha,\beta}(A*B))]^C = \boxtimes[H^*_{\alpha,\beta}(A*B)]^C$

10. $[\boxtimes(H^*_{\alpha,\beta}(A*B))]^C = \boxplus[H^*_{\alpha,\beta}(A*B)]^C$

11. $[\boxplus(J_{\alpha,\beta}(A*B))]^C = \boxtimes[J_{\alpha,\beta}(A*B)]^C$

12. $[\boxtimes(J_{\alpha,\beta}(A*B))]^C = \boxplus[J_{\alpha,\beta}(A*B)]^C$

13. $[\boxplus(J^*_{\alpha,\beta}(A*B))]^C = \boxtimes[J^*_{\alpha,\beta}(A*B)]^C$

14. $[\boxtimes(J^*_{\alpha,\beta}(A*B))]^C = \boxplus[J^*_{\alpha,\beta}(A*B)]^C$

Proof 1. Now $D_\alpha(A*B)$ = $\langle \mu_{A*B}(x) + \alpha\pi_{A*B}(x), \nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x) \rangle$
$\boxplus(D_\alpha(A*B))$ = $\langle \frac{1}{2}(\mu_{A*B}(x) + \alpha\pi_{A*B}(x)), \frac{1}{2}(\nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x)) + 1) \rangle$
$[\boxplus(D_\alpha(A*B))]^C$ = $\langle \frac{1}{2}(\nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x)) + 1), \frac{1}{2}(\mu_{A*B}(x) + \alpha\pi_{A*B}(x)) \rangle$
Again, $[D_\alpha(A*B)]^C$ = $\langle \nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x), \mu_{A*B}(x) + \alpha\pi_{A*B}(x) \rangle$
$\boxtimes[D_\alpha(A*B)]^C$ = $\langle \frac{1}{2}[\nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x) + 1], \frac{1}{2}[\mu_{A*B}(x) + \alpha\pi_{A*B}(x)] \rangle$
Hence $[\boxplus(D_\alpha(A*B))]^C = \boxtimes[D_\alpha(A*B)]^C$
Similarly 2. to 14. can be proved.

Theorem 3.2 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus \boxtimes (D_\alpha(A*B))]^C = \boxtimes \boxplus [D_\alpha(A*B)]^C$

2. $[\boxtimes \boxplus (D_\alpha(A*B))]^C = \boxplus \boxtimes [D_\alpha(A*B)]^C$

3. $[\boxplus \boxtimes (F_{\alpha,\beta}(A*B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A*B)]^C$

4. $[\boxtimes \boxplus (F_{\alpha,\beta}(A*B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A*B)]^C$

5. $[\boxplus\boxtimes(G_{\alpha,\beta}(A*B))]^C = \boxtimes\boxplus[G_{\alpha,\beta}(A*B)]^C$

6. $[\boxtimes\boxplus(G_{\alpha,\beta}(A*B))]^C = \boxplus\boxtimes[G_{\alpha,\beta}(A*B)]^C$

7. $[\boxplus\boxtimes(H_{\alpha,\beta}(A*B))]^C = \boxtimes\boxplus[H_{\alpha,\beta}(A*B)]^C$

8. $[\boxtimes\boxplus(H_{\alpha,\beta}(A*B))]^C = \boxplus\boxtimes[H_{\alpha,\beta}(A*B)]^C$

9. $[\boxplus\boxtimes(H^*_{\alpha,\beta}(A*B))]^C = \boxtimes\boxplus[H^*_{\alpha,\beta}(A*B)]^C$

10. $[\boxtimes\boxplus(H^*_{\alpha,\beta}(A*B))]^C = \boxplus\boxtimes[H^*_{\alpha,\beta}(A*B)]^C$

11. $[\boxplus\boxtimes(J_{\alpha,\beta}(A*B))]^C = \boxtimes\boxplus[J_{\alpha,\beta}(A*B)]^C$

12. $[\boxtimes\boxplus(J_{\alpha,\beta}(A*B))]^C = \boxplus\boxtimes[J_{\alpha,\beta}(A*B)]^C$

13. $[\boxplus\boxtimes(J^*_{\alpha,\beta}(A*B))]^C = \boxtimes\boxplus[J^*_{\alpha,\beta}(A*B)]^C$

14. $[\boxtimes\boxplus(J^*_{\alpha,\beta}(A*B))]^C = \boxplus\boxtimes[J^*_{\alpha,\beta}(A*B)]^C$

Proof 2. Now $D_\alpha(A*B)$ = $\langle \mu_{A*B}(x)+\alpha\pi_{A*B}(x), \nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x)\rangle$
$\boxplus(D_\alpha(A*B))$ = $\langle \frac{1}{2}(\mu_{A*B}(x)+\alpha\pi_{A*B}(x)), \frac{1}{2}(\nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x)+1)\rangle$
$[\boxtimes\boxplus(D_\alpha(A*B))]$ = $\langle \frac{1}{2}[\frac{1}{2}(\mu_{A*B}(x)+\alpha\pi_{A*B}(x))+1], \frac{1}{4}(\nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x)+1)\rangle$
$[\boxtimes\boxplus(D_\alpha(A*B))]^C$ = $\langle \frac{1}{4}(\nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x)+1), \frac{1}{2}[\frac{1}{2}(\mu_{A*B}(x)+\alpha\pi_{A*B}(x))+1]\rangle$
Again, $[D_\alpha(A*B)]^C$ = $\langle \nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x), \mu_{A*B}(x)+\alpha\pi_{A*B}(x)\rangle$
$\boxtimes[D_\alpha(A*B)]^C$ = $\langle \frac{1}{2}[\nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x)+1], \frac{1}{2}[\mu_{A*B}(x)+\alpha\pi_{A*B}(x)]\rangle$
$\boxplus\boxtimes[D_\alpha(A*B)]^C$ = $\langle \frac{1}{4}[\nu_{A*B}(x)+(1-\alpha)\pi_{A*B}(x)+1], \frac{1}{2}[\frac{1}{2}(\mu_{A*B}(x)+\alpha\pi_{A*B}(x))+1]\rangle$
Hence $[\boxtimes\boxplus(D_\alpha(A*B))]^C = \boxplus\boxtimes[D_\alpha(A*B)]^C$
Similarly, other parts can be proved.

Theorem 3.3 Let X **be a nonempty set. If** A **and** B **be any two IFSs drawn from** X **and** $\alpha, \beta \in [0,1]$**, where** $\alpha+\beta \leq 1$**, then**

1. $[\boxplus(D_\alpha(A\odot B))]^C = \boxtimes[D_\alpha(A\odot B)]^C$

2. $[\boxtimes(D_\alpha(A \odot B))]^C = \boxplus[D_\alpha(A \odot B)]^C$

3. $[\boxplus(F_{\alpha,\beta}(A \odot B))]^C = \boxtimes[F_{\alpha,\beta}(A \odot B)]^C$

4. $[\boxtimes(F_{\alpha,\beta}(A \odot B))]^C = \boxplus[F_{\alpha,\beta}(A \odot B)]^C$

5. $[\boxplus(G_{\alpha,\beta}(A \odot B))]^C = \boxtimes[G_{\alpha,\beta}(A \odot B)]^C$

6. $[\boxtimes(G_{\alpha,\beta}(A \odot B))]^C = \boxplus[G_{\alpha,\beta}(A \odot B)]^C$

7. $[\boxplus(H_{\alpha,\beta}(A \odot B))]^C = \boxtimes[H_{\alpha,\beta}(A \odot B)]^C$

8. $[\boxtimes(H_{\alpha,\beta}(A \odot B))]^C = \boxplus[H_{\alpha,\beta}(A \odot B)]^C$

9. $[\boxplus(H^*_{\alpha,\beta}(A \odot B))]^C = \boxtimes[H^*_{\alpha,\beta}(A \odot B)]^C$

10. $[\boxtimes(H^*_{\alpha,\beta}(A \odot B))]^C = \boxplus[H^*_{\alpha,\beta}(A \odot B)]^C$

11. $[\boxplus(J_{\alpha,\beta}(A \odot B))]^C = \boxtimes[J_{\alpha,\beta}(A \odot B)]^C$

12. $[\boxtimes(J_{\alpha,\beta}(A \odot B))]^C = \boxplus[J_{\alpha,\beta}(A \odot B)]^C$

13. $[\boxplus(J^*_{\alpha,\beta}(A \odot B))]^C = \boxtimes[J^*_{\alpha,\beta}(A \odot B)]^C$

14. $[\boxtimes(J^*_{\alpha,\beta}(A \odot B))]^C = \boxplus[J^*_{\alpha,\beta}(A \odot B)]^C$

Proof 9. Let us suppose that, $\alpha + \beta \leq 1$
Now $H^*_{\alpha,\beta}(A \odot B) = \langle \alpha\mu_{A\odot B}(x), \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x))\rangle$
$\boxplus(H^*_{\alpha,\beta}(A \odot B)) = \langle \frac{1}{2}(\alpha\mu_{A\odot B}(x), \frac{1}{2}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) + 1]\rangle$
$[\boxplus(H^*_{\alpha,\beta}(A \odot B))]^C = \langle \frac{1}{2}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) + 1], \frac{1}{2}(\alpha\mu_{A\odot B}(x)\rangle$
Again, $[H^*_{\alpha,\beta}(A \odot B)]^C = \langle \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x)), \alpha\mu_{A\odot B}(x)\rangle$
$\boxtimes[H^*_{\alpha,\beta}(A \odot B)]^C = \langle \frac{1}{2}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) + 1], \frac{1}{2}(\alpha\mu_{A\odot B}(x)\rangle$
Hence $[\boxplus(H^*_{\alpha,\beta}(A \odot B))]^C = \boxtimes[H^*_{\alpha,\beta}(A \odot B)]^C$
Similarly other parts can be proved.

Theorem 3.4 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus \boxtimes (D_\alpha(A \odot B))]^C = \boxtimes \boxplus [D_\alpha(A \odot B)]^C$

2. $[\boxtimes \boxplus (D_\alpha(A \odot B))]^C = \boxplus \boxtimes [D_\alpha(A \odot B)]^C$

3. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \odot B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A \odot B)]^C$

4. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A \odot B)]^C$

5. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \odot B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A \odot B)]^C$

6. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A \odot B)]^C$

7. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \odot B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A \odot B)]^C$

8. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A \odot B)]^C$

9. $[\boxplus \boxtimes (H^*_{\alpha,\beta}(A \odot B))]^C = \boxtimes \boxplus [H^*_{\alpha,\beta}(A \odot B)]^C$

10. $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A \odot B)]^C$

11. $[\boxplus \boxtimes (J_{\alpha,\beta}(A \odot B))]^C = \boxtimes \boxplus [J_{\alpha,\beta}(A \odot B)]^C$

12. $[\boxtimes \boxplus (J_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [J_{\alpha,\beta}(A \odot B)]^C$

13. $[\boxplus \boxtimes (J^*_{\alpha,\beta}(A \odot B))]^C = \boxtimes \boxplus [J^*_{\alpha,\beta}(A \odot B)]^C$

14. $[\boxtimes \boxplus (J^*_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [J^*_{\alpha,\beta}(A \odot B)]^C$

Proof 10. Let us suppose that, $\alpha + \beta \leq 1$
Now $H^*_{\alpha,\beta}(A \odot B) = \langle \alpha\mu_{A\odot B}(x), \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x)) \rangle$
$\boxplus(H^*_{\alpha,\beta}(A \odot B)) = \langle \frac{1}{2}(\alpha\mu_{A\odot B}(x), \frac{1}{2}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) \lrcorner \quad \llcorner /$

$\boxtimes \boxplus (H^*_{\alpha,\beta}(A \odot B)) = \langle \frac{1}{2}[\frac{1}{2}(\alpha\mu_{A\odot B}(x) + 1], \frac{1}{4}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) + 1]\rangle$

$[\boxtimes\boxplus(H^*_{\alpha,\beta}(A\odot B))]^C = \langle \frac{1}{4}[\nu_{A\odot B}(x)+\beta(1-\alpha\mu_{A\odot B}(x))-\nu_{A\odot B}(x))+1], \frac{1}{2}[\frac{1}{2}(\alpha\mu_{A\odot B}(x)+1]\rangle$
Again,$[H^*_{\alpha,\beta}(A \odot B)]^C = \langle \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x)), \alpha\mu_{A\odot B}(x)\rangle$
$\boxtimes[H^*_{\alpha,\beta}(A \odot B)]^C = \langle \frac{1}{2}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) + 1], \frac{1}{2}(\alpha\mu_{A\odot B}(x)\rangle$
$\boxplus \boxtimes [H^*_{\alpha,\beta}(A \odot B)]^C = \langle \frac{1}{4}[\nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x)) - \nu_{A\odot B}(x)) + 1], \frac{1}{2}[\frac{1}{2}(\alpha\mu_{A\odot B}(x) + 1]\rangle$
Hence $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A \odot B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A \odot B)]^C$
Similarly other parts can be proved.

Theorem 3.5 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus(D_\alpha(A \bowtie B))]^C = \boxtimes[D_\alpha(A \bowtie B)]^C$

2. $[\boxtimes(D_\alpha(A \bowtie B))]^C = \boxplus[D_\alpha(A \bowtie A)]^C$

3. $[\boxplus(F_{\alpha,\beta}(A \bowtie B))]^C = \boxtimes[F_{\alpha,\beta}(A \bowtie B)]^C$

4. $[\boxtimes(F_{\alpha,\beta}(A \bowtie B))]^C = \boxplus[F_{\alpha,\beta}(A \bowtie B)]^C$

5. $[\boxplus(G_{\alpha,\beta}(A \bowtie B))]^C = \boxtimes[G_{\alpha,\beta}(A \bowtie B)]^C$

6. $[\boxtimes(G_{\alpha,\beta}(A \bowtie B))]^C = \boxplus[G_{\alpha,\beta}(A \bowtie B)]^C$

7. $[\boxplus(H_{\alpha,\beta}(A \bowtie B))]^C = \boxtimes[H_{\alpha,\beta}(A \bowtie B)]^C$

8. $[\boxtimes(H_{\alpha,\beta}(A \bowtie B))]^C = \boxplus[H_{\alpha,\beta}(A \bowtie B)]^C$

9. $[\boxplus(H^*_{\alpha,\beta}(A \bowtie B))]^C = \boxtimes[H^*_{\alpha,\beta}(A \bowtie B)]^C$

10. $[\boxtimes(H^*_{\alpha,\beta}(A \bowtie B))]^C = \boxplus[H^*_{\alpha,\beta}(A \bowtie B)]^C$

11. $[\boxplus(J_{\alpha,\beta}(A \bowtie B))]^C = \boxtimes[J_{\alpha,\beta}(A \bowtie B)]^C$

12. $[\boxtimes(J_{\alpha,\beta}(A \bowtie B))]^C = \boxplus[J_{\alpha,\beta}(A \bowtie B)]^C$

13. $[\boxplus(J^{*}_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes[J^{*}_{\alpha,\beta}(A \bowtie B)]^{C}$

14. $[\boxtimes(J^{*}_{\alpha,\beta}(A \bowtie B))]^{C} = \boxplus[J^{*}_{\alpha,\beta}(A \bowtie B)]^{C}$

Proof Similar to Theorem 3.3.

Theorem 3.6 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus \boxtimes (D_{\alpha}(A \bowtie B))]^{C} = \boxtimes \boxplus [D_{\alpha}(A \bowtie B)]^{C}$

2. $[\boxtimes \boxplus (D_{\alpha}(A \bowtie B))]^{C} = \boxplus \boxtimes [D_{\alpha}(A \bowtie B)]^{C}$

3. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes \boxplus [F_{\alpha,\beta}(A \bowtie B)]^{C}$

4. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \bowtie B))]^{C} = \boxplus \boxtimes [F_{\alpha,\beta}(A \bowtie B)]^{C}$

5. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes \boxplus [G_{\alpha,\beta}(A \bowtie B)]^{C}$

6. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \bowtie B))]^{C} = \boxplus \boxtimes [G_{\alpha,\beta}(A \bowtie B)]^{C}$

7. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes \boxplus [H_{\alpha,\beta}(A \bowtie B)]^{C}$

8. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \bowtie B))]^{C} = \boxplus \boxtimes [H_{\alpha,\beta}(A \bowtie B)]^{C}$

9. $[\boxplus \boxtimes (H^{*}_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes \boxplus [H^{*}_{\alpha,\beta}(A \bowtie B)]^{C}$

10. $[\boxtimes \boxplus (H^{*}_{\alpha,\beta}(A \bowtie B))]^{C} = \boxplus \boxtimes [H^{*}_{\alpha,\beta}(A \bowtie B)]^{C}$

11. $[\boxplus \boxtimes (J_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes \boxplus [J_{\alpha,\beta}(A \bowtie B)]^{C}$

12. $[\boxtimes \boxplus (J_{\alpha,\beta}(A \bowtie B))]^{C} = \boxplus \boxtimes [J_{\alpha,\beta}(A \bowtie B)]^{C}$

13. $[\boxplus \boxtimes (J^{*}_{\alpha,\beta}(A \bowtie B))]^{C} = \boxtimes \boxplus [J^{*}_{\alpha,\beta}(A \bowtie B)]^{C}$

14. $[\boxtimes \boxplus (J^*_{\alpha,\beta}(A \bowtie B))]^C = \boxplus \boxtimes [J^*_{\alpha,\beta}(A \bowtie B)]^C$

Proof Similar to theorem 3.4.

Theorem 3.7 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus(D_\alpha(A \infty B))]^C = \boxtimes[D_\alpha(A \infty B)]^C$

2. $[\boxtimes(D_\alpha(A \infty B))]^C = \boxplus[D_\alpha(A \infty B)]^C$

3. $[\boxplus(F_{\alpha,\beta}(A \infty B))]^C = \boxtimes[F_{\alpha,\beta}(A \infty B)]^C$

4. $[\boxtimes(F_{\alpha,\beta}(A \infty B))]^C = \boxplus[F_{\alpha,\beta}(A \infty B)]^C$

5. $[\boxplus(G_{\alpha,\beta}(A \infty B))]^C = \boxtimes[G_{\alpha,\beta}(A \infty B)]^C$

6. $[\boxtimes(G_{\alpha,\beta}(A \infty B))]^C = \boxplus[G_{\alpha,\beta}(A \infty B)]^C$

7. $[\boxplus(H_{\alpha,\beta}(A \infty B))]^C = \boxtimes[H_{\alpha,\beta}(A \infty B)]^C$

8. $[\boxtimes(H_{\alpha,\beta}(A \infty B))]^C = \boxplus[H_{\alpha,\beta}(A \infty B)]^C$

9. $[\boxplus(H^*_{\alpha,\beta}(A \infty B))]^C = \boxtimes[H^*_{\alpha,\beta}(A \infty B)]^C$

10. $[\boxtimes(H^*_{\alpha,\beta}(A \infty B))]^C = \boxplus[H^*_{\alpha,\beta}(A \infty B)]^C$

11. $[\boxplus(J_{\alpha,\beta}(A \infty B))]^C = \boxtimes[J_{\alpha,\beta}(A \infty B)]^C$

12. $[\boxtimes(J_{\alpha,\beta}(A \infty B))]^C = \boxplus[J_{\alpha,\beta}(A \infty B)]^C$

13. $[\boxplus(J^*_{\alpha,\beta}(A \infty B))]^C = \boxtimes[J^*_{\alpha,\beta}(A \infty B)]^C$

14. $[\boxtimes(J^*_{\alpha,\beta}(A \infty B))]^C = \boxplus[J^*_{\alpha,\beta}(A \infty B)]^C$

Proof Similar to Theorem 3.3.

Theorem 3.8 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus \boxtimes (D_\alpha(A \infty B))]^C = \boxtimes \boxplus [D_\alpha(A \infty B)]^C$
2. $[\boxtimes \boxplus (D_\alpha(A \infty B))]^C = \boxplus \boxtimes [D_\alpha(A \infty B)]^C$
3. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \infty B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A \infty B)]^C$
4. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \infty B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A \infty B)]^C$
5. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \infty B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A \infty B)]^C$
6. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \infty B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A \infty B)]^C$
7. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \infty B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A \infty B)]^C$
8. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \infty B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A \infty B)]^C$
9. $[\boxplus \boxtimes (H^*_{\alpha,\beta}(A \infty B))]^C = \boxtimes \boxplus [H^*_{\alpha,\beta}(A \infty B)]^C$
10. $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A \infty B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A \infty B)]^C$
11. $[\boxplus \boxtimes (J_{\alpha,\beta}(A \infty B))]^C = \boxtimes \boxplus [J_{\alpha,\beta}(A \infty B)]^C$
12. $[\boxtimes \boxplus (J_{\alpha,\beta}(A \infty B))]^C = \boxplus \boxtimes [J_{\alpha,\beta}(A \infty B)]^C$
13. $[\boxplus \boxtimes (J^*_{\alpha,\beta}(A \infty B))]^C = \boxtimes \boxplus [J^*_{\alpha,\beta}(A \infty B)]^C$
14. $[\boxtimes \boxplus (J^*_{\alpha,\beta}(A \infty B))]^C = \boxplus \boxtimes [J^*_{\alpha,\beta}(A \infty B)]^C$

Proof Similar to theorem 3.4.

Theorem 3.9 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus(D_\alpha(A \lhd B))]^C = \boxtimes[D_\alpha(A \lhd B)]^C$

2. $[\boxtimes(D_\alpha(A \lhd B))]^C = \boxplus[D_\alpha(A \lhd B)]^C$

3. $[\boxplus(F_{\alpha,\beta}(A \lhd B))]^C = \boxtimes[F_{\alpha,\beta}(A \lhd B)]^C$

4. $[\boxtimes(F_{\alpha,\beta}(A \lhd B))]^C = \boxplus[F_{\alpha,\beta}(A \lhd B)]^C$

5. $[\boxplus(G_{\alpha,\beta}(A \lhd B))]^C = \boxtimes[G_{\alpha,\beta}(A \lhd B)]^C$

6. $[\boxtimes(G_{\alpha,\beta}(A \lhd B))]^C = \boxplus[G_{\alpha,\beta}(A \lhd B)]^C$

7. $[\boxplus(H_{\alpha,\beta}(A \lhd B))]^C = \boxtimes[H_{\alpha,\beta}(A \lhd B)]^C$

8. $[\boxtimes(H_{\alpha,\beta}(A \lhd B))]^C = \boxplus[H_{\alpha,\beta}(A \lhd B)]^C$

9. $[\boxplus(H^*_{\alpha,\beta}(A \lhd B))]^C = \boxtimes[H^*_{\alpha,\beta}(A \lhd B)]^C$

10. $[\boxtimes(H^*_{\alpha,\beta}(A \lhd B))]^C = \boxplus[H^*_{\alpha,\beta}(A \lhd B)]^C$

11. $[\boxplus(J_{\alpha,\beta}(A \lhd B))]^C = \boxtimes[J_{\alpha,\beta}(A \lhd B)]^C$

12. $[\boxtimes(J_{\alpha,\beta}(A \lhd B))]^C = \boxplus[J_{\alpha,\beta}(A \lhd B)]^C$

13. $[\boxplus(J^*_{\alpha,\beta}(A \lhd B))]^C = \boxtimes[J^*_{\alpha,\beta}(A \lhd B)]^C$

14. $[\boxtimes(J^*_{\alpha,\beta}(A \lhd B))]^C = \boxplus[J^*_{\alpha,\beta}(A \lhd B)]^C$

Proof Similar to Theorem 3.3.

Theorem 3.10 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus \boxtimes (D_{\alpha}(A \lhd B))]^C = \boxtimes \boxplus [D_{\alpha}(A \lhd B)]^C$

2. $[\boxtimes \boxplus (D_{\alpha}(A \lhd B))]^C = \boxplus \boxtimes [D_{\alpha}(A \lhd B)]^C$

3. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \lhd B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A \lhd B)]^C$

4. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \lhd B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A \lhd B)]^C$

5. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \lhd B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A \lhd B)]^C$

6. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \lhd B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A \lhd B)]^C$

7. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \lhd B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A \lhd B)]^C$

8. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \lhd B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A \lhd B)]^C$

9. $[\boxplus \boxtimes (H^{*}_{\alpha,\beta}(A \lhd B))]^C = \boxtimes \boxplus [H^{*}_{\alpha,\beta}(A \lhd B)]^C$

10. $[\boxtimes \boxplus (H^{*}_{\alpha,\beta}(A \lhd B))]^C = \boxplus \boxtimes [H^{*}_{\alpha,\beta}(A \lhd B)]^C$

11. $[\boxplus \boxtimes (J_{\alpha,\beta}(A \lhd B))]^C = \boxtimes \boxplus [J_{\alpha,\beta}(A \lhd B)]^C$

12. $[\boxtimes \boxplus (J_{\alpha,\beta}(A \lhd B))]^C = \boxplus \boxtimes [J_{\alpha,\beta}(A \lhd B)]^C$

13. $[\boxplus \boxtimes (J^{*}_{\alpha,\beta}(A \lhd B))]^C = \boxtimes \boxplus [J^{*}_{\alpha,\beta}(A \lhd B)]^C$

14. $[\boxtimes \boxplus (J^{*}_{\alpha,\beta}(A \lhd B))]^C = \boxplus \boxtimes [J^{*}_{\alpha,\beta}(A \lhd B)]^C$

Proof Similar to theorem 3.4.

Theorem 3.11 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus(D_\alpha(A \rhd B))]^C = \boxtimes[D_\alpha(A \rhd B)]^C$

2. $[\boxtimes(D_\alpha(A \rhd B))]^C = \boxplus[D_\alpha(A \rhd B)]^C$

3. $[\boxplus(F_{\alpha,\beta}(A \rhd B))]^C = \boxtimes[F_{\alpha,\beta}(A \rhd B)]^C$

4. $[\boxtimes(F_{\alpha,\beta}(A \rhd B))]^C = \boxplus[F_{\alpha,\beta}(A \rhd B)]^C$

5. $[\boxplus(G_{\alpha,\beta}(A \rhd B))]^C = \boxtimes[G_{\alpha,\beta}(A \rhd B)]^C$

6. $[\boxtimes(G_{\alpha,\beta}(A \rhd B))]^C = \boxplus[G_{\alpha,\beta}(A \rhd B)]^C$

7. $[\boxplus(H_{\alpha,\beta}(A \rhd B))]^C = \boxtimes[H_{\alpha,\beta}(A \rhd B)]^C$

8. $[\boxtimes(H_{\alpha,\beta}(A \rhd B))]^C = \boxplus[H_{\alpha,\beta}(A \rhd B)]^C$

9. $[\boxplus(H^*_{\alpha,\beta}(A \rhd B))]^C = \boxtimes[H^*_{\alpha,\beta}(A \rhd B)]^C$

10. $[\boxtimes(H^*_{\alpha,\beta}(A \rhd B))]^C = \boxplus[H^*_{\alpha,\beta}(A \rhd B)]^C$

11. $[\boxplus(J_{\alpha,\beta}(A \rhd B))]^C = \boxtimes[J_{\alpha,\beta}(A \rhd B)]^C$

12. $[\boxtimes(J_{\alpha,\beta}(A \rhd B))]^C = \boxplus[J_{\alpha,\beta}(A \rhd B)]^C$

13. $[\boxplus(J^*_{\alpha,\beta}(A \rhd B))]^C = \boxtimes[J^*_{\alpha,\beta}(A \rhd B)]^C$

14. $[\boxtimes(J^*_{\alpha,\beta}(A \rhd B))]^C = \boxplus[J^*_{\alpha,\beta}(A \rhd B)]^C$

Proof Similar to Theorem 3.3.

Theorem 3.12 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[\boxplus \boxtimes (D_{\alpha}(A \rhd B))]^{C} = \boxtimes \boxplus [D_{\alpha}(A \rhd B)]^{C}$

2. $[\boxtimes \boxplus (D_{\alpha}(A \rhd B))]^{C} = \boxplus \boxtimes [D_{\alpha}(A \rhd B)]^{C}$

3. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \rhd B))]^{C} = \boxtimes \boxplus [F_{\alpha,\beta}(A \rhd B)]^{C}$

4. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \rhd B))]^{C} = \boxplus \boxtimes [F_{\alpha,\beta}(A \rhd B)]^{C}$

5. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \rhd B))]^{C} = \boxtimes \boxplus [G_{\alpha,\beta}(A \rhd B)]^{C}$

6. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \rhd B))]^{C} = \boxplus \boxtimes [G_{\alpha,\beta}(A \rhd B)]^{C}$

7. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \rhd B))]^{C} = \boxtimes \boxplus [H_{\alpha,\beta}(A \rhd B)]^{C}$

8. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \rhd B))]^{C} = \boxplus \boxtimes [H_{\alpha,\beta}(A \rhd B)]^{C}$

9. $[\boxplus \boxtimes (H^{*}_{\alpha,\beta}(A \rhd B))]^{C} = \boxtimes \boxplus [H^{*}_{\alpha,\beta}(A \rhd B)]^{C}$

10. $[\boxtimes \boxplus (H^{*}_{\alpha,\beta}(A \rhd B))]^{C} = \boxplus \boxtimes [H^{*}_{\alpha,\beta}(A \rhd B)]^{C}$

11. $[\boxplus \boxtimes (J_{\alpha,\beta}(A \rhd B))]^{C} = \boxtimes \boxplus [J_{\alpha,\beta}(A \rhd B)]^{C}$

12. $[\boxtimes \boxplus (J_{\alpha,\beta}(A \rhd B))]^{C} = \boxplus \boxtimes [J_{\alpha,\beta}(A \rhd B)]^{C}$

13. $[\boxplus \boxtimes (J^{*}_{\alpha,\beta}(A \rhd B))]^{C} = \boxtimes \boxplus [J^{*}_{\alpha,\beta}(A \rhd B)]^{C}$

14. $[\boxtimes \boxplus (J^{*}_{\alpha,\beta}(A \rhd B))]^{C} = \boxplus \boxtimes [J^{*}_{\alpha,\beta}(A \rhd B)]^{C}$

Proof Similar to theorem 3.4.

Theorem 3.13 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[(\boxplus(D_\alpha(A))) * (\boxtimes(D_\alpha(B)))]^C = (\boxtimes[D_\alpha(A)]^C) * (\boxplus[D_\alpha(B)]^C)$

2. $[(\boxtimes(D_\alpha(A))) * (\boxplus(D_\alpha(B)))]^C = (\boxplus[D_\alpha(A)]^C) * (\boxtimes[D_\alpha(B)]^C)$

3. $[(\boxplus(F_{\alpha,\beta}(A))) * (\boxtimes(F_{\alpha,\beta}(B)))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) * (\boxplus[F_{\alpha,\beta}(B)]^C)$

4. $[(\boxtimes(F_{\alpha,\beta}(A))) * (\boxplus(F_{\alpha,\beta}(B)))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) * (\boxtimes[F_{\alpha,\beta}(B)]^C)$

5. $[(\boxplus(G_{\alpha,\beta}(A))) * (\boxtimes(G_{\alpha,\beta}(B)))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) * (\boxplus[G_{\alpha,\beta}(B)]^C)$

6. $[(\boxtimes(G_{\alpha,\beta}(A))) * (\boxplus(G_{\alpha,\beta}(B)))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) * (\boxtimes[G_{\alpha,\beta}(B)]^C)$

7. $[(\boxplus(H_{\alpha,\beta}(A))) * (\boxtimes(H_{\alpha,\beta}(B)))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) * (\boxplus[H_{\alpha,\beta}(B)]^C)$

8. $[(\boxtimes(H_{\alpha,\beta}(A))) * (\boxplus(H_{\alpha,\beta}(B)))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) * (\boxtimes[H_{\alpha,\beta}(B)]^C)$

9. $[(\boxplus(H^*_{\alpha,\beta}(A))) * (\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) * (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

10. $[(\boxtimes(H^*_{\alpha,\beta}(A))) * (\boxplus(H^*_{\alpha,\beta}(B)))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) * (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(J_{\alpha,\beta}(A))) * (\boxtimes(J_{\alpha,\beta}(B)))]^C = (\boxtimes[J_{\alpha,\beta}(A)]^C) * (\boxplus[J_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(J_{\alpha,\beta}(A))) * (\boxplus(J_{\alpha,\beta}(B)))]^C = (\boxplus[J_{\alpha,\beta}(A)]^C) * (\boxtimes[J_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(J^*_{\alpha,\beta}(A))) * (\boxtimes(J^*_{\alpha,\beta}(B)))]^C = (\boxtimes[J^*_{\alpha,\beta}(A)]^C) * (\boxplus[J^*_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(J^*_{\alpha,\beta}(A))) * (\boxplus(J^*_{\alpha,\beta}(B)))]^C = (\boxplus[J^*_{\alpha,\beta}(A)]^C) * (\boxtimes[J^*_{\alpha,\beta}(B)]^C)$

Proof

1. Now $\boxplus(D_\alpha(A))$ **=** $\langle\frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)], \frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]\rangle$
$\boxtimes(D_\alpha(B))$ **=** $\langle\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1], \frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]\rangle$
$[\boxplus(D_\alpha(A))*(\boxtimes(D_\alpha(B))]$ **=** $\langle\frac{\frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)]+\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]}{(2(\frac{1}{2}[\mu_A(x)+\alpha\pi_B(x)]+\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]+1))}$,
$\frac{\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]+\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]}{(2(\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]+\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]+1))}\rangle$
$[\boxplus(D_\alpha(A))*(\boxtimes(D_\alpha(B))]^C$ **=** $\langle\frac{\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]+\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]}{(2(\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]+\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]+1))}$,
$\frac{\frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)]+\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]}{(2(\frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)]+\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]+1)}\rangle$
Again, $[D_\alpha(A)]^C$ **=** $\langle\nu_A(x)+(1-\alpha)\pi_A(x), \mu_A(x)+\alpha\pi_A(x)\rangle$
$\boxtimes[D_\alpha(A)]^C$ **=** $\langle\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1], \frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)]\rangle$
And $[D_\alpha(B)]^C$ **=** $\langle\nu_B(x)+(1-\alpha)\pi_B(x), \mu_B(x)+\alpha\pi_B(x)\rangle$
$\boxplus[D_\alpha(B)]^C$ **=** $\langle\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)], \frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]\rangle$
So, $(\boxtimes[D_\alpha(A)]^C)*(\boxplus[D_\alpha(B)]^C)$ **=** $\langle\frac{\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]+\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]}{(2(\frac{1}{2}[\nu_A(x)+(1-\alpha)\pi_A(x)+1]+\frac{1}{2}[\nu_B(x)+(1-\alpha)\pi_B(x)]+1))}$,
$\frac{\frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)]+\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]}{(2(\frac{1}{2}[\mu_A(x)+\alpha\pi_A(x)]+\frac{1}{2}[\mu_B(x)+\alpha\pi_B(x)+1]+1))}\rangle$
Hence $[(\boxplus(D_\alpha(A)))*(\boxtimes(D_\alpha(B)))]^C = (\boxtimes[D_\alpha(A)]^C)*(\boxplus[D_\alpha(B)]^C)$
Similarly, 2. to 14. can be proved.

Theorem 3.14 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha+\beta \leq 1$, then

1. $[(\boxplus(D_\alpha(A)))\odot(\boxtimes(D_\alpha(B)))]^C = (\boxtimes[D_\alpha(A)]^C)\odot(\boxplus[D_\alpha(B)]^C)$

2. $[(\boxtimes(D_\alpha(A)))\odot(\boxplus(D_\alpha(B)))]^C = (\boxplus[D_\alpha(A)]^C)\odot(\boxtimes[D_\alpha(B)]^C)$

3. $[(\boxplus(F_{\alpha,\beta}(A)))\odot(\boxtimes(F_{\alpha,\beta}(B)))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C)\odot(\boxplus[F_{\alpha,\beta}(B)]^C)$

4. $[(\boxtimes(F_{\alpha,\beta}(A)))\odot(\boxplus(F_{\alpha,\beta}(B)))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C)\odot(\boxtimes[F_{\alpha,\beta}(B)]^C)$

5. $[(\boxplus(G_{\alpha,\beta}(A)))\odot(\boxtimes(G_{\alpha,\beta}(B)))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C)\odot(\boxplus[G_{\alpha,\beta}(B)]^C)$

6. $[(\boxtimes(G_{\alpha,\beta}(A)))\odot(\boxplus(G_{\alpha,\beta}(B)))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C)\odot(\boxtimes[G_{\alpha,\beta}(B)]^C)$

7. $[(\boxplus(H_{\alpha,\beta}(A)))\odot(\boxtimes(H_{\alpha,\beta}(B)))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C)\odot(\boxplus[H_{\alpha,\beta}(B)]^C)$

8. $[(\boxtimes(H_{\alpha,\beta}(A)))\odot(\boxplus(H_{\alpha,\beta}(B)))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C)\odot(\boxtimes[H_{\alpha,\beta}(B)]^C)$

9. $[(\boxplus(H^*_{\alpha,\beta}(A))) \odot (\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \odot (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

10. $[(\boxtimes(H^*_{\alpha,\beta}(A))) \odot (\boxplus(H^*_{\alpha,\beta}(B)))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \odot (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(J_{\alpha,\beta}(A))) \odot (\boxtimes(J_{\alpha,\beta}(B)))]^C = (\boxtimes[J_{\alpha,\beta}(A)]^C) \odot (\boxplus[J_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(J_{\alpha,\beta}(A))) \odot (\boxplus(J_{\alpha,\beta}(B)))]^C = (\boxplus[J_{\alpha,\beta}(A)]^C) \odot (\boxtimes[J_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(J^*_{\alpha,\beta}(A))) \odot (\boxtimes(J^*_{\alpha,\beta}(B)))]^C = (\boxtimes[J^*_{\alpha,\beta}(A)]^C) \odot (\boxplus[J^*_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(J^*_{\alpha,\beta}(A))) \odot (\boxplus(J^*_{\alpha,\beta}(B)))]^C = (\boxplus[J^*_{\alpha,\beta}(A)]^C) \odot (\boxtimes[J^*_{\alpha,\beta}(B)]^C)$

Proof 9. Let us suppose that, $\alpha + \beta \leq 1$

Now $\boxplus(H^*_{\alpha,\beta}(A))$ **=** $\langle \frac{1}{2}[\alpha\mu_A(x)], \frac{1}{2}[\nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x)) + 1] \rangle$

$\boxtimes(H^*_{\alpha,\beta}(B))$ **=** $\langle \frac{1}{2}[\alpha\mu_B(x) + 1], \frac{1}{2}[\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x))] \rangle$

$[\boxplus(H^*_{\alpha,\beta}(A)) \odot (\boxtimes(H^*_{\alpha,\beta}(B))]$ **=** $\langle \frac{\frac{1}{2}[\alpha\mu_A(x)][\frac{1}{2}[\alpha\mu_B(x)+1]]}{(2(\frac{1}{2}[\alpha\mu_A(x)]\frac{1}{2}[\alpha\mu_B(x)+1]+1]}$**,**
$\frac{\frac{1}{2}[\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))+1]\frac{1}{2}[\nu_B(x)+\beta(1-\alpha\mu_B(x)-\nu_B(x))]}{(2([\frac{1}{2}[\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))+1]\frac{1}{2}[\nu_B(x)+\beta(1-\alpha\mu B(x)-nu_B(x))]+1]))} \rangle$

So, $[\boxplus(H^*_{\alpha,\beta}(A)) \odot (\boxtimes(H^*_{\alpha,\beta}(B))]^C$
= $\langle \frac{\frac{1}{2}[\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))+1]\frac{1}{2}[\nu_B(x)+\beta(1-\alpha\mu_B(x)-\nu_B(x))]}{([\frac{1}{2}[\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))+1]\frac{1}{2}[\nu_B(x)+\beta(1-\alpha\mu B(x)-nu_B(x))]+1])}$**,**
$\frac{\frac{1}{2}[\alpha\mu_A(x)][\frac{1}{2}[\alpha\mu_B(x)+1]]}{(2(\frac{1}{2}[\alpha\mu_A(x)]\frac{1}{2}[\alpha\mu_B(x)+1]+1]} \rangle$

Again, $[H^*_{\alpha,\beta}(A))]^C$ **=** $\langle \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x)), \alpha\mu_A(x) \rangle$

$\boxtimes[H^*_{\alpha,\beta}(A))]^C$ **=** $\langle \frac{1}{2}[\nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x)) + 1], \frac{1}{2}[\alpha\mu_A(x)] \rangle$

$[H^*_{\alpha,\beta}(B))]^C$ **=** $\langle \nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)), \alpha\mu_B(x) \rangle$

$\boxplus[H^*_{\alpha,\beta}(B))]^C$ **=** $\langle \frac{1}{2}[\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x))], \frac{1}{2}[\alpha\mu_B(x) + 1] \rangle$

$(\boxtimes[H^*_{\alpha,\beta}(A)]^C) \odot (\boxplus[H^*_{\alpha,\beta}(B)]^C)$
= $\langle \frac{\frac{1}{2}[\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))+1]\frac{1}{2}[\nu_B(x)+\beta(1-\alpha\mu_B(x)-\nu_B(x))]}{(2([\frac{1}{2}[\nu_A(x)+\beta(1-\alpha\mu_A(x)-\nu_A(x))+1]\frac{1}{2}[\nu_B(x)+\beta(1-\alpha\mu B(x)-nu_B(x))]+1]))}$**,**
$\frac{\frac{1}{2}[\alpha\mu_A(x)][\frac{1}{2}[\alpha\mu_B(x)+1]]}{(2(\frac{1}{2}[\alpha\mu_A(x)]\frac{1}{2}[\alpha\mu_B(x)+1]+1]} \rangle$

Hence $[(\boxplus(H^*_{\alpha,\beta}(A))) \odot (\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \odot (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

Similarly, other parts can be proved.

Theorem 3.15 Let X **be a nonempty set. If** A **and** B **be any two IFSs drawn from** X **and** $\alpha, \beta \in [0, 1]$**, where** $\alpha + \beta \leq 1$**, then**

1. $[(\boxplus(D_\alpha(A))) \bowtie (\boxtimes(D_\alpha(B)))]^C = (\boxtimes[D_\alpha(A)]^C) \bowtie (\boxplus[D_\alpha(B)]^C)$

2. $[(\boxtimes(D_\alpha(A))) \bowtie (\boxplus(D_\alpha(B)))]^C = (\boxplus[D_\alpha(A)]^C) \bowtie (\boxtimes[D_\alpha(B)]^C)$

3. $[(\boxplus(F_{\alpha,\beta}(A))) \bowtie (\boxtimes(F_{\alpha,\beta}(B)))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \bowtie (\boxplus[F_{\alpha,\beta}(B)]^C)$

4. $[(\boxtimes(F_{\alpha,\beta}(A))) \bowtie (\boxplus(F_{\alpha,\beta}(B)))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \bowtie (\boxtimes[F_{\alpha,\beta}(B)]^C)$

5. $[(\boxplus(G_{\alpha,\beta}(A))) \bowtie (\boxtimes(G_{\alpha,\beta}(B)))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \bowtie (\boxplus[G_{\alpha,\beta}(B)]^C)$

6. $[(\boxtimes(G_{\alpha,\beta}(A))) \bowtie (\boxplus(G_{\alpha,\beta}(B)))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \bowtie (\boxtimes[G_{\alpha,\beta}(B)]^C)$

7. $[(\boxplus(H_{\alpha,\beta}(A))) \bowtie (\boxtimes(H_{\alpha,\beta}(B)))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \bowtie (\boxplus[H_{\alpha,\beta}(B)]^C)$

8. $[(\boxtimes(H_{\alpha,\beta}(A))) \bowtie (\boxplus(H_{\alpha,\beta}(B)))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \bowtie (\boxtimes[H_{\alpha,\beta}(B)]^C)$

9. $[(\boxplus(H^*_{\alpha,\beta}(A))) \bowtie (\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \bowtie (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

10. $[(\boxtimes(H^*_{\alpha,\beta}(A))) \bowtie (\boxplus(H^*_{\alpha,\beta}(B)))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \bowtie (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(J_{\alpha,\beta}(A))) \bowtie (\boxtimes(J_{\alpha,\beta}(B)))]^C = (\boxtimes[J_{\alpha,\beta}(A)]^C) \bowtie (\boxplus[J_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(J_{\alpha,\beta}(A))) \bowtie (\boxplus(J_{\alpha,\beta}(B)))]^C = (\boxplus[J_{\alpha,\beta}(A)]^C) \bowtie (\boxtimes[J_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(J^*_{\alpha,\beta}(A))) \bowtie (\boxtimes(J^*_{\alpha,\beta}(B)))]^C = (\boxtimes[J^*_{\alpha,\beta}(A)]^C) \bowtie (\boxplus[J^*_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(J^*_{\alpha,\beta}(A))) \bowtie (\boxplus(J^*_{\alpha,\beta}(B)))]^C = (\boxplus[J^*_{\alpha,\beta}(A)]^C) \bowtie (\boxtimes[J^*_{\alpha,\beta}(B)]^C)$

Proof Similar to the theorem 3.14.

Theorem 3.16 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[(\boxplus(D_{\alpha}(A)))\infty(\boxtimes(D_{\alpha}(B)))]^C = (\boxtimes[D_{\alpha}(A)]^C)\infty(\boxplus[D_{\alpha}(B)]^C)$

2. $[(\boxtimes(D_{\alpha}(A)))\infty(\boxplus(D_{\alpha}(B)))]^C = (\boxplus[D_{\alpha}(A)]^C)\infty(\boxtimes[D_{\alpha}(B)]^C)$

3. $[(\boxplus(F_{\alpha,\beta}(A)))\infty(\boxtimes(F_{\alpha,\beta}(B)))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C)\infty(\boxplus[F_{\alpha,\beta}(B)]^C)$

4. $[(\boxtimes(F_{\alpha,\beta}(A)))\infty(\boxplus(F_{\alpha,\beta}(B)))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C)\infty(\boxtimes[F_{\alpha,\beta}(B)]^C)$

5. $[(\boxplus(G_{\alpha,\beta}(A)))\infty(\boxtimes(G_{\alpha,\beta}(B)))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C)\infty(\boxplus[G_{\alpha,\beta}(B)]^C)$

6. $[(\boxtimes(G_{\alpha,\beta}(A)))\infty(\boxplus(G_{\alpha,\beta}(B)))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C)\infty(\boxtimes[G_{\alpha,\beta}(B)]^C)$

7. $[(\boxplus(H_{\alpha,\beta}(A)))\infty(\boxtimes(H_{\alpha,\beta}(B)))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C)\infty(\boxplus[H_{\alpha,\beta}(B)]^C)$

8. $[(\boxtimes(H_{\alpha,\beta}(A)))\infty(\boxplus(H_{\alpha,\beta}(B)))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C)\infty(\boxtimes[H_{\alpha,\beta}(B)]^C)$

9. $[(\boxplus(H^*_{\alpha,\beta}(A)))\infty(\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C)\infty(\boxplus[H^*_{\alpha,\beta}(B)]^C)$

10. $[(\boxtimes(H^*_{\alpha,\beta}(A)))\infty(\boxplus(H^*_{\alpha,\beta}(B)))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C)\infty(\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(J_{\alpha,\beta}(A)))\infty(\boxtimes(J_{\alpha,\beta}(B)))]^C = (\boxtimes[J_{\alpha,\beta}(A)]^C)\infty(\boxplus[J_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(J_{\alpha,\beta}(A)))\infty(\boxplus(J_{\alpha,\beta}(B)))]^C = (\boxplus[J_{\alpha,\beta}(A)]^C)\infty(\boxtimes[J_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(J^*_{\alpha,\beta}(A)))\infty(\boxtimes(J^*_{\alpha,\beta}(B)))]^C = (\boxtimes[J^*_{\alpha,\beta}(A)]^C)\infty(\boxplus[J^*_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(J^*_{\alpha,\beta}(A)))\infty(\boxplus(J^*_{\alpha,\beta}(B)))]^C = (\boxplus[J^*_{\alpha,\beta}(A)]^C)\infty(\boxtimes[J^*_{\alpha,\beta}(B)]^C)$

Proof Similar to the theorem 3.14.

Theorem 3.17 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[(\boxplus(D_\alpha(A))) \lhd (\boxtimes(D_\alpha(B)))]^C = (\boxtimes[D_\alpha(A)]^C) \lhd (\boxplus[D_\alpha(B)]^C)$

2. $[(\boxtimes(D_\alpha(A))) \lhd (\boxplus(D_\alpha(B)))]^C = (\boxplus[D_\alpha(A)]^C) \lhd (\boxtimes[D_\alpha(B)]^C)$

3. $[(\boxplus(F_{\alpha,\beta}(A))) \lhd (\boxtimes(F_{\alpha,\beta}(B)))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \lhd (\boxplus[F_{\alpha,\beta}(B)]^C)$

4. $[(\boxtimes(F_{\alpha,\beta}(A))) \lhd (\boxplus(F_{\alpha,\beta}(B)))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \lhd (\boxtimes[F_{\alpha,\beta}(B)]^C)$

5. $[(\boxplus(G_{\alpha,\beta}(A))) \lhd (\boxtimes(G_{\alpha,\beta}(B)))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \lhd (\boxplus[G_{\alpha,\beta}(B)]^C)$

6. $[(\boxtimes(G_{\alpha,\beta}(A))) \lhd (\boxplus(G_{\alpha,\beta}(B)))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \lhd (\boxtimes[G_{\alpha,\beta}(B)]^C)$

7. $[(\boxplus(H_{\alpha,\beta}(A))) \lhd (\boxtimes(H_{\alpha,\beta}(B)))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \lhd (\boxplus[H_{\alpha,\beta}(B)]^C)$

8. $[(\boxtimes(H_{\alpha,\beta}(A))) \lhd (\boxplus(H_{\alpha,\beta}(B)))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \lhd (\boxtimes[H_{\alpha,\beta}(B)]^C)$

9. $[(\boxplus(H^*_{\alpha,\beta}(A))) \lhd (\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \lhd (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

10. $[(\boxtimes(H^*_{\alpha,\beta}(A))) \lhd (\boxplus(H^*_{\alpha,\beta}(B)))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \lhd (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(J_{\alpha,\beta}(A))) \lhd (\boxtimes(J_{\alpha,\beta}(B)))]^C = (\boxtimes[J_{\alpha,\beta}(A)]^C) \lhd (\boxplus[J_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(J_{\alpha,\beta}(A))) \lhd (\boxplus(J_{\alpha,\beta}(B)))]^C = (\boxplus[J_{\alpha,\beta}(A)]^C) \lhd (\boxtimes[J_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(J^*_{\alpha,\beta}(A))) \lhd (\boxtimes(J^*_{\alpha,\beta}(B)))]^C = (\boxtimes[J^*_{\alpha,\beta}(A)]^C) \lhd (\boxplus[J^*_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(J^*_{\alpha,\beta}(A))) \lhd (\boxplus(J^*_{\alpha,\beta}(B)))]^C = (\boxplus[J^*_{\alpha,\beta}(A)]^C) \lhd (\boxtimes[J^*_{\alpha,\beta}(B)]^C)$

Proof Similar to the theorem 3.14.

Theorem 3.18 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $[(\boxplus(D_{\alpha}(A))) \rhd (\boxtimes(D_{\alpha}(B)))]^C = (\boxtimes[D_{\alpha}(A)]^C) \rhd (\boxplus[D_{\alpha}(B)]^C)$

2. $[(\boxtimes(D_{\alpha}(A))) \rhd (\boxplus(D_{\alpha}(B)))]^C = (\boxplus[D_{\alpha}(A)]^C) \rhd (\boxtimes[D_{\alpha}(B)]^C)$

3. $[(\boxplus(F_{\alpha,\beta}(A))) \rhd (\boxtimes(F_{\alpha,\beta}(B)))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \rhd (\boxplus[F_{\alpha,\beta}(B)]^C)$

4. $[(\boxtimes(F_{\alpha,\beta}(A))) \rhd (\boxplus(F_{\alpha,\beta}(B)))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \rhd (\boxtimes[F_{\alpha,\beta}(B)]^C)$

5. $[(\boxplus(G_{\alpha,\beta}(A))) \rhd (\boxtimes(G_{\alpha,\beta}(B)))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \rhd (\boxplus[G_{\alpha,\beta}(B)]^C)$

6. $[(\boxtimes(G_{\alpha,\beta}(A))) \triangleright (\boxplus(G_{\alpha,\beta}(B)))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \triangleright (\boxtimes[G_{\alpha,\beta}(B)]^C)$

7. $[(\boxplus(H_{\alpha,\beta}(A))) \triangleright (\boxtimes(H_{\alpha,\beta}(B)))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \triangleright (\boxplus[H_{\alpha,\beta}(B)]^C)$

8. $[(\boxtimes(H_{\alpha,\beta}(A))) \triangleright (\boxplus(H_{\alpha,\beta}(B)))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \triangleright (\boxtimes[H_{\alpha,\beta}(B)]^C)$

9. $[(\boxplus(H^*_{\alpha,\beta}(A))) \triangleright (\boxtimes(H^*_{\alpha,\beta}(B)))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \triangleright (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

10. $[(\boxtimes(H^*_{\alpha,\beta}(A))) \triangleright (\boxplus(H^*_{\alpha,\beta}(B)))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \triangleright (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(J_{\alpha,\beta}(A))) \triangleright (\boxtimes(J_{\alpha,\beta}(B)))]^C = (\boxtimes[J_{\alpha,\beta}(A)]^C) \triangleright (\boxplus[J_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(J_{\alpha,\beta}(A))) \triangleright (\boxplus(J_{\alpha,\beta}(B)))]^C = (\boxplus[J_{\alpha,\beta}(A)]^C) \triangleright (\boxtimes[J_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(J^*_{\alpha,\beta}(A))) \triangleright (\boxtimes(J^*_{\alpha,\beta}(B)))]^C = (\boxtimes[J^*_{\alpha,\beta}(A)]^C) \triangleright (\boxplus[J^*_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(J^*_{\alpha,\beta}(A))) \triangleright (\boxplus(J^*_{\alpha,\beta}(B)))]^C = (\boxplus[J^*_{\alpha,\beta}(A)]^C) \triangleright (\boxtimes[J^*_{\alpha,\beta}(B)]^C)$

Proof Similar to the theorem 3.14.

4 Some Observations

Let $A = \langle .7, .2, .1\rangle$, $B = \langle .6, .2, .2\rangle$ be two intuitionistic fuzzy sets and # be any operation defined in definition 2.5. Let us also consider $\alpha = .4$ and $\beta = .5$. Now we construct the following tables.

Table - 1

$\#$	$D_\alpha(A\#B)$	$\boxplus D_\alpha(A\#B)$	$\boxplus\boxplus D_\alpha(A\#B)$	$\boxtimes D_\alpha(A\#B)$	$\boxtimes\boxtimes D_\alpha(A\#B)$
$*$	$\langle .5124, .4876\rangle$	$\langle .2562, .7438\rangle$	$\langle .1281, .8719\rangle$	$\langle .7562, .2438\rangle$	$\langle .8781, .1219\rangle$
$\odot$	$\langle .4811, .5189\rangle$	$\langle .2405, .7595\rangle$	$\langle .1202, .8798\rangle$	$\langle .7406, .2594\rangle$	$\langle .8703, .1297\rangle$
$\bowtie$	$\langle .5278, .4722\rangle$	$\langle .2639, .7361\rangle$	$\langle .1320, .8680\rangle$	$\langle .7639, .2361\rangle$	$\langle .8820, .1180\rangle$
∞	$\langle .5222, .4778\rangle$	$\langle .2611, .7389\rangle$	$\langle .1305, .8695\rangle$	$\langle .7611, .2389\rangle$	$\langle .8805, .1195\rangle$
$\triangleleft$	$\langle .6248, .3752\rangle$	$\langle .3124, .6876\rangle$	$\langle .1562, .8438\rangle$	$\langle .8124, .1876\rangle$	$\langle .9062, .0938\rangle$
$\triangleright$	$\langle .5621, .4379\rangle$	$\langle .2810, .7190\rangle$	$\langle .1405, .8595\rangle$	$\langle .7810, .2190\rangle$	$\langle .8905, .1095\rangle$

Table - 2

$\#$	$F_{\alpha,\beta}(A\#B)$	$\boxplus F_{\alpha,\beta}(A\#B)$	$\boxplus\boxplus F_{\alpha,\beta}(A\#B)$	$\boxtimes F_{\alpha,\beta}(A\#B)$	$\boxtimes\boxtimes F_{\alpha,\beta}(A\#B)$
$*$	$\langle .5124, .4301\rangle$	$\langle .2562, .7151\rangle$	$\langle .1281, .8575\rangle$	$\langle .7562, .2150\rangle$	$\langle .8781, .1075\rangle$
$\odot$	$\langle .4811, .4356\rangle$	$\langle .2405, .7178\rangle$	$\langle .1202, .8589\rangle$	$\langle .7406, .2178\rangle$	$\langle .8703, .1089\rangle$
$\bowtie$	$\langle .5278, .4305\rangle$	$\langle .2639, .7152\rangle$	$\langle .1320, .8576\rangle$	$\langle .7639, .2152\rangle$	$\langle .8820, .1076\rangle$
∞	$\langle .5222, .4044\rangle$	$\langle .2611, .7022\rangle$	$\langle .1305, .8511\rangle$	$\langle .7611, .2022\rangle$	$\langle .8805, .1011\rangle$
$\lhd$	$\langle .6248, .3602\rangle$	$\langle .3124, .6801\rangle$	$\langle .1562, .8401\rangle$	$\langle .8124, .1801\rangle$	$\langle .9062, .0901\rangle$
$\rhd$	$\langle .5621, .3714\rangle$	$\langle .2810, .6857\rangle$	$\langle .1405, .8429\rangle$	$\langle .7810, .1857\rangle$	$\langle .8905, .0929\rangle$

Table - 3

$\#$	$G_{\alpha,\beta}(A\#B)$	$\boxplus G_{\alpha,\beta}(A\#B)$	$\boxplus\boxplus G_{\alpha,\beta}(A\#B)$	$\boxtimes G_{\alpha,\beta}(A\#B)$	$\boxtimes\boxtimes G_{\alpha,\beta}(A\#B)$
$*$	$\langle .1130, .0714\rangle$	$\langle .0565, .5357\rangle$	$\langle .0282, .7678\rangle$	$\langle .5565, .0357\rangle$	$\langle .7782, .0178\rangle$
$\odot$	$\langle .0592, .0096\rangle$	$\langle .0296, .5048\rangle$	$\langle .0148, .7524\rangle$	$\langle .5296, .0048\rangle$	$\langle .7648, .0024\rangle$
$\bowtie$	$\langle .1444, .1111\rangle$	$\langle .0722, .5556\rangle$	$\langle .0361, .7778\rangle$	$\langle .5722, .0556\rangle$	$\langle .7861, .0278\rangle$
∞	$\langle .0913, .0185\rangle$	$\langle .0456, .5092\rangle$	$\langle .0228, .7546\rangle$	$\langle .5456, .0092\rangle$	$\langle .7728, .0046\rangle$
$\lhd$	$\langle .2261, .1428\rangle$	$\langle .1131, .5714\rangle$	$\langle .0565, .7857\rangle$	$\langle .6131, .0714\rangle$	$\langle .8065, .0357\rangle$
$\rhd$	$\langle .1183, .0193\rangle$	$\langle .0592, .5097\rangle$	$\langle .0296, .7548\rangle$	$\langle .5592, .0096\rangle$	$\langle .7796, .0048\rangle$

Table - 4

$\#$	$H_{\alpha,\beta}(A\#B)$	$\boxplus H_{\alpha,\beta}(A\#B)$	$\boxplus\boxplus H_{\alpha,\beta}(A\#B)$	$\boxtimes H_{\alpha,\beta}(A\#B)$	$\boxtimes\boxtimes H_{\alpha,\beta}(A\#B)$
$*$	$\langle .1130, .4301\rangle$	$\langle .0565, .7151\rangle$	$\langle .0282, .8575\rangle$	$\langle .5565, .2150\rangle$	$\langle .7782, .1075\rangle$
$\odot$	$\langle .0592, .4356\rangle$	$\langle .0296, .7178\rangle$	$\langle .0148, .8589\rangle$	$\langle .5296, .2178\rangle$	$\langle .7648, .1089\rangle$
$\bowtie$	$\langle .1444, .4305\rangle$	$\langle .0722, .7152\rangle$	$\langle .0361, .8576\rangle$	$\langle .5722, .2152\rangle$	$\langle .7861, .1076\rangle$
∞	$\langle .0913, .4044\rangle$	$\langle .0456, .7022\rangle$	$\langle .0228, .8511\rangle$	$\langle .5456, .2022\rangle$	$\langle .7728, .1011\rangle$
$\lhd$	$\langle .2261, .3602\rangle$	$\langle .1131, .6801\rangle$	$\langle .0565, .8401\rangle$	$\langle .6131, .1801\rangle$	$\langle .8065, .0901\rangle$
$\rhd$	$\langle .1183, .3714\rangle$	$\langle .0592, .6857\rangle$	$\langle .0296, .8429\rangle$	$\langle .5592, .1857\rangle$	$\langle .7796, .0929\rangle$

Table - 5

$\#$	$H^*_{\alpha,\beta}(A\#B)$	$\boxplus H^*_{\alpha,\beta}(A\#B)$	$\boxplus\boxplus H^*_{\alpha,\beta}(A\#B)$	$\boxtimes H^*_{\alpha,\beta}(A\#B)$	$\boxtimes\boxtimes H^*_{\alpha,\beta}(A\#B)$
$*$	$\langle .1130, .3721\rangle$	$\langle .0565, .6861\rangle$	$\langle .0282, .8431\rangle$	$\langle .5565, .1861\rangle$	$\langle .7782, .0931\rangle$
$\odot$	$\langle .0592, .4608\rangle$	$\langle .0296, .7304\rangle$	$\langle .0148, .8652\rangle$	$\langle .5296, .2304\rangle$	$\langle .7648, .1152\rangle$
$\bowtie$	$\langle .1444, .3167\rangle$	$\langle .0722, .6584\rangle$	$\langle .0361, .8292\rangle$	$\langle .5722, .1584\rangle$	$\langle .7861, .0792\rangle$
∞	$\langle .0913, .4358\rangle$	$\langle .0456, .7179\rangle$	$\langle .0228, .8590\rangle$	$\langle .5456, .2179\rangle$	$\langle .7728, .1090\rangle$
$\lhd$	$\langle .2261, .2441\rangle$	$\langle .1131, .6221\rangle$	$\langle .0565, .8111\rangle$	$\langle .6131, .1221\rangle$	$\langle .8065, .0611\rangle$
$\rhd$	$\langle .1183, .4216\rangle$	$\langle .0592, .7108\rangle$	$\langle .0296, .8554\rangle$	$\langle .5592, .2108\rangle$	$\langle .7796, .1054\rangle$

Table - 6

$\#$	$J_{\alpha,\beta}(A\#B)$	$\boxplus J_{\alpha,\beta}(A\#B)$	$\boxplus\boxplus J_{\alpha,\beta}(A\#B)$	$\boxtimes J_{\alpha,\beta}(A\#B)$	$\boxtimes\boxtimes J_{\alpha,\beta}(A\#B)$
$*$	$\langle .5124, .0714\rangle$	$\langle .2562, .5357\rangle$	$\langle .1281, .7678\rangle$	$\langle .7562, .0357\rangle$	$\langle .8781, .0178\rangle$
$\odot$	$\langle .4811, .0096\rangle$	$\langle .2405, .5048\rangle$	$\langle .1202, .7524\rangle$	$\langle .7406, .0048\rangle$	$\langle .8703, .0024\rangle$
$\bowtie$	$\langle .5278, .1111\rangle$	$\langle .2639, .5556\rangle$	$\langle .1320, .7778\rangle$	$\langle .7639, .0556\rangle$	$\langle .8820, .0278\rangle$
∞	$\langle .5222, .0185\rangle$	$\langle .2611, .5092\rangle$	$\langle .1305, .7546\rangle$	$\langle .7611, .0092\rangle$	$\langle .8805, .0046\rangle$
$\lhd$	$\langle .6248, .1428\rangle$	$\langle .3124, .5714\rangle$	$\langle .1562, .7857\rangle$	$\langle .8124, .0714\rangle$	$\langle .9062, .0357\rangle$
$\rhd$	$\langle .5621, .0193\rangle$	$\langle .2810, .5097\rangle$	$\langle .1405, .7548\rangle$	$\langle .7810, .0096\rangle$	$\langle .8905, .0048\rangle$

Table - 7

$\#$	$J^*_{\alpha,\beta}(A\#B)$	$\boxplus J^*_{\alpha,\beta}(A\#B)$	$\boxplus\boxplus J^*_{\alpha,\beta}(A\#B)$	$\boxtimes J^*_{\alpha,\beta}(A\#B)$	$\boxtimes\boxtimes J^*_{\alpha,\beta}(A\#B)$
$*$	$\langle .5410, .0714\rangle$	$\langle .2705, .5357\rangle$	$\langle .1353, .7678\rangle$	$\langle .7705, .0357\rangle$	$\langle .8853, .0178\rangle$
$\odot$	$\langle .4849, .0096\rangle$	$\langle .2425, .5048\rangle$	$\langle .1213, .7524\rangle$	$\langle .7425, .0048\rangle$	$\langle .8713, .0024\rangle$
$\bowtie$	$\langle .5722, .1111\rangle$	$\langle .2861, .5556\rangle$	$\langle .1431, .7778\rangle$	$\langle .7861, .0556\rangle$	$\langle .8931, .0278\rangle$
∞	$\langle .5296, .0185\rangle$	$\langle .2648, .5094\rangle$	$\langle .1324, .7546\rangle$	$\langle .7648, .0092\rangle$	$\langle .8824, .0046\rangle$
$\lhd$	$\langle .6820, .1428\rangle$	$\langle .3410, .5714\rangle$	$\langle .1705, .7857\rangle$	$\langle .8410, .0714\rangle$	$\langle .9205, .0357\rangle$
$\rhd$	$\langle .5698, .0193\rangle$	$\langle .2849, .5097\rangle$	$\langle .1425, .7548\rangle$	$\langle .7849, .0096\rangle$	$\langle .8925, .0048\rangle$

From the above tables, some properties of IFSs on the basis of different operations are observed. The properties are:

I. $\boxplus\boxplus D_\alpha(A\#B) \subset \boxplus D_\alpha(A\#B) \subset D_\alpha(A\#B) \subset \boxtimes D_\alpha(A\#B) \subset \boxtimes\boxtimes D_\alpha(A\#B)$
II. $\boxplus\boxplus F_{\alpha,\beta}(A\#B) \subset \boxplus F_{\alpha,\beta}(A\#B) \subset F_{\alpha,\beta}(A\#B) \subset \boxtimes F_{\alpha,\beta}(A\#B) \subset \boxtimes\boxtimes F_{\alpha,\beta}(A\#B)$
III. $\boxplus\boxplus G_{\alpha,\beta}(A\#B) \subset \boxplus G_{\alpha,\beta}(A\#B) \subset G_{\alpha,\beta}(A\#B) \subset \boxtimes G_{\alpha,\beta}(A\#B) \subset \boxtimes\boxtimes G_{\alpha,\beta}(A\#B)$
IV. $\boxplus\boxplus H_{\alpha,\beta}(A\#B) \subset \boxplus H_{\alpha,\beta}(A\#B) \subset H_{\alpha,\beta}(A\#B) \subset \boxtimes H_{\alpha,\beta}(A\#B) \subset \boxtimes\boxtimes H_{\alpha,\beta}(A\#B)$
V. $\boxplus\boxplus H^*_{\alpha,\beta}(A\#B) \subset \boxplus H^*_{\alpha,\beta}(A\#B) \subset H^*_{\alpha,\beta}(A\#B) \subset \boxtimes H^*_{\alpha,\beta}(A\#B) \subset \boxtimes\boxtimes H^*_{\alpha,\beta}(A\#B)$
VI.$\boxplus\boxplus J_{\alpha,\beta}(A\#B) \subset \boxplus J_{\alpha,\beta}(A\#B) \subset J_{\alpha,\beta}(A\#B) \subset \boxtimes J_{\alpha,\beta}(A\#B) \subset \boxtimes\boxtimes J_{\alpha,\beta}(A\#B)$
VII. $\boxplus\boxplus J^*_{\alpha,\beta}(A\#B) \subset \boxplus J^*_{\alpha,\beta}(A\#B) \subset J^*_{\alpha,\beta}(A\#B) \subset \boxtimes J^*_{\alpha,\beta}(A\#B) \subset \boxtimes\boxtimes J^*_{\alpha,\beta}(A\#B)$

5 Conclusion

In this paper, some properties of modal operators along with some special operators defined on intuitionistic fuzzy sets have been investigated. Some intuitionistic fuzzy operators are also included in our investigation. As a result some new equalities are obtained. In the near future, these equalities will undoubtedly aid in our investigation of numerous additional features related to intuitionistic fuzzy operators. There are numerous application areas in which these results could find utility.

References

[1] **Atanassov, K. T. (1983). Intuitionistic fuzzy sets. *VII ITKR's Session, Sofia*,(Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian) Reprinted:International Journal of Bioautomation, 2016, 20(1), S1–S6**

[2] **Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3**

[3] **Atanassov, K. T. (1994). New operations defined over Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 61(2), 137-142.https://doi.org/10.1016/0165-0114(94)90229-1**

[4] Atanassov, K. T. (1997). Some operators in Intuitionistic fuzzy sets. *First Int. Conf. on IFS, Sofia, NIFS*, 3(4), 28–33.

[5] Atanassov, K. T. (2017). Type-I fuzzy sets and intuitionistic fuzzy sets. *Algorithms*, 10(3) 1-12. https://doi.org/10.3390/a10030106

[6] Bhattacharya, J. (2016). A few more on Intuitionistic fuzzy set. *Journal of Fuzzy set valued Analysis*, 3, 214-222. https://doi:10.5899/2016/jfsva-00322

[7] Bhattacharya, J. (2021). Some special operations and related results on Intuitionistic fuzzy sets. *Int. Journal of Scientific Research in Mathematical and Statistical Sciences*, 8(4), 10–13. DOI: https://doi.org/10.26438/ijsrmss/v8i4.1013

[8] Bhattacharya, J. (2021). Some results on certain properties of Intuitionistic fuzzy sets. *Journal of Fuzzy Extension and Applications*, 34(5), 377–387. https://doi.org/10.22105/jfea.2021.290202.1155

[9] Citil, M., & Tugrul, F. (2018). Some new equalities on the intuitionistic fuzzy modal operators. *Sakarya University Journal of Science*, 22(6), 1524–1531. https://doi.org/10.16984/saufenbilder.338899

[10] Ejegwa, P.A., Akowe, S.O., Otene, P. M. & Ikyule, J.M. (2014). An overview on intuitionistic fuzzy sets. *Int. Journal of Scientific and Technology Research*, 3(3), 142–145.

[11] Eker, E., Tagrul, F. & Citil, M. (2015). New equalities on the intuitionistic fuzzy operators and operations. *Notes on Intuitionistic Fuzzy Sets*, 21(4), 124–128.

[12] Kozae, A.M., Elshenawy, A. & Omran, M. (2015). Intuitionistic fuzzy set and its Application in selecting specialization: a case study for engineering students. *Int. Journal of Mathematical Analysis and Applications*,2(6), 74–78.

[13] Tarsuslu, S. , Citil, M., Dimirbas, E. & Aydin, M. (2017). Some modal operators with intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 23(5), 20–28.

[14] Zadeh, L.A. (2018). Fuzzy sets. *Information and Control*, 8, 338–353.

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